6 Subspace

A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

1. The zero vector is in H.
2. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
3. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Remark 4. A subspace is closed under addition and scalar multiplication.

6.1 Column space

The column space of a matrix A is the set Col (A) of all linear combinations of the columns of A. If $A = [a_1, a_2, ..., a_n]$, then $col(A) = span\{a_1, a_2, ..., a_n\}$.

6.2 Null space

The null space of a matrix A is the set null(A) of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

* all possible linear combinations of cols of A.

$col(A) = \{ Ax, \forall x \in \mathbb{R}^n \}$

* $T: \mathbb{R}^n \to \mathbb{R}^m$ linear, $T(x) = Ax$, $\forall x \in \mathbb{R}^n$, $range(T) = \{T(x), x \in \mathbb{R}^n\}$

$\Rightarrow \quad col(A) = range(A)$

6.3 Row space

The row space of a matrix A is the set $row(A)$ of all linear combinations of the rows of A.

Theorem 6.1. If two matrices A and B are row equivalent, their row spaces are the same.

Proof. Row operations are indeed the linear combinations of rows. If B is obtained from A by the EROs, the rows of B are the linear combinations of rows of A. As a result, the row space of B is in the row space of A. The other way is the same. $\square$

What about the column space?

Remark 5.
- The column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.
- The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

for any $v \in H$, $\exists\, c_1\, c_2 \cdots c_p$ s.t. $v = \sum_{i=1}^{p} c_i b_i$, where $\{b_1 \dots b_p\}$ is a basis of H.

6.4 Basis

A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

Theorem 6.2. The pivot columns of a matrix A form a basis for the column space of A.

Theorem 6.3. The nonzero rows of the $ref(A)$ form a basis for $row(A)$.

*. basis is the pivot cols of A but not the pivot col of ref(A).

6.5 Dimension

The dimension of a nonzero subspace H, denoted by $\dim(H)$, is the number of vectors on any basis for H. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined as zero.

6.6 Rank

$\text{rank}(A) = \dim(\text{col}(A)) = \text{col rank}(A)$

The rank of a matrix A, denoted by $\text{rank}(A)$, is the dimension of the column space of A. In addition, the dimension of the null space is called nullity. In addition, the row space dimension is called the matrix's row rank. $\text{row rank}(A) = \dim(\text{row}(A))$

Remark 6. The EROs do not change the dimension of the column space; hence the EROs do not change the rank of the matrix. $A \in \mathbb{R}^{m \times n}$

Theorem 6.4 (Rank theorem)**.** If a matrix A has n columns, then $\underbrace{rank(A)}_{=\dim(\text{col}(A))} + dim(null(A)) = n$. $= \#$ cols of A.

$$A \xrightarrow{\text{EROs}} ref(A) = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix} \begin{array}{l} \leftarrow \text{row 1} \\ \leftarrow \text{row 2} \\ \\ \end{array}$$

leading 1 = leading col

leading 1 leading col

RMK6

EROs ↘

⟹ EROs → DO NOT CHANGE

Thm 6.8 ⟹ EROs ↗ CHANGE

the row space (Thm 6.1)

the row rank

the col rank [dim of col(A)].

Example 6.5. Find the rank, column space, and null space of the matrix.

Theorem 6.6. Some facts about the rank.

1. $rank(AB) \leq \min(rank(A), rank(B))$.
2. $rank(A + B) \leq rank(A) + rank(B)$.

3. $rank(AA^T) = rank(A^TA) = rank(A) = rank(A^T)$.

Proof. I will only show the first statement and leave the other two as the homework questions. Since the columns of AB are the linear combinations of columns of A by B, this implies that $dim(col(AB) \leq dim(col(A))$. It follows that $rank(AB) \leq rank(A)$.

Suppose $x \in null(B)$, this implies that $Bx = 0$, consequently, $ABx = 0$, or, $x \in null(AB)$. This indeed shows that $null(B) \subset null(AB)$, or $dim(null(B)) \leq dim(null(AB))$. Since B and AB have the same number of columns, it follows from the Rank theorem that $rank(AB) \leq rank(B)$. □

pf of $rank(AB) \leq \min(rank(A), rank(B))$

I. $rank(AB) \leq rank(A)$.

The cols of AB = linear combination of cols A with B.

$\Rightarrow \quad col(AB) \subseteq col(A)$

$\Rightarrow \quad \dim(col(AB)) \leq \dim(col(A))$

$rank(AB) \leq rank(A)$

II. $rank(AB) \leq rank(B)$.

let $\forall x \in null(B)$,

assumption of the linear transformation.

def of null space $\Rightarrow \quad Bx = 0 \quad \Rightarrow \quad ABx = 0$.

$\Rightarrow \quad x \in null(AB) \Rightarrow null(B) \subseteq null(AB)$

$\Rightarrow \quad \dim(null(B)) \leq \dim(null(AB))$,

cols of B = # cols of AB, by the Rank Theorem $\Rightarrow$ $rank(AB) \leq rank(B)$.

6.6.1 Rank decomposition [factorization].

Every rank r matrix $A \in \mathbb{R}^{m\times n}$ matrix has a rank decomposition $A = CR$, where $C \in \mathbb{R}^{m\times r}$, $R \in \mathbb{R}^{r\times n}$ and columns of C form a basis for $col(A)$. One can construct C by taking all linearly independent columns of A. Because each column of A is the linear combination of columns of C by weights from the corresponding columns of A, the R matrix can be constructed easily. One way is to remove all zero rows from $ref(A)$.

$A \in \mathbb{R}^{m\cdot n}$, $rank(A) = r$

$A = CR$,

① $C \in \mathbb{R}^{m\times r}$,

② $col(C) = col(A)$.

③ $R \in \mathbb{R}^{r\cdot n}$

$$A = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 3 & 9 \\ 1 & 5 & 3 & 1 \\ 1 & 2 & 0 & 8 \end{pmatrix}$$

perform EROs to reduce A to its $ref(A)$.

$$ref(A) = \begin{pmatrix} \boxed{1} & 0 & -2 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By thm 6.2, basis of col (A)

$$C = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 7 & 9 \\ 1 & 5 & 1 \\ 1 & 2 & 8 \end{pmatrix}$$

R. = non-zero rows of ref (A).

$$= \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Theorem 6.7. For $A \in \mathbb{R}^{m\times n}$, we have $rank(A) = rank(A^t)$.

$A = CR$.

Proof. Suppose $rank(A) = r$ and admits the rank-decomposition $A = CR$. We have $A^t = R^tC^t$. Since the columns of A^t is the linear combination of columns of R^t, this implies that $col(A^t) \subset col(R^t)$, or $rank(A^t) \le rank(R^t) \le r = rank(A)$. Consider $A = (A^t)^t$ and complete the proof by yourself. □

Theorem 6.8. The row rank is equal to the column rank of a matrix.

Thm 6.7.

pf $\quad \text{rank}(A) = r, \quad A^{\in \mathbb{R}^{m \cdot n}}$ admits one Rank decomposition.

$A = CR.$

$A^t = R^t C^t, \quad$ cols of A^t = linear combination of cols of R^t using C^t.

$\Rightarrow \quad \text{col}(A^t) \subseteq \text{col}(R^t).$

$\Rightarrow \quad \text{rank}(A^t) \leq \text{rank}(R^t) \leq r = \text{rank}(A)$

$R \in \mathbb{R}^{r \cdot n} \Rightarrow R^t \in \mathbb{R}^{n \cdot r}, \Rightarrow R$ has r cols.

$\Rightarrow \text{rank}(R) \leq r$

Consider $A = (A^t)^t, \Rightarrow \text{rank}(A) \leq \text{rank}(A^t)$

$\Rightarrow \quad \text{rank}(A) = \text{rank}(A^t).$ $\blacksquare$

Thm 6.8. $\quad$ row rank (A) = col rank (A).

$$\text{col rank}(A) = \text{rank}(A) \overset{\text{Thm}}{\underset{6.7}{=}} \text{rank}(A^t) = \text{col rank}(A^t)$$

$$= \text{row rank}(A)$$