

6 Subspace

A subspace of \mathbb{R}^n is any set H in \mathbb{R}^n that has three properties:

1. The zero vector is in H .
2. For each \mathbf{u} and \mathbf{v} in H , the sum $\mathbf{u} + \mathbf{v}$ is in H .
3. For each \mathbf{u} in H and each scalar c , the vector $c\mathbf{u}$ is in H .

Remark 4. A subspace is closed under addition and scalar multiplication.

6.1 Column space

The column space of a matrix A is the set $\text{Col}(A)$ of all linear combinations of the columns of A . If $A = [a_1, a_2, \dots, a_n]$, then $\text{col}(A) = \text{span}\{a_1, a_2, \dots, a_n\}$.

6.2 Null space

The null space of a matrix A is the set $\text{null}(A)$ of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

* all possible linear combinations of cols of A .

$$\text{col}(A) = \{A\mathbf{x}, \forall \mathbf{x} \in \mathbb{R}^n\}$$

$$* T: \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ linear, } T(\mathbf{x}) = A\mathbf{x}, \forall \mathbf{x} \in \mathbb{R}^n, \text{ range}(T) = \left\{ \begin{matrix} T(\mathbf{x}), \\ \mathbf{x} \in \mathbb{R}^n \end{matrix} \right\}$$

$$\Rightarrow \text{col}(A) = \text{range}(A)$$

6.3 Row space

The row space of a matrix A is the set $\text{row}(A)$ of all linear combinations of the rows of A .

Theorem 6.1. If two matrices A and B are row equivalent, their row spaces are the same.

Proof. Row operations are indeed the linear combinations of rows. If B is obtained from A by the EROs, the rows of B are the linear combinations of rows of A . As a result, the row space of B is in the row space of A . The other way is the same. \square

What about the column space?

Remark 5. • The column space of an $m \times n$ matrix is a subspace of \mathbb{R}^m .

- The null space of an $m \times n$ matrix A is a subspace of \mathbb{R}^n .

6.4 Basis

for any $v \in H$, $\exists c_1 c_2 \dots c_p$
 s.t. $v = \sum_{i=1}^p c_i b_i$, where $\{b_1 \dots b_p\}$ is a basis of H .

A basis for a subspace H of \mathbb{R}^n is a linearly independent set in H that spans H .

Theorem 6.2. The pivot columns of a matrix A form a basis for the column space of A .

Theorem 6.3. The nonzero rows of the $\text{ref}(A)$ form a basis for $\text{row}(A)$. \downarrow \times . basis is the pivot cols of A but not the pivot col of $\text{ref}(A)$.

6.5 Dimension

The dimension of a nonzero subspace H , denoted by $\dim(H)$, is the number of vectors on any basis for H . The dimension of the zero subspace $\{0\}$ is defined as zero.

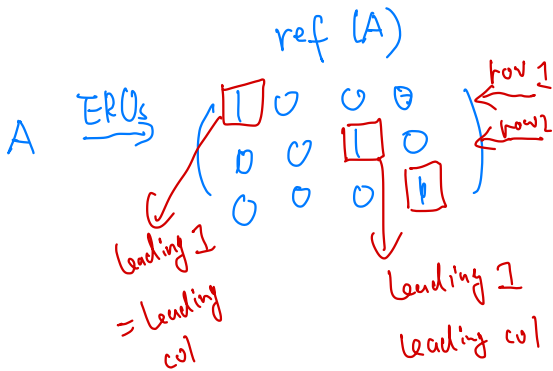
6.6 Rank

$$\text{rank}(A) = \dim(\text{col}(A)) = \text{col rank}(A)$$

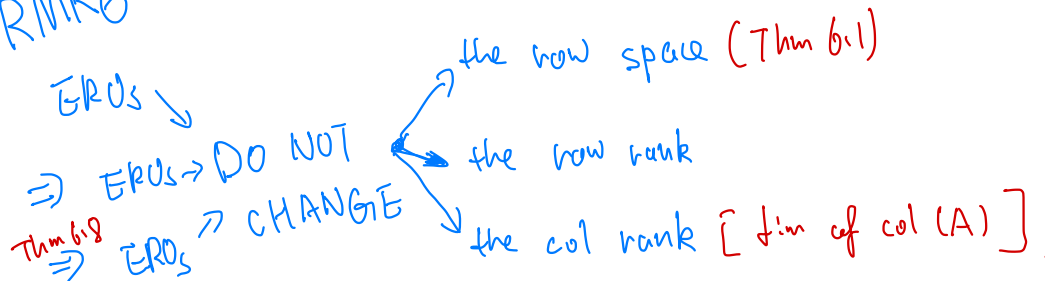
The rank of a matrix A , denoted by $\text{rank}(A)$, is the dimension of the column space of A . In addition, the dimension of the null space is called nullity. In addition, the row space dimension is called the matrix's row rank. $\text{row rank}(A) = \dim(\text{row}(A))$

Remark 6. The EROs do not change the dimension of the column space; hence the EROs do not change the rank of the matrix. $A \in \mathbb{R}^{m \times n}$

Theorem 6.4 (Rank theorem). If a matrix A has n columns, then $\text{rank}(A) + \dim(\text{null}(A)) = n$.
 \parallel
 $\dim(\text{col}(A)) = \# \text{ cols of } A$.



RANK



Example 6.5. Find the rank, column space, and null space of the matrix.

Theorem 6.6. Some facts about the rank.

- $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.
- $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.

$$3. \text{rank}(AA^T) = \text{rank}(A^T A) = \text{rank}(A) = \text{rank}(A^T).$$

Proof. I will only show the first statement and leave the other two as the homework questions. Since the columns of AB are the linear combinations of columns of A by B , this implies that $\dim(\text{col}(AB)) \leq \dim(\text{col}(A))$. It follows that $\text{rank}(AB) \leq \text{rank}(A)$.

Suppose $x \in \text{null}(B)$, this implies that $Bx = 0$, consequently, $ABx = 0$, or, $x \in \text{null}(AB)$. This indeed shows that $\text{null}(B) \subset \text{null}(AB)$, or $\dim(\text{null}(B)) \leq \dim(\text{null}(AB))$. Since B and AB have the same number of columns, it follows from the Rank theorem that $\text{rank}(AB) \leq \text{rank}(B)$. \square

pf of $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$

$$\text{I. } \text{rank}(AB) \leq \text{rank}(A).$$

The cols of AB = linear combination of cols A with B .

$$\Rightarrow \text{col}(AB) \subseteq \text{col}(A)$$

$$\Rightarrow \dim(\text{col}(AB)) \leq \dim(\text{col}(A))$$

$$\text{rank}(AB) \leq \text{rank}(A)$$

$$\text{II. } \text{rank}(AB) \leq \text{rank}(B).$$

let $\forall x \in \text{null}(B)$, assumption of the linear transformation.
def of null space $\Rightarrow Bx = 0 \Rightarrow ABx = 0$.

$$\Rightarrow x \in \text{null}(B) \Rightarrow \text{null}(B) \subseteq \text{null}(AB)$$

$$\Rightarrow \dim(\text{null}(B)) \leq \dim(\text{null}(AB)),$$

cols of B = # cols of AB , by the Rank Theorem \Rightarrow
 $\text{rank}(AB) \leq \text{rank}(B)$.

6.6.1 Rank decomposition [factorization]

Every rank r matrix $A \in \mathbb{R}^{m \times n}$ matrix has a rank decomposition $A = CR$, where $C \in \mathbb{R}^{m \times r}$, $R \in \mathbb{R}^{r \times n}$ and columns of C form a basis for $\text{col}(A)$. One can construct C by taking all linearly independent columns of A . Because each column of A is the linear combination of columns of C by weights from the corresponding columns of R , the R matrix can be constructed easily. One way is to remove all zero rows from $\text{ref}(A)$.

$$A \in \mathbb{R}^{m \times n}, \text{rank}(A) = r$$

$$A = CR,$$

$$\textcircled{1} C \in \mathbb{R}^{m \times r},$$

$$\textcircled{2} \text{col}(C) = \text{col}(A).$$

$$\textcircled{3} R \in \mathbb{R}^{r \times n}$$

$$A = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 3 & 9 \\ 1 & 5 & 3 & 1 \\ 1 & 2 & 0 & 8 \end{pmatrix}$$

perform EROs to reduce A to its $\text{ref}(A)$.

$$\text{ref}(A) = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By thm 6.2, basis of $\text{col}(A)$

$$C = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 7 & 3 \\ 1 & 5 & 3 \\ 1 & 2 & 0 \end{pmatrix}$$

$R =$ non-zero rows of $\text{ref}(A)$.

$$= \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A = CR.$$

Theorem 6.7. For $A \in \mathbb{R}^{m \times n}$, we have $\text{rank}(A) = \text{rank}(A^t)$.

Proof. Suppose $\text{rank}(A) = r$ and admits the rank-decomposition $A = CR$. We have $A^t = R^t C^t$. Since the columns of A^t is the linear combination of columns of R^t , this implies that $\text{col}(A^t) \subset \text{col}(R^t)$, or $\text{rank}(A^t) \leq \text{rank}(R^t) \leq r = \text{rank}(A)$. Consider $A = (A^t)^t$ and complete the proof by yourself. \square

Theorem 6.8. The row rank is equal to the column rank of a matrix.

Thm 6.7.

pf $\text{rank}(A) = r$, $A \in \mathbb{R}^{m \times n}$ admits one Rank decomposition.

$$A = CR.$$

$A^t = R^t C^t$, cols of $A^t =$ linear combination of cols of R^t using C^t .

$$\Rightarrow \text{col}(A^t) \subseteq \text{col}(R^t).$$

$$\Rightarrow \text{rank}(A^t) \leq \text{rank}(R^t) \leq r = \text{rank}(A)$$

$$\leftarrow R \in \mathbb{R}^{n \times r} \Rightarrow R^t \in \mathbb{R}^{r \times n}, \Rightarrow R \text{ has } r \text{ cols.}$$

$$\Rightarrow \text{rank}(R) \leq r$$

Consider $A = (A^t)^t$, $\Rightarrow \text{rank}(A) \leq \text{rank}(A^t)$

$$\Rightarrow \text{rank}(A) = \text{rank}(A^t).$$



Thm 6.8. $\text{row rank}(A) = \text{col rank}(A)$.

$$\text{col rank}(A) = \text{rank}(A) \stackrel{\text{Thm 6.7}}{=} \text{rank}(A^t) = \text{col rank}(A^t)$$

$$= \text{row rank}(A)$$