Orthogonality

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1 Projection

Definition 1.1. A projector is a square matrix P $\in \mathbb{R}^{n \times n}$ which satisfies $P^2 = P$.

We want to study the behavior of a vector in the range space of P. It is an important topic in approximation theory. If $v \in range(P)$, or $v = Pu$ for some u, then the projection is its own shadow, i.e., $P(Pu) = Pu$.

Definition 1.2. $I - P$ is also a projector if P is a projector. $I - P$ is called the complementary projector of P.

One can verify this as follows. $(I-P)^2 = I - 2P + P^2 = I - P$. The next theorem tells us that: onto which space does the $I - P$ project? Or, what is the $\underbrace{\text{range}}$ of the complementary projector?

col(I-P) = range(I-P)

Theorem 1.3. Let P be any projector;

$$range(I - P) = null(P).$$

Proof. Let $v \in null(P)$, it follows that $(I - P)v = v$, i.e., $v \in range(I - P)$. On the other side, $range(I - P) = \{(I - P)v, \forall v\}$. It follows that $(I - P)v = v - Pv$, but $P(v - Pv) = 0$ since P is a projector. Hence $range(I - P) \subset null(P)$. $\square$

Remark 1. Let $P = I - (I - P)$, since $(I - P)$ is also a projector, it follows from the last theorem that $range(P) = null(I - P)$.

Theorem 1.4.

$\Leftrightarrow$ null(I-P) $\cap$ null(P) = 0

$$range(P) \cap null(P) = 0. \tag{1}$$

Proof. We first claim that $null(I - P) \cap null(P) = 0$. Assume not, i.e., let $v \neq 0$, $Pv = 0$ and $(I - P)v = 0$, but this implies that $v = 0$ which is the contradiction. Since $null(I - P) = range(P)$, we have $range(P)$ intersects $null(P)$ trivially. $\square$

Remark 2. The Rank theorem shows that $dim(null(P)) + rank(P) = n$, but since $null(P)$ and $rank(P)$ intersect trivially, this implies that $range(P)$ and $null(P)$ separate $\mathbb{R}^n$. A projector will separate $\mathbb{R}^n$ into two spaces. Conversely, let S_1 and S_2 be two subspaces of $\mathbb{R}^n$, where $S_1 \cap S_2 = 0$ and $S_1 + S_2 = \mathbb{R}^n$. Can we find a projector P such that $range(P) = S_1$ and $range(I - P) = null(P) = S_2$? This is an important question, and one solution to this problem is the orthogonal projection, which we will discuss later.

Def 1.2. verification

$$(I-P)^2 = I^2 - IP - PI + \underbrace{P^2}_{P}$$

$$= I - P$$

Thm 1.3.

I. $\text{null}(P) \subseteq \text{range}(I-P)$,

$v \in \text{null}(P)$. $Pv = 0$

$$\underbrace{(I-P)v}_{\in \text{range}(I-P)} = v - Pv = v \in \text{range}(I-P)$$

$\Rightarrow$ $\text{null}(P) \subseteq \text{range}(I-P)$.

II. $\text{range}(I-P) \subseteq \text{null}(P)$.

$\text{range}(I-P) = \{(I-P)v, \ \forall v \in \mathbb{R}^n\}$

$$P(I-P)v = Pv - P^2 v = 0$$

$\Rightarrow$ $(I-P)v \in \text{null}(P)$ $\Rightarrow$ $\text{range}(I-P) \subseteq \text{null}(P)$.

Thm 1.4.

$\text{null}(I-P) \cap \text{null}(P) = 0$

Assume not. or $\exists v \neq 0$, $Pv = 0$ & $(I-P)v = 0$

$\Rightarrow$ $Iv - Pv = 0$ $\Rightarrow$ $v = 0$ contradiction.

Remark 2.

From Thm 1.4.

$$\text{range}(P) \cap \text{null}(P) = 0$$

Rank theorem.

$$\dim(\text{range}(P)) + \dim(\text{null}(P)) = n. = \dim \text{ of } \mathbb{R}^n$$

$\Rightarrow$ range (P) & null (P) will separate $\mathbb{R}^n$.

$\Leftrightarrow$ for $v \in \mathbb{R}^n$, there exist $x \in \text{null}(P)$, $y \in \text{range}(P)$

s.t. $v = x + y$.

Question,

suppose S_1 & S_2 be two subspaces of $\mathbb{R}^n$,

① $S_1 \cap S_2 = 0$

② $S_1 + S_2 = \mathbb{R}^n$

Can we find a projector P, s.t.

$$\begin{cases} \text{range}(P) = S_1 \\ \text{null}(P) = \text{range}(I-P) = S_2 \end{cases}$$

?

2 Inner product

We will review the orthogonality. We will revisit this topic later.

Definition 2.1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$,

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{2}$$

then the inner product, also referred to as a dot product, of u and v is

$$\langle u, v \rangle = [u_1 \ u_2 \ \cdots \ u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n \tag{3}$$

Theorem 2.2. Let $\mathbf{u}$, v and w be vectors in $\boxed{\mathbb{R}^n}$, and let c be a scalar. Then

a. $u \cdot v = v \cdot u$

b. $(u + v) \cdot w = u \cdot w + v \cdot w$

c. $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$, $c \in \mathbb{R}$

d. $u \cdot u \geq 0$, and $u \cdot u = 0$ if and only if $u = 0$

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$$

$$\|v\|_2 = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

$$\|v\|_1 = |v_1| + |v_2| + \ldots + |v_n|$$

Definition 2.3. The length (or norm) of a v is the nonnegative scaler $||v||$ defined by

$$||v|| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \quad \text{and} \quad ||v||^2 = v \cdot v. \tag{4}$$

For any scalar c,

$$||cv|| = |c|||v||. \tag{5}$$

3 Orthogonality

Definition 3.1. Two vectors u and v in $\mathbb{R}^n$ are orthogonal to each other if $u \cdot v = 0$ ($\langle u, v \rangle = 0$). Zero vector is orthogonal to every vector in $\mathbb{R}^n$. $\langle 0, u \rangle = 0$, for all $u \in \mathbb{R}^n$.

Theorem 3.2 (Pythagorean)**.** Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal to each other if and only if $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

Theorem 3.3. If $\{q_1, q_2, ..., q_k\}$ is a set of nonzero orthogonal vectors, then the set is linearly independent.

Proof. Assume not. There is $j \leq k$ such that:

$$q_j = c_0q_0 + ... + c_{j-1}q_{j-1} + c_{j+1}q_{j+1} + ... + c_kq_k,$$

where c_i not all zero. Now,

$$q_j \cdot q_j = 0.$$

This implies that $\|q_j\| = 0$, which is a contradiction. $\square$

Thm 3.2.

$$\|u+v\|^2 = \langle u+v, u+v \rangle = \langle u, v\rangle + 2\langle u, v\rangle + \langle v, v\rangle$$

$$= \langle u, u\rangle + \langle v, v\rangle$$

$$= \|u\|^2 + \|v\|^2$$

Def orthogonal set.

$\{q_1\, q_2 \ldots q_k\}$, $q_i \in \mathbb{R}^n$,

$$\langle q_i, q_j \rangle = \begin{cases} 0, & i \neq j \\ \neq 0, & i = j. \end{cases}$$

$\Rightarrow$ $\{q_1 \ldots q_k\}$ is an orthogonal set.

Thm 3.3.

pf: $\{q_1 \ldots q_k\}$ is an orthogonal set.

Assume, $\{q_1 \ldots q_k\}$ is not linearly indep.

There exists $j \leq k$. s.t.

$$q_j = c_0 q_0 + \ldots c_{j-1} q_{j-1} + c_{j+1} q_{j+1} + \ldots c_k q_k. \quad (*)$$

c_i not all = zero.

Consider the inner product of $(*)$ with q_i.

$$\Rightarrow \langle \xi_j , \xi_i \rangle \underset{\substack{\uparrow \\ \text{orthogonal} \\ \text{set}}}{=0} = c_0 \underbrace{\langle \xi_0, \xi_i \rangle}_{=0} + \dots + \underbrace{c_i \langle \xi_i , \xi_i \rangle}_{\neq 0} + \dots c_k \underbrace{\langle \xi_k, \xi_i \rangle}_{=0}$$

$$\Rightarrow \quad 0 = c_i \underbrace{\langle \xi_i , \xi_i \rangle}_{\neq 0} \quad \Rightarrow \quad c_i = 0.$$

Since i can be any index which is not $\neq j$

$$\Rightarrow \quad c_1 = c_2 = c_3 = \dots c_{j-1} = c_{j+1} = \dots c_k = 0$$

$\lightning$

* linearly indep $\not\Rightarrow$ orthogonality.

Eg. $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\langle v_1 , v_2 \rangle \neq 0$$

Orthonormal set.

If $\{v_1 \; v_2 \dots v_k\}$ is an orthonormal set,

① $\{v_1 \dots v_k\}$ must be orthogonal.

② $\langle v_i , v_i \rangle = \|v_i\|^2 = 1.$