Unitary, normal and Hermitian

Zecheng Zhang

February 3, 2023

1 Unitary

Unitary matrices are the generalization of orthogonal matrices to complex field. The following theorems are equivalent and can all be regarded as the definitions of the unitary matrices.

Theorem 1.1. If $U \in \mathbb{C}^{n\times n}$, the following are equivalent:

1. U is unitary, i.e., $U^*U = I$.
2. U is nonsingular and $U^* = U^{-1}$.
3. $UU^* = I$.
4. The columns of U form an orthonormal set.
5. The rows of U form an orthonormal set.
6. For all $x \in \mathbb{C}^n$, the Euclidean length of $y = Ux$ is the same as that of x, that is $y^*y = x^*x$.

Definition 1.2. $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ are unitary equivalent if there exists a unitary matrix U such that $A = U^*BU$. $\Rightarrow$ A & B are unitary equivalent $\Rightarrow$ A & B are similar to each other.

Theorem 1.3. If A, B are unitary equivalent and are of size n, then

$$\sum_{i,j=1}^{n} |A_{ij}|^2 = \sum_{i,j=1}^{n} |B_{ij}|^2. \tag{1}$$

Proof. We have $\sum_{ij} |A_{ij}|^2 = tr(A^*A)$, by carrying out the matrix multiplication. It then suffices to check $trace(B^*B) = trace(A^*A)$. However $B = U^*AU$, it follows that,

$$trace(B^*B) = trace(U^*A^*UU^*AU) = trace(U^*A^*AU) = trace(AUU^*A^*) = trace(A^*A). \tag{2}$$

□

2 Normal

Definition 2.1. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be normal if $A^*A = AA^*$.

Theorem 2.2. Suppose $A \in \mathbb{C}^{n\times n}$ has eigenvalues $\lambda_1, ..., \lambda_n$, the following statements are equivalent:

Thm 1.3.

$$\sum_{ij} |A_{ij}|^2 = tr(A^* A)$$

It suffices to prove $tr(A^* A) = tr(B^* B)$.

Because $B = U^* A U$ (Schur factorization)

$$tr(B^* B) = tr(\underbrace{U^* A^* U}_{B^*} U^* A U)$$

U is unitary

$$= tr(U^* A^* A U)$$

$tr(AB) = tr(BA)$

$$= tr(A U U^* A^*)$$

$$= tr(A A^*) = tr(A^* A). \quad \blacksquare$$

Similarity between A & B $\not\Rightarrow$ unitary equivalent.

(check $\sum |\,|^2 \neq \sum |\,|^2$)

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

1. A is normal.
2. A is unitarily diagonalizable.
3. $\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i=1}^{n} |\lambda_i|^2$.
4. There is an orthonormal set of n eigenvectors of A.

Proof. By the Schur factorization, there exist a unitary matrix U and an upper-triangular matrix T such that:

$$A = UTU^*. \tag{3}$$

Let us show 1 to 2. To show A is unitarily diagonalizable, we only need to show T is diagonal. Since A is normal, we have

$$TT^* = U^*AUU^*A^*U = U^*AA^*U = U^*A^*AU = U^*A^*UU^*AU = T^*T. \tag{4}$$

This implies that T is normal. Since T is triangular, the homework question implies that T is diagonal.

Let us now prove 2 to 4 and leave the others as exercise. From the second argument,

$$A = UTU^*, \tag{5}$$

where T is diagonal and $U = [u_1, ..., u_n]$ is unitary. It follows that,

$$AU = UT. \tag{6}$$

This is equivalent to $Au_i = \lambda_i u_i$ for all i, i.e., u_i are the orthonormal eigenvectors.

□

3 Hermitian

Definition 3.1. A matrix A if $A^* = A$, where $A^* = \bar{A}^T$.

Theorem 3.2. *A is Hermitian if and only if at least one of the following holds:*

1. *x^*Ax is real for all $x \in \mathbb{C}^{n \times n}$.*
2. *A is normal and all the eigenvalues of A are real.*
3. *S^*AS is Hermitian for all $S \in \mathbb{C}^{n \times n}$.*

Proof. Let us first prove the first statement. Take the complex conjugate of x^*Ax, we have $(x^*Ax)^* = x^*A^*x$, since $A = A^*$, x^*Ax is real for all x. Now suppose x^*Ax is real for all x., we have

$$(x^* + y^*)A(x + y) = (x^*Ax) + (y^*Ay) + (x^*Ay + y^*Ax), \tag{7}$$

is real for all x, y. The first two terms of 7 are real; we conclude that the sum of the last two terms is real. Now let $x = e_k$ and $y = e_j$, this implies that $a_{kj} + a_{jk}$ is real, i.e., $img(a_kj) = img(a_{jk})$.

Thm 2.2.

$1 \Rightarrow 2.$ $\qquad \Leftrightarrow T = U^* A U$

Pf. $\quad A = UTU^*$, U is unitary & T is upper triangular

We need to show T is diagonal.

$$
\begin{aligned}
TT^* &= U^*AU \cdot U^*A^*U \\
&= U^*AA^*U \\
&\overset{\text{normal}}{=} U^*A^*AU \\
&= U^*A^*\underbrace{UU^*}_{I}AU = T^*T
\end{aligned}
$$

$\Rightarrow$ T is normal. $\quad$ B/c T is triangular, T is diagonal (HW)