1. A is normal.
2. A is unitarily diagonalizable.
3. $\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i=1}^{n} |\lambda_i|^2$.
4. There is an orthonormal set of n eigenvectors of A.

Proof. By the Schur factorization, there exist a unitary matrix U and an upper-triangular matrix T such that:

$$A = UTU^*. \tag{3}$$

Let us show 1 to 2. To show A is unitarily diagonalizable, we only need to show T is diagonal. Since A is normal, we have

$$TT^* = U^*AUU^*A^*U = U^*AA^*U = U^*A^*AU = U^*A^*UU^*AU = T^*T. \tag{4}$$

This implies that T is normal. Since T is triangular, the homework question implies that T is diagonal.

Let us now prove 2 to 4 and leave the others as exercise. From the second argument,

$$A = UTU^*, \tag{5}$$

where T is diagonal and $U = [u_1, ..., u_n]$ is unitary. It follows that,

$$AU = UT. \tag{6}$$

This is equivalent to $Au_i = \lambda_i u_i$ for all i, i.e., u_i are the orthonormal eigenvectors.

□

3 Hermitian

Definition 3.1. A matrix A if $A^* = A$, where $A^* = \bar{A}^T$.

Theorem 3.2. *A is Hermitian if and only if at least one of the following holds:*

1. *x^*Ax is real for all $x \in \mathbb{C}^{n\times n}$.*
2. *A is normal and all the eigenvalues of A are real.*
3. *S^*AS is Hermitian for all $S \in \mathbb{C}^{n\times n}$.*

Proof. Let us first prove the first statement. Take the complex conjugate of x^*Ax, we have $(x^*Ax)^* = x^*A^*x$, since $A = A^*$, x^*Ax is real for all x. Now suppose x^*Ax is real for all x., we have

$$(x^* + y^*)A(x + y) = (x^*Ax) + (y^*Ay) + (x^*Ay + y^*Ax), \tag{7}$$

is real for all x, y. The first two terms of 7 are real; we conclude that the sum of the last two terms is real. Now let $x = e_k$ and $y = e_j$, this implies that $a_{kj} + a_{jk}$ is real, i.e., $img(a_kj) = img(a_{jk})$.

Thm 3.2.

Suppose A is Hermitian,

$$\overline{(x^* A x)} = x^* A^* x = x^* A x \Rightarrow \text{real}.$$

2. $\Rightarrow$ ✓ (HW)

$\Leftarrow$

Assume A is normal & all eigenvalues are real.

Since A is normal,

$$\Rightarrow \quad A = UDU^*, \quad U \text{ is unitary}, \quad D = \begin{pmatrix} \lambda_1 \in \mathbb{R} & & \\ & \ddots & \\ & & \lambda_n \in \mathbb{R} \end{pmatrix}.$$

$$A^* = U D^* U^* = U D U^* = A.$$

$\Rightarrow$ A is Hermitian.

3. $\Rightarrow$ ✓ (by the definition)

$\Leftarrow$ Take $S = I$.

Let $x = ie_k$ and $y = e_j$, this implies that $ia_{kj} + ia_{jk}$ is real, i.e., $real(a_{kj}) = real(a_{jk})$. It follows that $a_{kj} = a_{jk}$, or, A is Hermitian.

The second argument. Let us assume A is normal and all eigenvalues are real. By Theorem 2.2, A is unitary diagonalizable., i.e., there exist unitary matrix U and diagonal matrix $D = diag(\lambda_1, ..., \lambda_n)$ such that $A = UDU^*$. Now take the complex conjugate; we have $A^* = UD^*U^*$. Since D is real, this implies $A = A^*$.

The last statement is trivial. $\square$

Theorem 3.2 implies the following important result.

Theorem 3.3 (Spectral Theorem)**.** Let $A \in \mathbb{C}^{n\times n}$. Then A is Hermitian if and only if there is a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a real diagonal matrix $D \in \mathbb{R}^{n\times n}$ such that $A = UDU^*$.

Proof. Suppose A is Hermitian. By Theorem 3.2, A is normal, and all the eigenvalues of A are real. It follows that, A is unitary diagonalizable., i.e., there exist unitary matrix U and a real diagonal matrix $D = diag(\lambda_1, ..., \lambda_n)$ such that $A = UDU^*$. The other direction is trivial. $\square$

Theorem 3.4. Every matrix $A \in \mathbb{C}^{n\times n}$ is uniquely determined by its Hermitian form x^*Ax. Specifically, $A = B$ if and only if $x^*Ax = x^*Bx$ for all $x \in \mathbb{C}^n$.

Proof. Homework exercise. $\square$

We now discuss another important result of the Hermitian matrices. It is called the variational characterization of eigenvalues.

Theorem 3.5 (Rayleigh Ritz)**.** Let $A \in \mathbb{C}^{n\times n}$ be Hermitian, and denote all eigenvalues of A as $\lambda_1 \leq ... \leq \lambda_n$. Then, we have,

x^*Ax is real (Thm 3.2)

1. $\lambda_1 x^*x \leq x^*Ax \leq \lambda_n x^*x$.
2. $\lambda_{max} = \lambda_n = \max_{x\neq 0} \frac{x^*Ax}{x^*x} = max_{x^*x=1} x^*Ax$.
3. $\lambda_{min} = \lambda_1 = \min x \neq 0 \frac{x^*Ax}{x^*x} = min_{x^*x=1} x^*Ax$.

Proof. Since A is Hermitian, it admits the unitary diagonalization $A = UDU^*$, where U is unitary and $D = diag(\lambda_1, ..., \lambda_n)$. Here we assume $\lambda_1 \leq ... \leq \lambda_n$. For $x \in \mathbb{C}^n$, we have,

$$x^*Ax = (U^*x)^*D(U^*x) = \sum_{i=1}^{n} \lambda_i |(U^*x)_i|^2, \tag{8}$$

where $(U^*x)_i$ is the $i-$th entry of the vector (U^*x). We then have,

$$\lambda_{min} \sum_{i=1}^{n} |(U^*x)_i|^2 \leq x^*Ax \leq \lambda_{max} \sum_{i=1}^{n} |(U^*x)_i|^2. \tag{9}$$

Since U is unitary,

$$\sum_{i=1}^{n} |(U^*x)_i|^2 = x^*x. \tag{10}$$

It follows that,

$$\lambda_1 = \lambda_{min} x^*x \leq x^*Ax \leq \lambda_{max} x^*x = \lambda_n \tag{11}$$

The estimation is sharp, for if x satisfies $Ax = \lambda_1 x$, the equal sign holds. The other side is similar. $\square$

Thm 3.3.

Suppose A is Hermitian.

$\Rightarrow$ A is normal with real eigenvalues.

$A = UDU^*$ (b/c spectral thm for normal matrix) Thm 2.2.

$\hookrightarrow$ D is real.

The other direction ✓

Thm 3.5 (Rayleigh Ritz).

Pf. Since A is Hermitian,

$A = UDU^*$, U is unitary, $D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$ $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$.

For any $x \in \mathbb{C}^n$

$$x^* A x = x^* \underbrace{UDU^*}_{A} x = (U^* x)^* D (U^* x)$$

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad y^* D y = \underbrace{(\bar{y}_1 \cdots \bar{y}_n)}_{y^*} \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \\ \cdots \\ \lambda_n y_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i \bar{y}_i y_i = \sum_{i=1}^{n} \lambda_i |y_i|^2$$

$$x^* A x = \sum_{i=1}^{n} \lambda_i \left| \underbrace{(U^* x)_i}_{} \right|^2$$

$\rightarrow$ ith entry of $U^* x$.

$$\lambda_{min} \sum_{i=1}^{n} |(U^*x)_i|^2 \le x^*Ax \le \lambda_{max} \sum_{i=1}^{n} |(U^*x)_i|^2$$

$$= \lambda_{max}\, x^*x.$$

Because U^* is unitary, by Thm 1.1,

$$\sum_{i=1}^{n} |(U^*x)_i|^2 = (U^*x)^* Ux$$

$$= x^*x$$

Now we have $x^*Ax \le \lambda_{max}\, x^*x$.

case 1, $x = 0$, $\Rightarrow$ $0 \le 0$

Case 2. $x \ne 0$ $\Rightarrow$ $x^*x > 0$

$$\max_{\substack{x \ne 0 \\ x \in \mathbb{C}^n}} \boxed{\frac{x^*Ax}{x^*x}} \le \lambda_{max}$$

Rayleigh quotient.

Now if we take x s.t. $Ax = \lambda_{max} x$.

$$\Rightarrow \quad \frac{x^*Ax}{x^*x} = \frac{x^*\lambda_{max}x}{x^*x} = \lambda_{max} \frac{x^*x}{x^*x} = \lambda_{max}.$$

$\Rightarrow$ The estimation is sharp.

Or. $$\max_{\substack{x \ne 0 \\ x \in \mathbb{C}^n}} \frac{x^*Ax}{x^*x} = \lambda_{max}$$

$$\lambda_{max} = \max_{\substack{x \neq 0 \\ x \in \mathbb{C}^n}} \frac{x^* A x}{x^* x} = \max_{\substack{x \neq 0 \\ x \in \mathbb{C}^n}} \frac{x^*}{\sqrt{x^* x}} A \cdot \boxed{\frac{x}{\sqrt{x^* x}}}$$

$y, \quad y^* y = 1.$

$$= \max_{y^* y = 1} y^* A y .$$

Thm 3.4.

If $x^t A x = x^* B x$ for all $x \in \mathbb{R}^n$

$\Leftrightarrow$ $x^t C x = 0$, $\Rightarrow C = 0$.

Check if $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $x \in \mathbb{R}^n$.

Unitary $\Rightarrow$ normal.

Hermitian $\Rightarrow$ normal

$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is normal, but not U or H.

Summary

T upper triangular

1. $U^* T U$, U is unitary. , $\forall A \in \mathbb{C}^{n \cdot n}$.

2. $U^* D U$, $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, , A is normal $\in \mathbb{C}^{n \cdot n}$.

3. $U^* D U$, $\boxed{\lambda_i \text{ are real.}}$, A is Hermitian.