



Note

On the maximum number of edges in a hypergraph with a unique perfect matching

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ABSTRACT

In this note, we determine the maximum number of edges of a k -uniform hypergraph, $k \geq 3$, with a unique perfect matching. This settles a conjecture proposed by Snevily.

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1. Introduction

Let $\mathcal{H} = (V, \mathcal{E})$, $\mathcal{E} \subseteq \binom{V}{k}$, be a k -uniform hypergraph (or k -graph) on km vertices for $m \in \mathbb{N}$. A perfect matching in \mathcal{H} is a collection of edges $\{M_1, M_2, \dots, M_m\} \subseteq \mathcal{E}$ such that $M_i \cap M_j = \emptyset$ for all $i \neq j$ and $\bigcup_i M_i = V$. In this note we are interested in the maximum number of edges of a hypergraph \mathcal{H} with a unique perfect matching. Heteyi observed (see, e.g., [1–3]) that for ordinary graphs (i.e. $k = 2$), this number cannot exceed m^2 . To see this, note that at most two edges may join any pair of edges from the matching. Thus the number of edges is bounded from above by $m + 2 \binom{m}{2} = m^2$. Heteyi also provides a unique graph satisfying the above conditions. His construction can be easily generalized to uniform hypergraphs (see Section 2 for details). Snevily [4] anticipated that such generalization is optimal. Here we present our main result.

Theorem 1.1. For integers $k \geq 2$ and $m \geq 1$ let

$$f(k, m) = m + b_{k,2} \binom{m}{2} + b_{k,3} \binom{m}{3} + \dots + b_{k,k} \binom{m}{k},$$

where

$$b_{k,\ell} = \frac{\ell-1}{\ell} \sum_{i=0}^{\ell-1} (-1)^i \binom{\ell}{i} \binom{k(\ell-i)}{k}.$$

Let $\mathcal{H} = (V, \mathcal{E})$ be a k -graph of order km with a unique perfect matching. Then

$$|\mathcal{E}| \leq f(k, m). \quad (1.1)$$

Moreover, (1.1) is tight.

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In particular, if $\mathcal{H} = (V, \mathcal{E})$ is a 3-uniform hypergraph of order $3m$ with a unique perfect matching, then

$$|\mathcal{E}| \leq f(3, m) = m + 9 \binom{m}{2} + 18 \binom{m}{3} = \frac{5m}{2} - \frac{9m^2}{2} + 3m^3.$$

2. Construction

In this section, we provide a recursive construction of a hypergraph \mathcal{H}_m^* of order km with a unique perfect matching and containing exactly $f(k, m)$ edges.

Let \mathcal{H}_1^* be a k -graph on k vertices with exactly one edge. Trivially, this graph has a unique perfect matching. Suppose we already constructed a k -graph \mathcal{H}_{m-1}^* on $k(m-1)$ vertices with a unique perfect matching. To construct the graph \mathcal{H}_m^* on km vertices, add $k-1$ new vertices to \mathcal{H}_{m-1}^* and add all edges containing at least one of these new vertices. Then, add another new vertex and draw the edge containing the k new vertices. Formally, let

$$M_i = \{k(i-1) + 1, \dots, ki\} \quad \text{for } i = 1, \dots, m. \quad (2.1)$$

Let $\mathcal{H}_m^* = (V_m, \mathcal{E}_m)$, $m \geq 1$, be a k -graph on km vertices with the vertex set

$$V_m = \{1, \dots, km\} = \bigcup_{i=1}^m M_i$$

and the edge set (defined recursively)

$$\mathcal{E}_m = \mathcal{E}_{m-1} \cup \left\{ E \in \binom{V_m}{k} : E \cap M_m \neq \emptyset, km \notin E \right\} \cup \{M_m\},$$

where $\mathcal{E}_0 = \emptyset$.

Note that \mathcal{H}_m^* has a unique perfect matching, namely, $\mathcal{M}_m = \{M_1, M_2, \dots, M_m\}$. To see this, observe that the vertex km is only included in edge M_m . Hence, any matching must include M_m . Removing all vertices in M_m , we see that M_{m-1} must be also included and so on. We call the elements of \mathcal{M}_m , *matching edges*.

Claim 2.1. The k -graph $\mathcal{H}_m^* = (V_m, \mathcal{E}_m)$ satisfies $|\mathcal{E}_m| = f(k, m)$.

Proof. For $\ell = 1, 2, \dots, k$, let \mathcal{B}_ℓ be the set of edges that intersect exactly ℓ matching edges, i.e.,

$$\mathcal{B}_\ell = \left\{ E \in \mathcal{E}_m : \sum_{i=1}^m \mathbf{1}_{E \cap M_i \neq \emptyset} = \ell \right\}.$$

Note that $\mathcal{E}_m = \bigcup_{\ell} \mathcal{B}_\ell$. Clearly, $|\mathcal{B}_1| = |\{M_1, \dots, M_m\}| = m$, giving us the first term in $f(k, m)$. Now we show that $|\mathcal{B}_\ell| = b_{k,\ell} \binom{m}{\ell}$ for $\ell = 2, \dots, k$. Let $\mathcal{L} = \{M_{i_1}, M_{i_2}, \dots, M_{i_\ell}\} \subseteq \mathcal{M}_m$ be any set of ℓ matching edges with $1 \leq i_1 < i_2 < \dots < i_\ell \leq m$. Let \mathcal{G} be the collection of k -sets on the vertex set of \mathcal{L} which intersect all of $M_{i_1}, \dots, M_{i_\ell}$. The principle of inclusion and exclusion (conditioning on the number of k -sets that do not intersect a given subset of matching edges) yields that

$$|\mathcal{G}| = \sum_{i=0}^{\ell-1} (-1)^i \binom{\ell}{i} \binom{k(\ell-i)}{k}.$$

Now note that due to the symmetry of the roles of the vertices in \mathcal{G} , each vertex belongs to the same number of edges of \mathcal{G} , say η . Consequently, the number of pairs (x, E) , $x \in E \in \mathcal{G}$ equals $k\ell\eta$. On the other hand, since every edge of \mathcal{G} consists of k vertices we get that the number of pairs is equal to $|\mathcal{G}|k$, implying that $\eta = |\mathcal{G}|/\ell$.

By construction, $E \in \mathcal{G}$ implies $E \in \mathcal{B}_\ell$ unless vertex ki_ℓ is in E . As

$$|\{E \in \mathcal{G} : ki_\ell \in E\}| = \eta = |\mathcal{G}|/\ell,$$

the number of edges of \mathcal{B}_ℓ on the vertex set of \mathcal{L} equals

$$\frac{\ell-1}{\ell} |\mathcal{G}| = b_{k,\ell}. \quad (2.2)$$

As this argument applies to any choice of ℓ matching edges, we have $|\mathcal{B}_\ell| = b_{k,\ell} \binom{m}{\ell}$, thus proving the claim. \square

Corollary 2.2. For all integers $k \geq 2$ and $m \geq 1$,

$$f(k, m) = m + \sum_{i=1}^{m-1} \left[\binom{k(i+1)-1}{k} - \binom{ki}{k} \right].$$

Proof. We prove this by counting the edges of $\mathcal{H}_m^* = (V_m, \mathcal{E}_m)$ in a different way. Let $a_m = |\mathcal{E}_m|$, $m \geq 1$. Then it is easy to see that the following recurrence relation holds: $a_1 = 1$ and

$$a_m = a_{m-1} + \binom{km-1}{k} - \binom{k(m-1)}{k} + 1 \quad \text{for } m \geq 2, \quad (2.3)$$

where the first binomial coefficient counts all the edges that do not contain vertex km ; the second coefficient counts all the edges which do not intersect the matching edge M_m (cf. (2.1)); and the term 1 stands for M_m itself. Summing (2.3) over $m, m-1, \dots, 2$ gives the desired formula. \square

Note that \mathcal{H}_m^* proves that (1.1) is tight. However, in contrast to the case of $k=2$, there are hypergraphs on km vertices containing a unique perfect matching and $f(k, m)$ edges which are not isomorphic to \mathcal{H}_m^* . For example, if $m=2$, consider an edge $E \in \mathcal{H}_2^*$, $E \neq M_1, M_2$. Let \bar{E} be the complement of E , i.e., $\bar{E} = \{1, \dots, 2k\} \setminus E$. Then, the hypergraph obtained from \mathcal{H}_2^* by replacing E with \bar{E} provides a non-isomorphic example for the tightness of (1.1).

3. Proof of Theorem 1.1

We start with some definitions. We use the terms “edge” and “ k -set” interchangeably.

Definition 3.1. Given any collection of $2 \leq \ell \leq k$ disjoint edges $\mathcal{L} = \{M_1, \dots, M_\ell\}$, we call a collection of edges $\mathcal{C} = \{C_1, \dots, C_\ell\}$ a *covering* of \mathcal{L} if

- $C_i \cap M_j \neq \emptyset$ for all $i, j \in \{1, \dots, \ell\}$, and
- $\bigcup_i C_i = \bigcup_i M_i$.

Note that the second condition forces the edges in a covering to be disjoint.

Definition 3.2. Let \mathcal{L} be as in Definition 3.1, let \mathcal{C} be a covering of \mathcal{L} and let $C \in \mathcal{C}$. We say C is of type \vec{a} if

- $\vec{a} = (a_1, \dots, a_\ell) \in \mathbb{N}^\ell$, $\sum_i a_i = k$ and $a_1 \geq a_2 \geq \dots \geq a_\ell \geq 1$, and
- there exists a permutation σ of $\{1, 2, \dots, \ell\}$ such that $|C \cap M_{\sigma(i)}| = a_i$ for each $1 \leq i \leq \ell$.

Let $\mathcal{A}_{k,\ell} = \{\vec{a} = (a_1, \dots, a_\ell) \in \mathbb{N}^\ell : a_1 \geq a_2 \geq \dots \geq a_\ell \geq 1 \text{ and } a_1 + \dots + a_\ell = k\}$.

Given a vector $\vec{a} \in \mathcal{A}_{k,\ell}$, let $\mathcal{C}_{\vec{a}}$ be the collection of all coverings \mathcal{C} of \mathcal{L} such that every $C \in \mathcal{C}$ is of type \vec{a} . In other words, $\mathcal{C}_{\vec{a}}$ consists of coverings using only edges of type \vec{a} . We claim that $\mathcal{C}_{\vec{a}}$ is not empty for every $\vec{a} \in \mathcal{A}_{k,\ell}$. Indeed, for $i = 0, \dots, \ell-1$ let σ_i be a permutation of $\{1, 2, \dots, \ell\}$ (clockwise rotation) obtained by a cyclic shift by i , i.e., $\sigma_i(j) = j+i \pmod{\ell}$. We form C_i by picking $a_{\sigma_i(j)}$ items from M_j for each $1 \leq j \leq \ell$. As $\sum_i a_{\sigma_i(j)} = k$, we may pick the ℓ edges C_i to be disjoint, thereby obtaining a covering.

Proof of Theorem 1.1. Let $\mathcal{H} = (V, \mathcal{E})$ be a k -graph of order km with the unique perfect matching $\mathcal{M} = \{M_1, \dots, M_m\}$. We show that $|\mathcal{E}| \leq f(k, m)$.

We partition the edges into collections of edges which intersect exactly ℓ of the matching edges. That is, for $\ell = 1, \dots, k$, we set

$$\mathcal{B}_\ell = \left\{ E \in \mathcal{E} : \sum_{i=1}^m \mathbf{1}_{E \cap M_i \neq \emptyset} = \ell \right\}.$$

Clearly, $|\mathcal{E}| = \sum_{\ell=1}^k |\mathcal{B}_\ell|$. Once again, $|\mathcal{B}_1| = m$. We will show, by contradiction, that $|\mathcal{B}_\ell| \leq b_{k,\ell} \binom{m}{\ell}$ for all $2 \leq \ell \leq k$.

Suppose that $|\mathcal{B}_\ell| > b_{k,\ell} \binom{m}{\ell}$ for some $2 \leq \ell \leq k$. Then, by the pigeonhole principle, there exists some set of ℓ matching edges, say, without loss of generality, $\mathcal{L} = \{M_1, \dots, M_\ell\}$ such that

$$|\mathcal{B}_\ell \cap \mathcal{H}[\mathcal{L}]| \geq b_{k,\ell} + 1, \quad (3.1)$$

where $\mathcal{H}[\mathcal{L}]$ denotes the sub-hypergraph of \mathcal{H} spanned by the vertices in $\bigcup_{i=1}^\ell M_i$. Let \mathcal{G} be the collection of all k -sets on $\bigcup_i M_i$ that intersect every $M_i \in \mathcal{L}$. That is

$$\mathcal{G} = \left\{ A : |A| = k, A \cap M_i \neq \emptyset \text{ for each } 1 \leq i \leq \ell \text{ and } A \subseteq \bigcup_i M_i \right\}.$$

As in (2.2), we have

$$b_{k,\ell} = \frac{\ell-1}{\ell} |\mathcal{G}| = \frac{\ell-1}{\ell} \sum_{\vec{a} \in \mathcal{A}_{k,\ell}} |\mathcal{G}_{\vec{a}}|,$$

where $\mathcal{G}_{\vec{a}}$ is the collection of k -sets of type \vec{a} . Hence, by Eq. (3.1) we get

$$|\mathcal{B}_\ell \cap \mathcal{H}[\mathcal{L}]| \geq \frac{\ell-1}{\ell} \sum_{\vec{a} \in \mathcal{A}_{k,\ell}} |\mathcal{G}_{\vec{a}}| + 1,$$

and consequently, there exists some type \vec{a} such that

$$|\mathcal{B}_\ell \cap \mathcal{G}_{\vec{a}}| \geq \frac{\ell-1}{\ell} |\mathcal{G}_{\vec{a}}| + 1. \quad (3.2)$$

Recall that $|\mathcal{C}| = \ell$ and that $\mathcal{C}_{\vec{a}}$ is the nonempty collection of all coverings \mathcal{C} of \mathcal{L} such that every $C \in \mathcal{C}$ is of type \vec{a} . By symmetry, every k -set $A \in \mathcal{G}_{\vec{a}}$ belongs to exactly

$$\frac{|\mathcal{C}_{\vec{a}}| \ell}{|\mathcal{G}_{\vec{a}}|}$$

coverings $\mathcal{C} \in \mathcal{C}_{\vec{a}}$. Since no $\mathcal{C} \in \mathcal{C}_{\vec{a}}$ is contained in \mathcal{H} (otherwise we could replace \mathcal{L} by \mathcal{C} to obtain a different perfect matching, contradicting the uniqueness of \mathcal{M}), the number of k -sets in $\mathcal{G}_{\vec{a}}$ that are not in \mathcal{B}_ℓ is at least

$$|\mathcal{C}_{\vec{a}}| \left/ \frac{|\mathcal{C}_{\vec{a}}| \ell}{|\mathcal{G}_{\vec{a}}|} \right. = \frac{|\mathcal{G}_{\vec{a}}|}{\ell}.$$

That means,

$$|\mathcal{B}_\ell \cap \mathcal{G}_{\vec{a}}| \leq \frac{\ell-1}{\ell} |\mathcal{G}_{\vec{a}}|$$

which contradicts (3.2). Thus, $|\mathcal{B}_\ell| \leq b_{k,\ell} \binom{m}{\ell}$, as required. \square

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