

Singular value decomposition (SVD)

Zecheng Zhang

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This section will discuss singular value decomposition (SVD) of a matrix $A \in \mathbb{R}^{m \times n}$.

1 Construction

step 1:

The first singular value is defined as:

$$\sigma_1 = \sup_{\|v\|=1} \|Av\|.$$

Remark 1. The first singular value is well defined, i.e., such a $v_1 \in \mathbb{R}^n$ always exists. Non-rigorous argument: the function $v \rightarrow \|Av\|$ is continuous and with a compact domain.

Now one can find $u_1 \in \mathbb{R}^m$ with $\|u_1\| = 1$ such that $Av_1 = \sigma_1 u_1$.

One can follow the definition of the first singular value and define the second singular value as,

$$\sigma_2 = \sup_{\|v\|=1, v \perp v_1} \|Av\|.$$

The remark 1 implies that such a v_2 always exists and let us denote it as v_2 . In addition, we can find $u_2 \in \mathbb{R}^m$ with $\|u_2\| = 1$ such that $Av_2 = \sigma_2 u_2$.

Remark 2. $\sigma_2 \leq \sigma_1$ because v_2 is taken from a smaller subspace $\{v_1\}^\perp \subset \mathbb{R}^n$.

Theorem 1.1. u_1 and u_2 which are defined above are orthogonal.

The theorem implies that $u_1 \perp u_2$. Repeat the process, one can find a unit vector $v_3 \in W_2 = \{v_1, v_2\}^\perp$ such that it admits

$$\sigma_3 = \sup_{\|v\|=1, v \in V_2} \|Av\|.$$

In addition, one can find a unit vector u_3 such that $Av_3 = \sigma_3 u_3$. One can show that $\{u_1, u_2, u_3\}$ are orthogonal.

Remark 3. Let us define $W_p = \{v_1, v_2, \dots, v_p\}^\perp$. If $\sup_{v \in W_p} \|Av\| = 0$, or, $Av_{p+1} = 0$, we can make u_{p+1} (nonzero if possible) to be any vector which is orthogonal to $\{u_1, \dots, u_p\}$. If u_{p+1} has to be zero, $\text{span}\{u_1, \dots, u_p\} = \mathbb{R}^m$

Repeat the process for n times (why is n the maximum step of the process?), we then can construct an orthonormal matrix $V = [v_1, \dots, v_n] \in \mathbb{R}^{n \times n}$, another matrix with orthonormal columns $U = [u_1, \dots, u_n] \in \mathbb{R}^{m \times n}$ up to some 0 columns, and a diagonal matrix $\Sigma \in \mathbb{R}^{n \times n}$ with diagonal entries being $\sigma_1, \dots, \sigma_n$ (up to some 0). Recall the matrix multiplication we have,

$$AV = U\Sigma.$$

step 1. $\sigma_1 = \sup_{\substack{v \in \mathbb{R}^n \\ \|v\|=1}} \|Av\|$

Find $u_1 \in \mathbb{R}^n, \|u_1\|=1,$
 $Av_1 = \sigma_1 u_1, \quad \|Av_1\| = \sigma_1, \|v_1\| = 1.$

$$u_1 = \frac{Av_1}{\sigma_1}, \quad \|u_1\| = \frac{\|Av_1\|}{\sigma_1} = 1.$$

step 2. $\sigma_2 = \sup_{\substack{v \in \mathbb{R}^n \\ v \perp \{u_1\} \\ \|v\|=1}} \|Av\|$

Find $u_2 \in \mathbb{R}^n, \|u_2\|=1,$
 $Av_2 = \sigma_2 u_2, \quad \|Av_2\| = \sigma_2, \|v_2\|=1, v_2 \in \{v_1\}^\perp$

step 3. $\sigma_3 = \sup_{\substack{v \in \mathbb{R}^n \\ v \perp \{u_1, u_2\} \\ \|v\|=1}} \|Av\|,$

u_1 & u_2 are
orthogonal
Thm 1.1.

$$Av_3 = \sigma_3 u_3. \quad \|Av_3\| = \sigma_3, \|v_3\|=1, v_3 \perp \{u_1, u_2\}$$

u_1, u_2, u_3 are
orthogonal
Thm 1.1.

Remark.

$$1. \quad \sigma_1 \geq \sigma_2 \geq \sigma_3 \dots$$

2. Q: How many steps can we do?

A: $n - \text{steps}$.

3. Q.: suppose $V_p = \text{span}\{v_1 \dots v_p\}$

with $\sigma_1 \dots \sigma_p \neq 0$, but.

$$\sigma_{p+1} = \sup_{\substack{v \in \mathbb{R}^n \\ v \perp V_p \\ \|v\|=1}} \|Av\| = 0$$

$$\|Av\|=0 \Leftrightarrow Av=0, v \in \text{null}(A)$$

$$\text{or all candidates } [V_p^\perp] \subseteq \text{null}(A)$$

4. Q: If $\sigma_{p+1} = 0$, how to find U_{p+1}

A: construct a U_{p+1} which is orthogonal to $\{u_1 \dots u_p\}$

$$\text{Eg. } u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (\text{construct } u_3)$$

$$\xi: \quad \mathcal{Q}: \quad u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \sigma_3 = 0$$

$$u_3 = 0$$

6. Q: When $|p+1|$ must be 0?

$$A: \quad |p+1| > m = \dim(\mathbb{R}^m), \quad u_i \in \mathbb{R}^m$$

$$\begin{cases} AV_1 = \sigma_1 u_1 \\ AV_2 = \sigma_2 u_2 \\ \dots \\ AV_n = \sigma_n u_n \end{cases}$$

Denote $V = [v_1 \ v_2 \ \dots \ v_n] \in \mathbb{R}^{n \times n}$, V is orthogonal matrix

$$U = [u_1 \ \dots \ u_n] \in \mathbb{R}^{m \times n}$$

$$\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & \dots & \sigma_n \end{pmatrix} \in \mathbb{R}^{n \times n}$$

$$AV = U\Sigma$$

$$U^\dagger AV = \Sigma$$

$$AVV^\dagger = U\Sigma V^\dagger$$

$$A = U\Sigma V^\dagger$$

SVD A
← (reduced)

Full SVD. $A \in \mathbb{R}^{m \times n}$, $m > n$

$$A = U \Sigma V^t$$

$$U_F = \left(\underbrace{u_1, u_2, \dots, u_n}_{\mathbb{R}^{m \times n}} \mid u_{n+1}, \dots, u_m \right) \in \mathbb{R}^{m \times m}$$

\downarrow
 $u_{n+1}, \dots, u_m \neq 0$ are orthonormal to $\{u_1, \dots, u_n\}$

U_F is also unitary.

$$A = \underbrace{U_F}_{\mathbb{R}^{m \times m}} \underbrace{\Sigma}_{\mathbb{R}^{m \times n}} \underbrace{V^t}_{\mathbb{R}^{n \times n}}$$

$$A = U \Sigma V^t$$

Full SVD $\rightarrow A = \left[U \mid u_{n+1} \dots u_m \right] \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \\ \hline & & & & 0 \\ & & & & \vdots \\ & & & & 0 \end{pmatrix} V^t$

\downarrow U_F \downarrow Σ_F

Theorem 1.2. $rank(A)$ equals to the number of nonzero singular values.

Proof. Let us assume $\{\sigma_1, \dots, \sigma_p\}$ are all nonzero singular values but $\sigma_{p+1} = 0$. Let $V_p = \{v_1, \dots, v_p\}$ be the singular vector corresponding to $\sigma_1, \dots, \sigma_p$. We claim that $V_p \subset row(A)$. We have $AV = U\Sigma$, or $U^T A = \Sigma V^T$. The i -th row ($i \leq p$) of the right-hand side is $\sigma_i v_i^t$. The i -th row on the left-hand side is $(u_i)^t A$, it follows that $v_i^t = \frac{1}{\sigma_i} (u_i)^t A$. This implies that $V_p \subset row(A)$.

By theorem in the last section (Complement theorem), $null(A) = row(A)^\perp \subset V_p^\perp$. Now, for $v \in V_p^\perp$, we have $Av = 0$, otherwise contradicts with the definition of V_p . As a result, $V_p^\perp \subset null(A)$, i.e., $V_p^\perp = null(A)$. By the rank theorem, $dim(V_p) = rank(A)$. \square

As a corollary, $V_p = row(A)$. We summarize the results in the following theorem.

Theorem 1.3. Assume $\{\sigma_1, \dots, \sigma_p\}$ are all nonzero singular values, $\{v_1, \dots, v_p\}$ and $\{u_1, \dots, u_p\}$ are right and left singular vectors respectively, we denote the space spanned by them as V_p and U_p . The followings are true:

$$\begin{aligned} V_p &= row(A), \\ U_p &= col(A). \end{aligned}$$

Proof. The first one is proved in the last theorem and let us prove $U_p = col(A)$. We claim that for any $y \in \mathbb{R}^n$, there exists $x \in row(A)$ such that $Ax = Ay$. When $y \in row(A)$, it is true. When $y \notin row(A)$, assume not, i.e., for all $x \in row(A)$, $A(x - y) = 0$. This implies that $x - y \in null(A)^\perp$. It follows from the theorem we discussed in the last section that $x - y \in row(A)$. However, this is the contradiction. The claim is proved.

Now for any $y \in \mathbb{R}^n$, let $y_x \in row(A)$ such that $Ay_x = Ay$. Since $V_p = row(A)$, $y_x = \sum_{i=1}^p c_i^y v_i$ for some constants c_i . It follows that $Ay = \sum_i \sigma_i c_i^y u_i$. The other direction is also true, this implies that $col(A) = U_p$. \square

Full SVD: make U matrix orthonormal when $m > n$. One can append an additional $m - n$ orthonormal columns to fulfill this goal. Σ should change as well so that the product $AV = U\Sigma$ still holds. To do this, one can append $m - n$ zero rows to the bottom of Σ . As a result, we have $AV = U\Sigma$ where $V \in \mathbb{R}^{n \times n}$, $U \in \mathbb{R}^{m \times m}$ and $\Sigma \in \mathbb{R}^{m \times n}$. Since V is orthonormal, we have:

$$A = U\Sigma V^{-1}.$$

2 Revisit SVD

2.1 From SVD

Let $A \in \mathbb{R}^{m \times n}$. Suppose A admits an SVD $A = U\Sigma V^t$, where $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices and $\Sigma \in \mathbb{R}^{m \times n}$ is a diagonal matrix. Let us now consider $AA^t \in \mathbb{R}^{m \times m}$ and $A^t A \in \mathbb{R}^{n \times n}$, which are symmetric matrices.

$$\begin{aligned} A^t A &= V\Sigma^t U^t U \Sigma V^t = V\Sigma^t \Sigma V^t, \\ AA^t &= U\Sigma V^t V \Sigma^t U^t = U\Sigma \Sigma^t U^t. \end{aligned}$$

$\Sigma\Sigma^2$ and $\Sigma^2\Sigma$ are still diagonal, and nonzero entries of these two matrices are indeed singular values squared.

In addition, since U and V are orthogonal (hence invertible), this implies that $V\Sigma^t\Sigma V^t$ and $U\Sigma\Sigma^tU^t$ are the eigenvalue decomposition (diagonalization) of A^tA and AA^t (theorem 4.1 in diagonalization section).

2.2 From eigenvalue decomposition

A^tA is symmetric and by the Spectral theorem, let $\{v_i\}_{i=1}^n$ be the orthonormal eigenvectors of A^tA corresponding to eigenvalue $\lambda_1 \geq \lambda_2 \dots \geq \lambda_n$. We first claim that $\lambda_1 \geq \lambda_2 \dots \geq \lambda_n \geq 0$. We have,

$$\|Av_i\|^2 = (Av_i)^t Av_i = v_i^t A^t Av_i = \lambda_i \|v_i\|^2 \geq 0,$$

it implies that $\lambda_i \geq 0$.

Let $\sigma_1 = \sqrt{\lambda_1}$ for all i . We want to find $\{u_k\}_k$, which are orthonormal, such that,

$$Av_k = \sigma_k u_k.$$

When $\sigma_k \neq 0$, one can define $u_k = \frac{1}{\sigma_k} Av_k$. Let us claim all u_k are orthonormal. Let u_i, u_j be nonzero and defined as before. We have,

$$u_i^t u_j = \frac{1}{\sigma_i \sigma_j} v_i^t A^t Av_j = v_i^t v_j = \delta_{ij}.$$

The claim is proved. When $\lambda_p = 0$, for some $1 \leq p \leq n$, we can construct u_p which is orthogonal to u_1, u_2, \dots, u_{p-1} . If $\{u_1, \dots, u_{p-1}\}$ have formed a basis for \mathbb{R}^m , then set $u_p = 0$. Now we can construct an orthonormal matrix $V = [v_1, \dots, v_n] \in \mathbb{R}^{n \times n}$, another matrix with orthonormal columns $U = [u_1, \dots, u_n] \in \mathbb{R}^{m \times n}$ up to some 0 columns, and a diagonal matrix $\Sigma \in \mathbb{R}^{n \times n}$ with diagonal entries being $\sigma_1, \dots, \sigma_n$. The SVD follows: $AV = U\Sigma$. One can apply the same trick as before to make U a square matrix and obtain the full SVD.

Remark 4. Nonzero u_k constructed before are eigenvectors of AA^t . The proof is simple.

$$AA^t u_k = AA^t \frac{1}{\sigma_k} Av_k = A \frac{1}{\sigma_k} A^t Av_k = A \sigma_k v_k = \sigma_k^2 u_k.$$

Remark 5. If $S = S^t$ is symmetric and assumes all eigenvalues are non-negative, then the eigenvalue decomposition is the SVD (as long as the eigenvalues are in descending order).

3 Rank k approximation

Let us consider the SVD of $A \in \mathbb{R}^{m \times n}$, i.e., $A = U\Sigma V^t$. Recall the matrix multiplication, we have a decomposition for A as,

$$A = \sum_{i=1}^n \sigma_i u_i v_i^t = \sum_{i=1}^r \sigma_i u_i v_i^t,$$

where $\{\sigma_i\}_{i=1}^r$ are all nonzero singular values of A . Let us define an approximation A_k to A as:

$$A_k = \sum_{i=1}^k \sigma_i u_i v_i^t,$$

where $k \leq r$. It is easy to check $\text{rank}(A_k) = k$. We then can show A_k is the best approximation to A ; the result is summarized in the following theorem.

3.1 Preliminaries

Definition 3.1. L^2 norm of a matrix $A \in \mathbb{R}^{m \times n}$ is defined as:

$$\|A\|_2 = \max_{x \in \mathbb{R}^n, \|x\|=1} \|Ax\| = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|}{\|x\|} = \sigma_1.$$

In the rest of the notes, we sometimes write $\|\cdot\|_2$ as $\|\cdot\|$ for simplicity.

Remark 6. For any $x \in \mathbb{R}^n$ and $x \neq 0$, we have, $\frac{\|Ax\|}{\|x\|} \leq \|A\|_2$. This implies that $\|Ax\| \leq \|A\|\|x\|$.

3.2 Eckart-Young theorem

Theorem 3.2 (Eckart-Young). Suppose $A, B \in \mathbb{R}^{m \times n}$ and $\text{rank}(B) = k \leq \text{rank}(A) = r$. We then have:

$$\|A - B\| \geq \|A - A_k\| = \sigma_{k+1}.$$

That is A_k is the best $\text{rank } k$ approximation to A in the L^2 sense.

Proof. We first prove that $\|A - A_k\| = \sigma_{k+1}$. We have,

$$A - A_k = \sum_{i=k+1}^n \sigma_i u_i v_i^t = \sum_{i=1}^{n-k} \sigma_{i+k} u_{i+k} v_{i+k}^t + \sum_{i=n-k+1}^n \tilde{\sigma}_i \tilde{u}_i \tilde{v}_i^t,$$

where $\tilde{\sigma}_i = 0$, \tilde{u}_i are orthonormal to all u_{i+k} , and \tilde{v}_i are also orthonormal to all v_{i+k} . The summation is then an SVD of $A - A_k$. Since $\|A - A_k\|$ is equal to the first singular value of its SVD, we have $\|A - A_k\| = \sigma_{k+1}$.

Assume not, i.e., assume there is $B \in \mathbb{R}^{m \times n}$ with $\text{rank}(B) = k$ such that $\|A - B\| < \|A - A_k\| = \sigma_{k+1}$. For any $w \in \mathbb{R}^n$, we have $\|(A - B)w\| < \sigma_{k+1}\|w\|$. It follows that, for any $w \in \text{null}(B)$, we have,

$$\|(A - B)w\| = \|Aw\| < \sigma_{k+1}\|w\|. \quad (1)$$

Now for any $w \in V_{k+1} = \text{span}\{v_1, v_2, \dots, v_{k+1}\}$, we claim that $\|Aw\| \geq \sigma_{k+1}\|w\|$. Since $w \in V_{k+1}$, there exist c_1, \dots, c_{k+1} , such that $w = \sum_{i=1}^{k+1} c_i v_i$. It follows that

$$\|Aw\| = \left\| \sum_{i=1}^{k+1} c_i A v_i \right\| = \sum_{i=1}^{k+1} |c_i| \sigma_i \geq \sigma_{k+1} \|w\|, \quad (2)$$

where the last inequality is due to the orthogonality of v_i and $\sigma_1 \geq \dots \geq \sigma_{k+1}$.

The Rank theorem indicates that $\dim(\text{null}(B)) = n - k$, however $\dim(V_{k+1}) = k + 1$. We then have $\dim(\text{null}(B)) + \dim(V_{k+1}) > n$. Since $\text{null}(B)$ and V_{k+1} both are subspace of \mathbb{R}^n , this implies that there exists $w \neq 0$ such that $w \in \text{null}(B) \cap V_{k+1}$. However, [1](#) and [2](#) cannot hold simultaneously, which is the contradiction. □

Corollary 3.2.1. Suppose $A, B \in \mathbb{R}^{m \times n}$ and $\text{rank}(B) \leq k \leq \text{rank}(A) = r$. We then have:

$$\|A - B\| \geq \|A - A_k\| = \sigma_{k+1}.$$

Proof. Let $\text{rank}(B) = k - j$, $0 \leq j \leq k$, by Eckart-Young, we have $\|A - B\| \geq \|A - A_{k-j}\| = \sigma_{k-j+1} \geq \sigma_{k+1} = \|A - A_k\|$. □

3.3 Eckart-Young theorem (Frobenius)

Definition 3.3. The Frobenius norm of $A \in \mathbb{R}^{m \times n}$ is:

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2}.$$

Theorem 3.4. Frobenius norm can be calculated in the following way,

$$\|A\|_F^2 = \sum_i \sigma_i^2.$$

Corollary 3.4.1. Let the SVD of A be $A = U\Sigma V^t$, and $U = [u_1, \dots, u_n]$, $V = [v_1, \dots, v_n]$, and the diagonal entries of Σ are $\sigma_1, \dots, \sigma_n$. Let $A_k = \sum_{i=1}^k \sigma_i u_i v_i^t$, we have,

$$\|A - A_k\|_F^2 = \sum_{i=k+1}^n \sigma_i^2.$$

Theorem 3.5 (Courant Fisher min max). For $A \in \mathbb{R}^{m \times n}$, the singular value σ_i of A satisfy:

$$\begin{aligned} \sigma_k &= \max_{V \subset \mathbb{R}^n, \dim(V)=k} \min_{v \in V, \|v\|=1} \|Av\|, \\ \sigma_{k+1} &= \min_{V \subset \mathbb{R}^n, \dim V=n-k} \max_{v \in V, \|v\|=1} \|Av\|. \end{aligned}$$

Proof. Let us prove the first one first. Let V be any k -dimensional space. Since $\dim(\text{span}\{v_k, \dots, v_n\}) = n - k + 1$, V intersects $\text{span}\{v_k, \dots, v_n\}$ nontrivially. Let v be a unit vector in the intersection, i.e., there exist c_k, \dots, c_n such that, $v = \sum_{i=k}^n c_i v_i$. Moreover, $\|v\| = \sum_{i=k}^n |c_i| = 1$. We have, $Av = \sum_{i=k}^n c_i \sigma_i u_i$, it follows that, $\|Av\| = \sum_{i=k}^n |c_i| \sigma_i \|u_i\| \leq \sigma_k$. This implies that for any V of dimension k , there exists v such that $\|Av\| \leq \sigma_k$, i.e., $\min_{v \in V, \|v\|=1} \|Av\| \leq \sigma_k$. Now we need to find a V such that the equality sign holds, i.e., $\|Av\| = \sigma_k$. We claim that V can be $\text{span}\{v_1, \dots, v_k\}$, i.e., $V \cap \text{span}\{v_k, \dots, v_n\} = \text{span}\{v_k\}$. Now let $v = v_k$, it follows that $\|Av\| = \sigma_k$. The claim is proved, i.e., maximizing over all V , we can obtain the equal sign.

The second one can be derived similarly. Let V be any $(n - k)$ -dimensional subspace of \mathbb{R}^n , it intersects $V_{k+1} := \text{span}\{v_1, \dots, v_{k+1}\}$ nontrivially, i.e., there exists unit vector v in the intersection. It follows that, there exist c_1, \dots, c_{k+1} such that $v = \sum_{i=1}^{k+1} c_i v_i$. We have $Av = \sum_{i=1}^{k+1} c_i \sigma_i u_i$, it follows that, $\|Av\| \geq \sigma_{k+1}$. This implies that for any V of dimension $n - k$, there exists v such that $\|Av\| \geq \sigma_{k+1}$, i.e., $\max_{v \in V, \|v\|=1} \|Av\| \geq \sigma_{k+1}$. The equality holds when $V = \text{span}\{v_{k+1}, \dots, v_n\}$ and $v = v_{k+1}$. \square

Theorem 3.6 (Weyl). Let $A, B \in \mathbb{R}^{m \times n}$, and denote the singular values as $\sigma_i(A)$ and $\sigma_i(B)$. We then have:

$$\sigma_{i+j-1}(A + B) \leq \sigma_i(A) + \sigma_j(B). \quad (3)$$

Proof. Let V_A , and V_B be the subspace of \mathbb{R}^n of dimensions $n - k$ and $n - l$, which are orthogonal to the first k and l right singular vectors of A and B respectively. Let $W = V_A \cap V_B$, we have $\dim(W) \geq n - k - l$. It follows that,

$$\max_{v \in W, \|v\|=1} \|Av + Bv\| = \max_{v \in W, \|v\|=1} \|Av\| + \|Bv\| \leq \sigma_{k+1} + \sigma_{l+1}.$$

By Curant-Fisher,

$$\sigma_{k+l+1}(A+B) = \min_{V \subset \mathbb{R}^n, \dim V = n-k-l} \max_{v \in V, \|v\|=1} \|Av + Bv\| \leq \max_{v \in W, \|v\|=1} \|Av + Bv\| = \sigma_{k+1} + \sigma_{l+1}.$$

□

Weyl's inequality will help us prove the Eckart-Young for the Frobenius norm.

Theorem 3.7 (Eckart-Young Frobenius). Suppose $A, B \in \mathbb{R}^{m \times n}$ and $\text{rank}(B) = k \leq \text{rank}(A) = r$. We then have:

$$\|A - B\|_F^2 \geq \|A - A_k\|_F^2 = \sum_{i=k+1}^r \sigma_i^2.$$

That is A_k is the best $\text{rank } k$ approximation to A in the L^2 sense.

Proof. Let $X = A - B$ and $Y = B$ and apply Weyl's inequality [3.6](#),

$$\sigma_{i+k}(A) \leq \sigma_i(A - B) + \sigma_{k+1}(B) = \sigma_i(A - B),$$

where the last equality is due to $\text{rank}(B) = k$. Apply Corollary [3.4.1](#) it follows that,

$$\begin{aligned} & \|A - A_k\|_F^2 \\ &= \sum_{i=k+1}^r \sigma_i(A)^2 = \sum_{i=1}^{r-k} \sigma_{i+k}^2(A) \leq \sum_{i=1}^{r-k} \sigma_i^2(A - B) \leq \sum_{i=1}^{\min(m,n)} \sigma_i^2(A - B) = \|A - B\|_F^2. \end{aligned}$$

□

A direct consequence of the Eckart-Young for the Frobenius norm is the proper orthogonal decomposition (POD).

3.4 Proper orthogonal decomposition (POD)

Given $A = [y_1, y_2, \dots, y_n] \in \mathbb{R}^{m \times n}$, and a set of orthonormal vectors $Q = [x_1, \dots, x_k] \in \mathbb{R}^{m \times k}$, one wants to solve the following problem:

$$\min_Q \sum_{i=1}^n \left\| y_i - \sum_{j=1}^k \langle y_i, x_j \rangle x_j \right\|^2. \quad (4)$$

We claim that the equation [4](#) is equivalent to the matrix form,

$$\sum_{i=1}^n \left\| y_i - \sum_{j=1}^k \langle y_i, x_j \rangle x_j \right\|^2 = \|A - QQ^T A\|_F^2. \quad (5)$$

Denote the matrix as columns, i.e., $\|A - QQ^T A\|_F = \|[y_1 - QQ^T y_1, \dots, y_n - QQ^T y_n]\|_F$; and denote $y_i - QQ^T y_i$ as $z_i \in \mathbb{R}^m$, it follows that,

$$\|[y_1 - QQ^T y_1, \dots, y_n - QQ^T y_n]\|_F^2 = \sum_{i=1}^n \sum_{j=1}^m z_{ji}^2 = \sum_{i=1}^n \|z_i\|^2 = \sum_{i=1}^n \|y_i - QQ^T y_i\|^2.$$

It is not hard to see $QQ^T y_i = \sum_{j=1}^k \langle y_i, x_j \rangle x_j$. The claim is proved. Apply the Eckart-Young theorem for the Frobenius norm; we then have the POD theorem.

Theorem 3.8. Given $A = [y_1, y_2, \dots, y_n] \in \mathbb{R}^{m \times n}$ with rank r . For any $k \leq r$, we consider,

$$\min_Q \sum_{i=1}^n \left\| y_i - \sum_{j=1}^k \langle y_i, x_j \rangle x_j \right\|^2, \quad (6)$$

where $Q = [x_1, \dots, x_k] \in \mathbb{R}^{m \times k}$ is a set of orthonormal vectors. The minimum is given by the left singular vectors of A , which are also called proper orthogonal modes. Denote the singular values of A as σ_i , the minimum is equal to $\sum_{i=k+1}^r \sigma_i^2$.

Proof. The only statement left to prove is $QQ^T A = A_k$ if $Q = [u_1, \dots, u_k]$ and u_k are the singular vectors. We have,

$$QQ^T A = [u_1, u_2, \dots, u_k][u_1, u_2, \dots, u_k]^T A = \sum_{i=1}^k u_i u_i^T A.$$

□