Conditioning and stability

Zecheng Zhang

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In the abstract, we can view a problem as $f : X \to Y$ where X, Y are two spaces. A well-conditioned problem is one with the property that all small perturbation of x lead to only small changes in $f(x)$.

1 Relative condition number

Denote $\delta f = f(x + \delta x) - f(x)$. The relative conditioning number is defined as

$$\kappa(x) = \lim_{\delta \to 0} \sup_{\|\delta x\| \le \delta} \left(\frac{\|\delta f\|}{\|f(x)\|} \Big/ \frac{\|\delta x\|}{\|x\|}\right). \tag{1}$$

One can assume δx and δf are infinitesimal, then

$$\kappa(x) = \sup_{\|\delta x\|}\left(\frac{\|\delta f\|}{\|f(x)\|} \Big/ \frac{\|\delta x\|}{\|x\|}\right). \tag{2}$$

When f is differentiable, we can express the quantity in terms of the Jacobian of f,

$$\kappa = \frac{\|J(x)\|}{\|f(x)\|/\|x\|}. \tag{3}$$

A problem is well-conditioned if κ is small (e.g., 1, 10, 100), and a problem is ill-conditioned if κ is large (e.g., 10^6 or bigger).

Example 1.1. Consider $x \to x/2$.

Example 1.2. Consider $x \to \sqrt{x}$, $x > 0$.

Example 1.3. Consider $f(x) = x_1 - x_2$.

2 Conditioning of matrix multiplication

Let $A \in \mathbb{R}^{m \times n}$, we consider the problem of computing Ax given a x. We want to know how Ax will change if there is a perturbation in x. The conditioning number of A is defined as,

$$\kappa = \sup_{\delta x}\left(\frac{\|A(x+\delta x) - Ax\|}{\|Ax\|} \Big/ \frac{\|\delta x\|}{\|x\|}\right) = \sup_{\delta x} \frac{\|A\delta x\|}{\|\delta x\|} \Big/ \frac{\|Ax\|}{\|x\|}. \tag{4}$$

Note that sup is over all δx and $\frac{\|Ax\|}{\|x\|}$ is independent with respect to sup, it follows that,

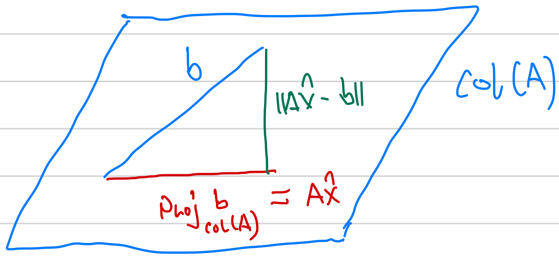
$$\kappa = \frac{\|x\|}{\|Ax\|} \sup_{\delta x} \frac{\|A\delta x\|}{\|\delta x\|} = \|A\| \frac{\|x\|}{\|Ax\|}, \tag{5}$$

where $\|A\|$ is the operator norm (it is L_2 norm if $\|\cdot\|$ is the L_2 vector norm). Note that, the condition number depends both on A and x.

$A \in \mathbb{R}^{m \cdot n}$

$$\|A\hat{x} - b\| \le \|Ax - b\|$$

$$A\hat{x} = \mathrm{Proj}_{col(A)}\, b$$ = Proj of b onto col(A)



Eg
$$A = \begin{pmatrix} 1 & -6 \\ 1 & -2 \\ 1 & +1 \\ 1 & 7 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 6 \end{pmatrix}$$

1. $A^{t}Ax = A^{t}b$ (normal equation)

2. QR, A must have linear indep cols.

$$\hat{x} = R^{-1}Q^{*}b$$

3. SVD

solution. $\langle a_1 \; a_2 \rangle = 0 \Rightarrow a_1$ & a_2 are orthogonal to each other.

$$\mathrm{Proj}_{col(A)}\, b = \frac{\langle a_1, b\rangle}{\langle a_1, a_1\rangle} a_1 + \frac{\langle a_2 \; b\rangle}{\langle a_2 \; a_2\rangle} a_2$$

$$= 2a_1 + \frac{1}{2}a_2 = \begin{pmatrix} -1 \\ 1 \\ 5/2 \\ 11/2 \end{pmatrix}$$

$$A\hat{x} = \text{Proj}_{\text{col}(A)} b = \underline{2}a_1 + \underline{\tfrac{1}{2}}a_2$$

$$\Rightarrow \hat{x} = \begin{pmatrix} 2 \\ 1/2 \end{pmatrix}.$$

$\blacksquare$

Eg:

$$A = \begin{pmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix}$$

Conditioning & stability.

$f : X \to Y$, X Y are two spaces.

A well-conditioning problem has the property:

all small perturbation in x lead to small change in $f(x)$.

Relative condition number.

$\delta f = f(x+\delta x) - f(x)$, $x \in X$

condition number of f at x,

$$\kappa(x) = \lim_{\delta \to 0} \sup_{\|\delta x\| < \delta} \left(\frac{\|\delta f\|}{\|f(x)\|} \Big/ \frac{\|\delta x\|}{\|x\|} \right)$$

suppose $\delta(x)$ is small enough.

$$\kappa(x) = \sup_{\|\delta x\|} \frac{\|\delta f\|}{\|f(x)\|} \Big/ \frac{\|\delta x\|}{\|x\|} \quad \checkmark$$

suppose f is differentiable.

$$\kappa(x) = \lim_{\delta \to 0} \sup_{\|\delta x\| \le \delta} \frac{\|\delta f\|}{\|\delta x\|} \frac{\|x\|}{\|f(x)\|} = \frac{\|J(x)\|}{\|f(x)\| / \|x\|}$$

$$[J(x)]_{ij} = \frac{\partial f_i}{\partial x_j} \quad \text{(Jacobian)}$$

Eg 1.1. $x \to \frac{x}{2}, \quad x > 0.$

$$\underline{X} = (0, \infty)$$

$$\underline{Y} = (0, \infty)$$

$$f(x) = \frac{x}{2}$$

$$\kappa(x) = \frac{\|J\|}{\|f(x)\| / \|x\|} = \frac{f'(x)}{x/2 \,/\, x} = \frac{1/2}{1/2} = 1$$

"small" contioning number

< 100, well condioning.

$1000, \cdots,$ ill - conditioned.

Eg. $x \to \sqrt{x}, \quad x > 0.$

Eg 1.3. $f(x) = x_1 - x_2$. $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$J(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\|x\|_\infty = \max_{\forall i} |x_i|$$

$$\|J(x)\|_\infty = 1$$

$$\kappa(x) = \frac{\|J(x)\|_\infty}{\|f(x)\| / \|x\|_\infty} = \frac{1}{|x_1 - x_2|} \cdot \max\{|x_1|, |x_2|\}$$

when $x_1 = x_2$, $\kappa(x) \to \infty$, ill-conditioned.

Eg. 1.4.

$$y^2 - 2y + 1 = 0,$$

consider peturb the coeffs for y & const.

f: coeff or y & const, $\longrightarrow$ solution of the equation.

$$x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \|x\|_\infty = 2$$

$$y^2 - 2y + 0.9999 = 0$$

$$\delta x = \begin{pmatrix} 0 \\ , \\ 0.0001 \end{pmatrix}, \quad \|\delta x\|_\infty = 0.0001$$

$$f(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \|f\|_\infty = 1$$

$$\delta f = f(x+\delta x) - f(x) = \begin{pmatrix} 0.99 \\ 1.01 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.01 \\ +0.01 \end{pmatrix}$$

$$\|\delta f\| = 0.01$$

$$k(x) > \frac{\|\delta f\|}{\|f(x)\|} \Big/ \frac{\|\delta(x)\|}{\|x\|} = 200$$

(a little bit big)

Remark 1. Suppose A is nonsingular square matrix. We have $\|x\| = \|A^{-1}Ax\| \leq \|A^{-1}\|\|Ax\|$, this further implies that,

$$\kappa \leq \|A\|\|A^{-1}\|, \tag{6}$$

or

$$\kappa = c\|A\|\|A^{-1}\|, \tag{7}$$

for some positive constant $c = \frac{\|x\|}{\|Ax\|}/\|A^{-1}\|$.

Theorem 2.1. Let $A \in \mathbb{R}^{m\times n}$ be invertiable and let us consider $Ax = b$. The problem of computing b given x has conditioning number,

$$\kappa = \|A\|\frac{\|x\|}{\|b\|} \leq \|A\|\|A^{-1}\|, \tag{8}$$

with the perturbation in x. The problem of computing x given b has the conditioning number,

$$\kappa = \|A^{-1}\|\frac{\|b\|}{\|x\|} \leq \|A^{-1}\|\|A\|, \tag{9}$$

with the perturbation in b. If we use the L_2 norm, the first equality holds if x is a multiple of a right singular vector of A corresponding to the minimal singular value. The second equality holds if b is a multiple of a left singular vector of A corresponding to the largest singular value.

Definition 2.2. We will call $\kappa(A) = \|A\|\|A^{-1}\|$ the condition of A relative to norm $\|\cdot\|$ and denote it as $\kappa(A) = \|A\|\|A^{-1}\|$. The conditioning number is attached to matrix A not to the problem and x. If $\kappa(A)$ is small, A is called well-conditioned, otherwise, it is called ill-conditioned. If A is singular, we write $\kappa(A) = \infty$.

Remark 2. If $\|\cdot\| = \|\cdot\|_2$, $\|A\| = \sigma_1$ and $\|A^{-1}\| = 1/\sigma_m$, it follows that $\kappa(A) = \frac{\sigma_1}{\sigma_m}$