Eigenvalue

Zecheng Zhang

February 1, 2023

1 Eigenvalue and eigenvector

eigen-vector cannot be 0

but λ can be equal to 0

An eigenvector of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$, such an $\mathbf{x}$ is called an eigenvector corresponding to λ.

λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{1}$$

has a nontrivial solution. The set of all solutions of (1) is just the null space of the matrix $A - \lambda I$, so this set is a subspace of $\mathbb{R}^n$. It is called the eigenspace of A corresponding to λ, which consists of the zero vector and all the eigenvectors corresponding to λ.

$\overset{\in \mathbb{R}^{n\times n}}{A} x = \lambda x$

If $\lambda = 0$ is an eigenvalue

e. vector

$\Rightarrow$ there exists $x \neq 0$

s.t. $Ax = 0$

$\Leftrightarrow$ dim(null(A)) > 0

$\Leftrightarrow$ rank(A) < n

$\Leftrightarrow$ A is non-singular

$\Leftrightarrow$ ... ...

Remark 1. 0 is an eigenvalue of matrix A if and only if the equation

$$A\mathbf{x} = 0\mathbf{x} \tag{2}$$

has a nontrivial solution. Furthermore, the system has a nontrivial solution if and only if A is not invertible.

2 Characteristic polynomial

The scalar equation

$$det(A - \lambda I) = 0 \tag{3}$$

is called the characteristic equation of A. A scaler λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation.

Remark 2. $det(A - \lambda I)$ is a polynomial in λ. It can be shown that if A is an $n \times n$ matrix, then $det(A - \lambda I)$ is a polynomial of degree n. This polynomial is called the characteristic polynomial of A. The (algebraic) multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Remark 3. Matrix A always has n eigenvalues if count the multiplicity.

Theorem 2.1. If $\mathbf{v}_1, \cdots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \cdots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ is linearly independent.

Proof. Let $\lambda_1 \neq \lambda_2$ be two eigenvalues and v_1 and v_2 be the corresponding eignevectors. Assume v_1 and v_2 are linear dependent. That is, there exists $a \neq 0$ such that,

$$v_1 + av_2 = 0. \tag{4}$$

Multiply A on both sides of the equation,

$$Av_1 + aAv_2 = \lambda_1 v_1 + a\lambda_2 v_2 = 0. \tag{5}$$

Multiply 4 by λ_1, we have $\lambda_1 v_1 + a\lambda_1 v_2 = 0$. Subtract 4 from 5, it follows that,

$$a(\lambda_2 - \lambda_1)v_2 = 0.$$

This implies $\lambda_1 = \lambda_2$, which is the contradiction. □

A is similar to B, If $\exists$ P non singular s.t. $A = PBP^{-1}$

3 Similarity

A and B are $n \times n$ matrices, then A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{-1}$. We say that A and B are similar. Changing A into $P^{-1}AP$ is called a similarity transformation.

Theorem 3.1. If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Proof. A and B are similar, i.e., $A = PBP^{-1}$ for some P. Let λ and v be an eigen-pair of A, i.e., $Av = \lambda v$. We now show that λ is also an eigenvalue of B. Substitute $A = PBP^{-1}$ into $Av = \lambda v$, it follows that,

$$BP^{-1}v = \lambda P^{-1}v.$$

Let us denote $w = P^{-1}v$. w is not zero since P is invertible (because the null space of B is zero). This implies that λ is also an eigenvalue of B. □

4 Diagonalizable

A is diagonalizable if $A = PDP^{-1}$, where D is a diagonal matrix, & P is invertible.

Theorem 4.1. An $n \times n$ matrix A is diagonalizable if only if A has n linearly independent eigenvectors.

$P = [P_1 \cdots P_n]$

$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \Rightarrow AP_i = \lambda_i P_i$

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if

- the columns of P are n linearly independent eigenvectors of A.
- the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Remark 4. If $A = PDP^{-1}$ for some invertible P and diagonal D, then A^k is easy to compute.

Thm 3.1.

If A & B are similar, $\exists P$ (invertible)

s.t. $A = PBP^{-1}$

Suppose $(\lambda - v)$ is an eigen pair of A,

$$Av = \lambda v \quad (v \neq 0)$$

Replace A by PBP^{-1}

By the Rank theorem

$$PBP^{-1}v = \lambda v$$

$$\dim(\text{null}(P^{-1})) + \underbrace{\text{rank}(P^{-1})}_{n} = n$$

$$\Rightarrow \underbrace{BP^{-1}v}_{w} = \lambda \underbrace{P^{-1}v}_{w}$$

$$\Rightarrow \dim(\text{null}(P^{-1})) = 0$$

$$Bw = \lambda w$$

$$\Rightarrow \text{null}(P^{-1}) = \{0\}$$

Since $v \neq 0$, $\text{rank}(P^{-1}) = n$

$\Leftrightarrow \text{null}(P^{-1}) = \{0\} \Rightarrow \underbrace{P^{-1}v}_{= w} \neq 0 \Rightarrow (\lambda, w)$ is an eigen-pair of B.

A is Unitary Diagonalizable.

if $A = UDU^*$, where U is unitary & D is diagonal.

① $U^* = (\overline{U})^T$

U has orthonormal columns.

② U is unitary, if $U^*U = UU^* = I$ → We will discuss more about the unitary matrix next time.

③ D is a diagonal matrix.

5 Schur Factorization

A Schur factorization of a matrix A is a factorization

$$A = QTQ^{\cancel{T}*}, \tag{6}$$

where Q is unitary and T is upper-triangular. A and T are similar, the eigenvalues of A necessarily appear on the diagonal of T.

Theorem 5.1 (Schur)**.** Every square matrix $A \in \mathbb{C}^{n\times n}$ has a Schur factorization.

Proof. Let us prove by induction. Case $n = 1$ is trivial. Let us prove the result for $n > 1$. Assume the result is true for $n-1$ size square matrix. Let y be any eigenvector of A with eigenvalue λ, define an unitary matrix $U = [x, P_{n\times(n-1)}] \in \mathbb{C}^{n\times n}$ with the first column being normalized y (denoted as x).

$$\begin{bmatrix} x^* \\ P^* \end{bmatrix} A \begin{bmatrix} x & P \end{bmatrix} = \begin{bmatrix} x^*A \\ P^*A \end{bmatrix} \begin{bmatrix} x & P \end{bmatrix} = \begin{bmatrix} x^*Ax & x^*AP \\ P^*Ax & P^*AP \end{bmatrix} = \begin{bmatrix} \lambda & B \\ 0 & C \end{bmatrix}, \tag{7}$$

where $B \in \mathbb{R}^{1\times(n-1)}$ and $C \in \mathbb{R}^{(n-1)\times(n-1)}$. By the inductive hypothesis, there exists a Schur factorization of VRV^* of C, where V is unitary and R is upper-triangular. Define,

$$Q = U \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}. \tag{8}$$

It is easy to verify Q is unitary and we have,

$$Q^*AQ = \begin{bmatrix} 1 & 0 \\ 0 & V^* \end{bmatrix} \begin{bmatrix} \lambda & B \\ 0 & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} \lambda & B \\ 0 & V^*C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} \lambda & BV \\ 0 & R \end{bmatrix} := T. \tag{10}$$

Since A and T are similar, they have the same eigenvalues. $\square$

Thm 5.1. (Schur)

Let us prove by mathematical induction.

Step.1 when $n=1$, trivial.

Step 2. Assume the result is true for $n-1$ square matrix.

Step 3. Let $A \in \mathbb{C}^{n\times n}$ & (λ, x) be an eigen-pair of A,

& further assume that $\|x\|^2 = 1$.

unitary matrix : all cols are orthogonal to each other & $\| \ \| = 1$.

Define an unitary matrix $U = [\,x, \underset{n\cdot(n-1)}{P}\,] \in \mathbb{C}^{n\times n}$

$$x = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 0 & & 0 \\ 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}$$

$$U^* A U = \underbrace{\begin{pmatrix} x^* \\ P^* \end{pmatrix}}_{U^*} A \underbrace{\begin{pmatrix} x & P \end{pmatrix}}_{U}$$

$$= \begin{pmatrix} x^*A \\ P^*A \end{pmatrix}_{2\cdot 1} \begin{pmatrix} x & P \end{pmatrix}_{1\cdot 2}$$

$$= \begin{pmatrix} x^*Ax, & x^*AP \\ P^*Ax, & P^*AP \end{pmatrix}$$

$$= \begin{pmatrix} \lambda, & B_{1\cdot(n-1)} \\ 0_{1\cdot(n-1)}, & C^{(n-1)(n-1)} \end{pmatrix}$$

$B = x^*AP$, $C = P^*AP$

$$x^* \boxed{Ax} = x^* \boxed{\lambda x} = \lambda x^* x = \lambda \|x\|^2 = \lambda$$

$$P^*\lambda x = \lambda P^* x = 0$$

$C \in \mathbb{C}^{(n-1)\times(n-1)}$. By the hypothesis, there exist unitary matrix V & an upper triangular matrix R s.t. $C = VRV^*$

Define

$$Q = U_{n\cdot n}\begin{bmatrix} 1 & 0 \\ 0 & V_{(n-1)(n-1)} \end{bmatrix}$$

Now let us verify Q is unitary.

$$QQ^* = U\begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & V^* \end{bmatrix}U^*}_{Q^*}$$

$$= U\begin{bmatrix} 1 & 0 \\ 0 & \underbrace{VV^*}_{=I_{n-1}} \end{bmatrix}U^*$$

$$= UU^* = I$$

$$T =: Q^* A Q$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & V^* \end{bmatrix}U^* A U\begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & B \\ 0 & V^* C \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & BV \\ 0 & R \end{bmatrix}$$

① R is an upper triangular matrix

$\Rightarrow$ T is also upper-triangular.

② T & A have the same eigenvalues.

(b/c, T & A are similar & thm 3.1)

③ T is upper triangular

$\Rightarrow$ diagonal entries of T are e.values of T

$\Rightarrow$ diagonal entries of T are evalues of A.