5 Inner product

We will review the orthogonality. We will revisit this topic later.

Definition: Let **u** and **v** be vectors in $\mathbb{R}^n$,

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then the inner product, also referred to as a dot product, of **u** and **v** is

$$[u_1 \; u_2 \; \cdots \; u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

$$\langle u, v \rangle = u \cdot v = \underline{u^t} v$$

Theorem 5.1. Let **u**, **v** and **w** be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

norm

i. $\|u+v\| \leq \|u\| + \|v\|$

ii. $\|cu\| = |c| \cdot \|u\|$

(iii) $\|u\| \geq 0, \; \|u\| = 0 \Leftrightarrow u = 0$

Definition: The length (or norm) of a **v** is the nonnegative scaler $||\mathbf{v}||$ defined by

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \quad \text{and} \quad ||\mathbf{v}||^2 = \mathbf{v} \cdot \mathbf{v}$$

For any scalar c,

$$||c\mathbf{v}|| = |c|||\mathbf{v}||.$$

6 Orthogonality

$\langle u, v \rangle = 0$

Orthogonal vectors: Two vectors **u** and **v** in $\mathbb{R}^n$ are orthogonal to each other if $\mathbf{u} \cdot \mathbf{v} = 0$. Zero vector is orthogonal to every vector in $\mathbb{R}^n$.

Theorem 6.1 (Pythagorean)**.** Two vectors **u** and **v** in $\mathbb{R}^n$ are orthogonal to each other if and only if $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

$$\|u+v\| = \sqrt{\langle u+v, u+v \rangle} \Rightarrow \|u+v\|^2 = \langle u+v, u+v \rangle$$

$$= \underline{\langle u, u \rangle} + 2\underbrace{\cancel{\langle u, v \rangle}}_{\text{orthog.}}^{=0} + \underline{\langle v, v \rangle}$$

$$= \|u\|^2 + \|v\|^2$$

Theorem 6.2. if $\{q_1, q_2, ..., q_k\}$ is a set of $\boxed{\text{nonzero}}$ orthogonal vectors, then the set is linearly independent.

Proof. Assume not. There is $j \leq k$ such that:

$$q_j = c_0q_0 + ... + c_{j-1}q_{j-1} + c_{j+1}q_{j+1} + ... + c_kq_k,$$

where c_i not all zero. Now, test with q_j

$$\text{LHS} = q_j \cdot q_j = 0. \Rightarrow \|q_j\|^2 = 0$$

This implies that $\|q_j\| = 0$, which is a contradiction. ☐

$$\text{RHS} = c_0 \underbrace{\cancel{\langle q_0, q_j \rangle}}_{=0} + \ldots c_{j-1} \underbrace{\cancel{\langle q_{j-1}, q_j \rangle}}_{=0} + c_{j+1} \underbrace{\cancel{\langle q_{j+1}, q_j \rangle}}_{=0} + \ldots$$

$$= 0$$

linear indep $\not\Rightarrow$ orthogonal

counter example : $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ v_1 v_2 $v_1 \cdot v_2 = 1 \neq 0$ $\Rightarrow$ not orthogonal.

If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be orthogonal to W. The set of all vectors that are orthogonal to W is called the orthogonal ~~component~~ complement of W and is denoted by $W^{\perp}$. mathematically $W^{\perp} = \{u \in \mathbb{R}^n, \underbrace{u \perp v}_{u \cdot v = 0 \text{ (u \& v are orthogonal)}}, \forall v \in W\}$.

if $\underset{\text{vector}}{z} \boxed{\perp} \underset{\text{subspace}}{W}$ then it satisfies $z \cdot v = 0$, for all $v \in W$.

1. A vector $\mathbf{x}$ is in $W^{\perp}$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.
2. $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Theorem: Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row}A)^{\perp} = \text{Nul}A \text{ and } (\text{Col}A)^{\perp} = \text{Nul}A^T$$

Orthonormal Sets: A set $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an orthonormal set if it is an orthogonal set of unit vectors. If W is the subspace spanned by such a set, then $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an orthonormal basis for W, since the set is automatically linearly independent.

① orthogonal set
② $\|u_i\| = 1$. for all i

7 Orthogonal projection

Definition: An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

① basis
② orthogonal.

Theorem 7.1. Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p \quad \Leftrightarrow \quad \text{solve } [u_1 \dots u_p] \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix} = y$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j},$$

for all $j = 1, ..., p$. Moreover, if $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ is orthonormal, $c_j = \mathbf{y} \cdot \mathbf{u}_j$ for all j.

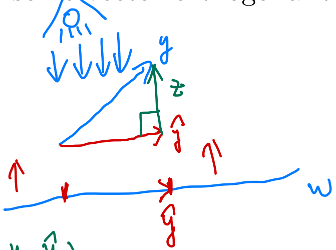
Now let assume $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Given a nonzero vector $\mathbf{y}$ in $\mathbb{R}^n$. One can decompose a vector $\mathbf{y}$ in $\mathbb{R}^n$ into the sum of two vectors

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \tag{6}$$

where $\hat{\mathbf{y}} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$, c_i are defined in theorem 7.1, and $\mathbf{z}$ is some vector orthogonal to $\mathbf{u}$ $\hat{y}$.

Remark 5. Let us show that z is orthogonal to $\hat{y}$. For any u_i,

$$\langle u_i, z \rangle = u_i \cdot z = y \cdot u_i - \sum_{i=1}^{p} c_i u_i \cdot u_i = 0,$$



where we use the theorem 7.1 in the last step.

$$u_i \cdot z \overset{(6)}{=} u_i \cdot (y - \hat{y})$$
$$= u_i \cdot y - u_i \cdot \hat{y}$$
$$\overset{\text{thm 7.1}}{=} u_i \cdot y - u_i \cdot (\cancel{c_1 u_1} + \dots c_i u_i + \dots \cancel{c_p u_p})$$
$$= u_i \cdot y - c_i \langle u_i, u_i \rangle$$
$$= u_i \cdot y - \frac{u_i \cdot y}{\cancel{u_i \cdot u_i}} \cancel{\langle u_i, u_i \rangle} = 0$$
$$\Rightarrow z \cdot \hat{y} = 0$$

Remark 6. if $y \in W$, $z = 0$.

$\hat{\mathbf{y}}$ is denoted by $\text{proj}_W \mathbf{y}$ and is called the orthogonal projection of $\mathbf{y}$ onto W.

Dimension of W & $W^{\perp}$.

$\dim(W) = k$, $\dim(W^{\perp}) = n - k$

let $\{\underbrace{b_1, b_2 \cdots b_k}_{\text{as the basis for } W} \,\|\, \underbrace{b_{k+1}, \cdots b_n}_{n-k}\}$ be an orthogonal basis for $\mathbb{R}^n$.

check : $b_j \perp b_i$ for $\forall j \geq k+1$
$\forall i \leq k$

$\Rightarrow$ $\{b_{k+1}, \dots b_n\}$ forms a basis of $W^{\perp}$

$\Rightarrow$ $\dim(W^{\perp}) = n - k$.

8 Orthogonal matrix

Theorem 8.1. An $m \times n$ matrix U has orthonormal columns $\overset{\Longleftrightarrow}{\text{if and only if}}$ $U^T U = I$.

An $\boxed{n \times n}$ matrix U is orthogonal if its columns are orthonormal. An equivalent definition: if $U^t U = UU^t = I$, i.e., $U^{-1} = U^t$, then U is called an orthogonal matrix.

or $UU^t = I$

def of inverse of a matrix.

Remark 7. The eigenvalues of an orthogonal matrix A. Suppose $Ax = \lambda x$, and let us consider the length of λx, i.e.,

(x, λ & U may be complex)

$$\|\lambda x\|^2 = \langle \lambda x, \lambda x \rangle = (\lambda x)^* \cdot \lambda x = \lambda\bar{\lambda}x^*x = |\lambda|^2 x^*x = x^*A^*Ax = x^*x.$$

This implies that $\lambda = e^{i\phi}$, or, λ has module 1 and lies on the unit circle.

Theorem 8.2. Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

1. $||U\mathbf{x}||^2 = ||\mathbf{x}||^2$
2. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$
3. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = \mathbf{0}$.

$$\langle Ux, Ux \rangle = x^t \underbrace{U^t U}_{I} x = x^t x = \|x\|^2$$

$$\lambda = re^{\phi i}$$

$$\bar{\lambda} = re^{-\phi i}$$

$$\lambda \cdot \bar{\lambda} = r^2 e^0 = r^2 = |\lambda|^2$$