1. A is normal.
2. A is unitarily diagonalizable.
3. $\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i=1}^{n} |\lambda_i|^2$.
4. There is an orthonormal set of n eigenvectors of A.

Proof. By the Schur factorization, there exist a unitary matrix U and an upper-triangular matrix T such that:

$$A = UTU^*. \tag{3}$$

Let us show 1 to 2. To show A is unitarily diagonalizable, we only need to show T is diagonal. Since A is normal, we have

$$TT^* = U^*AUU^*A^*U = U^*AA^*U = U^*A^*AU = U^*A^*UU^*AU = T^*T. \tag{4}$$

This implies that T is normal. Since T is triangular, the homework question implies that T is diagonal.

Let us now prove 2 to 4 and leave the others as exercise. From the second argument,

$$A = UTU^*, \tag{5}$$

where T is diagonal and $U = [u_1, ..., u_n]$ is unitary. It follows that,

$$AU = UT. \tag{6}$$

This is equivalent to $Au_i = \lambda_i u_i$ for all i, i.e., u_i are the orthonormal eigenvectors.

□

3 Hermitian (symmetric matrix, $\mathbb{R}$)

Definition 3.1. A matrix A is Hermitian if $A^* = A$, where $A^* = \bar{A}^T$.

Theorem 3.2. A is Hermitian if and only if at least one of the following holds:

1. x^*Ax is real for all $x \in \mathbb{C}^n$.
2. A is normal and all the eigenvalues of A are real.
3. S^*AS is Hermitian for all $S \in \mathbb{C}^{n\times n}$.

Proof. Let us first prove the first statement. Take the complex conjugate of x^*Ax, we have $(x^*Ax)^* = x^*A^*x$, since $A = A^*$, x^*Ax is real for all x. Now suppose x^*Ax is real for all x., we have

$$(x^* + y^*)A(x + y) = (x^*Ax) + (y^*Ay) + (x^*Ay + y^*Ax), \tag{7}$$

is real for all x, y. The first two terms of 7 are real; we conclude that the sum of the last two terms is real. Now let $x = e_k$ and $y = e_j$, this implies that $a_{kj} + a_{jk}$ is real, i.e., $img(a_kj) = img(a_{jk})$.

Thm 2.2.

$2 \Rightarrow 4$

$A = UTU^*$, U is unitary & T is diagonalizable. $T = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$\underbrace{AU}_{} = UTU^*U = \underbrace{UT}_{}$, $U = [u_1, u_2, \ldots, u_n]$

ith column of AU = $Au_i \Longrightarrow \lambda_i u_i$

$\Rightarrow$ $(u_i\ \lambda_i)$ is an eigenpair of A,

all eigenvectors are orthonormal to each other.

orthonormal

$4 \Rightarrow 2$ $(\lambda_i\ u_i)$ is an eigen-pair & $\{u_1 \ldots u_n\} := U$

$$Au_i = \lambda_i u_i, \quad \text{for all } i = 1, \ldots, n$$

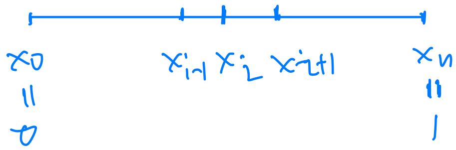
$$AU = UT, \quad T = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$A = UTU^*$$

Application of symmetric matrix.

$$\begin{cases} -u'' = f, & x \in [0,1]. \\ u(0) = u(1) = a. \end{cases}$$

$$(u'')_i = \frac{(u_i)' - (u_{i-1})'}{h} = \left(\frac{u_{i+1}-u_i}{h} - \frac{u_i - u_{i-1}}{h} \right) \Big/ h$$



$$(u'')_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

$h = x_{i+1} - x_i$, for $i = 0, \dots, n$.

$i=1$, $$-\frac{u_2 - 2u_1 + u_0}{h^2} = f(x_1)$$

$$-\frac{u_2 - 2u_1}{h^2} = f(x_1) + \frac{u_0}{h^2}$$

$i=2$ $$-\frac{u_3 - 2u_{\boxed{2}} + u_1}{h^2} = f(x_2)$$

$\| \; -u''$ at $x = x_2$

$i=n-1$, $$-\frac{u_n - 2u_{n-1} + u_{n-2}}{h^2} = f(x_{n-1})$$

$$\Longleftrightarrow \quad -\frac{-2u_{n-1} + u_{n-2}}{h^2} = f(x_{n-1}) + \frac{u_n}{h^2}$$

$$
\begin{array}{l}
1\text{st} \rightarrow \\
2\text{nd} \rightarrow \\
\frac{1}{h^2}
\end{array}
\underbrace{\begin{pmatrix} 2 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ & & & & & \\ & & & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}_{(n-1)(n-1)}}_{A \text{ (stiffness matrix)}, \quad A = A^t}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix}
=
\begin{pmatrix} f(x_1) + \frac{u_0}{h^2} \\ f(x_2) \\ \vdots \\ f(x_{n-1}) + \frac{u_n}{h^2} \end{pmatrix}
$$

Thm 3.2.

statement 1.

Assume $x^* A x$ is real for all x, want to prove A is Hermitian.

$$(x^* + y^*) A (x + y)$$

$$= \underbrace{x^* A x + y^* A y}_{\text{real}} + x^* A y + y^* A x = \text{real}$$

x^* & $y^* \in \mathbb{C}^n$

$\Rightarrow \quad x^* A y + y^* A x$ is real.

$(0, 0, \ldots 1, \cdots 0) \begin{pmatrix} \\ \\ \end{pmatrix}$

Let $x = e_k \quad y = e_j$

$$x^* A y = k\text{th row of } A \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{th} = A_{kj}$$

$y^* A x = A_{jk}$

$A_{kj} + A_{jk} = \text{real number}$

$\Rightarrow \quad \text{img}(A_{kj}) = -\text{img}(A_{jk})$

$A_{kj} = a + bi$

$A_{jk} = c + di$

$A_{kj} + A_{jk}$

$= (a+c) + (d+b)i$

$\Rightarrow b + d = 0$

$x = i e_k, \quad y = e_j$

$x^* A y = -i A_{kj} \qquad y^* A x = i A_{jk}$

$-i A_{kj} + i A_{jk} = \text{real number.}$

$\Rightarrow \text{real}(A_{kj}) = \text{real}(A_{jk})$

$$\begin{cases} p = a + bi \\ q = c + di \end{cases}$$

real part same $\Rightarrow a = c$

img part $\Rightarrow b = -d$

$\Rightarrow p = \overline{q}$

$\Rightarrow \quad A_{kj} = \overline{A_{jk}}$

$\Rightarrow \quad A = (\overline{A})^t$

$\Rightarrow \quad A$ is Hermitian matrix.