Definition 3.4. A set $\{u_1, \cdots, u_p\}$ in $\mathbb{R}^m$ is an orthonormal set if it is an orthogonal set of unit vectors. If W is the subspace spanned by such a set, then $\{u_1, \cdots, u_p\}$ in $\mathbb{R}^m$ is an orthonormal basis for W, since the set is automatically linearly independent.

4 Orthogonal matrix

$U = [u_1 \dots u_n]$

$\langle u_i, u_j \rangle = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Theorem 4.1. An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

Remark 3. However, UU^t may not be an identity matrix!

In general $UU^t \neq I$, where U has orthonormal columns.

Definition 4.2. An $\boxed{n \times n}$ matrix U is orthogonal if its columns are orthonormal. An equivalent definition: if $U^t U = UU^t = I$, i.e., $U^{-1} = U^t$, then U is called an orthogonal matrix.

Remark 4. The eigenvalues of an orthogonal matrix A. Suppose $Ax = \lambda x$, and let us consider the length of λx, i.e.,

$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\lambda \bar{\lambda} x^* x = |\lambda|^2 x^* x = x^* A^* A x = x^* x.$$

This implies that $\lambda = e^{i\phi}$, or, λ has module 1 and lies on the unit circle.

Theorem 4.3. Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

matrix with orthonormal columns preserve the norm, inner product, orthogonality.

1. $||U\mathbf{x}|| = ||\mathbf{x}||$
2. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$
3. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = \mathbf{0}$.

pf: $(Uy)^t Ux = y^t \underbrace{U^t U}_{I \text{ (Thm 4.1)}} x = y^t x = \langle y, x \rangle$

5 Orthogonal complement

$W^\perp = \{ w, \underbrace{\langle w, v \rangle}_{=0}, v \in W \}$

Definition 5.1. If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be orthogonal to W. The set of all orthogonal vectors to W is called the orthogonal complement of W and is denoted by $W^\perp$.

Theorem 5.2 (Complement Theorem)**.** Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row}A)^\perp = \text{Null}A \quad \text{and} \quad (\text{Col}A)^\perp = \text{Null}A^T. \tag{6}$$

Proof. Let $A = [a_1, ..., a_m]$, where $a_i \in \mathbb{R}^{1 \times n}$ is ith row of A. If $x \in null(A)$,

$$Ax = [a_1 x, a_2 x, ..., a_m x]^t = 0. \tag{7}$$

This implies that x is orthogonal to each row of A, hence x is perpendicular with $Row(A)$, or $Null(A) \subset Row(A)^\perp$. Similarly, let $x \in Row(A)^\perp$, we have $a_i x = 0$, where a_i is the $i-$th row of A, this implies that $Ax = 0$ or $x \in Null(A)$, we then have $Row(A)^\perp \subset Null(A)$.

The other result can be proved similarly.

□

Theorem 5.3. Some useful properties of complement space. Suppose W is a subspace of $\mathbb{R}^n$

(i) $0 \in W^{\perp}$, $\langle 0, v \rangle = 0$, for $\forall v \in W$.

(ii) $u \in W^{\perp}$
$v \in W^{\perp}$, $\Rightarrow$ $\langle u, y \rangle = 0$
$\langle v, y \rangle = 0$, $\forall y \in W$.

$\Downarrow$

$\langle u+v, y \rangle = 0$ $\Rightarrow$ $u+v$ is orthogonal to $\forall y \in W$ $\Rightarrow$ $u+v \in W^{\perp}$.

(iii) $u \in W^{\perp}$, $cu \in W^{\perp}$.

✓

Thm 5.2.

$$\text{Row}(A)^{\perp} = \text{null}(A).$$

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \in \mathbb{R}^{m \cdot n}, \quad a_i \in \mathbb{R}^{1 \cdot n}$$

Let $x \in \text{null}(A)$.

$$Ax = 0$$

$$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} x = 0$$

$\Updownarrow$ Row-column rule

$$\begin{bmatrix} a_1 x \\ a_2 x \\ \vdots \\ a_m x \end{bmatrix} = 0 \iff \langle a_i, x \rangle = 0, \text{ for all } i.$$

$\iff$ x is orthogonal to all rows of A.

$\iff$ x is orthogonal with row(A).

$$x \in \text{row}(A)^\perp \iff \text{null}(A) \subseteq \text{row}(A)^\perp$$

1. $W^\perp$ is a subspace.
2. $(W^\perp)^\perp = W$.
3. Let W be a subspace of R^n with dimension d, $W^\perp$ has dimension $n-d$. Moreover, W and $W^\perp$ separate $\mathbb{R}^n$.

Proof. Let $A \in \mathbb{R}^{n\times d}$ such that $col(A) = W$. By the Complement theorem, $col(A)^\perp = null(A^t)$. It follows from the Rank theorem that,

$$dim(null(A^t)) + dim(col(A^t)) = n. \tag{8}$$

Since $dim(col(A^t)) = dim(row(A)) = d$, this implies that $dim(null(A^t)) = dim(col(A)^\perp) = n-d$. $W \cap W^\perp = 0$, this implies that W and $W^\perp$ separate $\mathbb{R}^n$. □

6 Orthogonal projection

Definition 6.1. An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Theorem 6.2. Let $\{u_1, \cdots, u_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^m$. For each y in W, the weights in the linear combination

$U = [u_1 \cdots u_p]$, $c = \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix}$

$$y = c_1u_1 + \cdots + c_pu_p \tag{9}$$

solve the linear system

$U c = y$

are given by

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j}, \tag{10}$$

for all $j = 1, ..., p$. Moreover, if $\{u_1, \cdots, u_p\}$ is orthonormal, $c_j = y \cdot u_j$ for all j.

Proof. Inner product u_i on both side of the equation, this gives,

$$\langle y, u_i\rangle = c_i\langle u_i, u_i\rangle, \forall i = 1, ..., p. \tag{11}$$

It follows that $c_i = \frac{\langle y,u_i\rangle}{\langle u_i,u_i\rangle}$. When the set is orthonormal, $\langle u_i, u_i\rangle = 1$. □

Theorem 6.3. Let W be a subspace of $\mathbb{R}^n$. Then for each $y \in \mathbb{R}^n$, y can be uniquely written as:

$$y = \hat{y} + z,$$ (W & $W^\perp$ will separate $\mathbb{R}^n$)

where $\hat{y} \in W$ and $z \in W^\perp$. Moreover, let $\{u_1, \cdots, u_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$, then $\hat{y} = c_1u_1 + \cdots + c_pu_p$, c_i are defined in theorem 6.2.

Proof. Let us first show that z is orthogonal to $\hat{y}$. For any u_i,

$$u_i \cdot z = y \cdot u_i - \sum_{i=1}^{p} c_ju_j \cdot u_i = 0,$$

where we use the theorem 6.2 in the last step. Next, let us consider the uniqueness. Suppose $y = \hat{y}_1 + z_1$, where $\hat{y}_1 \in W$ and $z_1 \in W^\perp$. We have $\hat{y} - \hat{y}_1 = z - z_1$, but $\hat{y} - \hat{y}_1 \in W$ and $z - z_1 \in W^\perp$ due to the closedness of subspace. This shows that $\hat{y} - \hat{y}_1 = z - z_1 = 0$.

□

Pf thm 5.3 (iii).

$A \in \mathbb{R}^{n \times d}$, $\mathrm{col}(A) = W$, $\mathrm{rank}(A) = d$.

By the Complement thm,

$$W^{\perp} = \mathrm{col}(A)^{\perp} = \mathrm{null}(A^t)$$

By the rank theorem

$$\boxed{\dim(\mathrm{null}(A^t))} + \underbrace{\dim(\mathrm{col}(A^t))}_{= \dim(\mathrm{row}(A)) = \mathrm{rank}(A) = d} = \underset{\#\text{ cols of } A^t}{n}$$

$$\Rightarrow \underbrace{\dim(\mathrm{null}(A^t))}_{= \dim(W^{\perp})} = n - d$$

Assume not. $\exists\, v \neq 0$, $v \in W$ & $v \in W^{\perp}$

$\langle v, v \rangle = 0 \iff v = 0$. ↯

$\Rightarrow$ W & $W^{\perp}$ separate $\mathbb{R}^n$.

Thm 6.2.

$\{u_1 \cdots u_p\}$ orthogonal

$$y = \sum_{i=1}^{p} c_i u_i$$

$$\langle y, u_j \rangle = \langle \sum_{i=1}^{p} c_i u_i, u_j \rangle, \quad j = 1, \ldots, p$$

$$\langle y, u_j \rangle \overset{\text{Thm 2.2}}{=} \sum_{i=1}^{p} c_i \langle u_i, u_j \rangle$$

$$= c_j \langle u_j, u_j \rangle$$

$$\Rightarrow \quad c_j = \frac{\langle y, u_j \rangle}{\langle u_j, u_j \rangle}$$

Thm 6.3.

$y = \hat{y} + z$ is unique decomposition.

Assume not true,

$$\exists \; \hat{y}_1 \neq \hat{y}_1^{\in W} \quad z_1 \neq z_1^{\in W^\perp}$$

s.t. $\hat{y} + z = \hat{y}_1 + z_1 = y$

$$\Rightarrow \quad \underbrace{\hat{y} - \hat{y}_1}_{\in W} = \underbrace{z_1 - z}_{\in W^\perp} = 0$$

closedness of W & $W^\perp$

Σ