Problem Sheet 3

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Problem 1. (1) Let

$$\alpha = (2x + y\cos xy)dx + (x\cos xy)dy$$

on $\mathbb{R}^2$, show that α is exact.

(2) Show that $\mathbb{R}^m$ and $\mathbb{R}^n$ are diffeomorphic if and only if $m = n$.

(3) Define

$$\pi_*^k : \ \Omega_c^k(\mathbb{R}^n \times \mathbb{R}) \to \Omega_c^{k-1}(\mathbb{R}^n)$$

by mapping a compactly supported k form

$$\omega = \sum_{I=(i_1<\cdots<i_k)} a_I(x,t)dx^I + \sum_{J=(j_1<\cdots<j_{k-1})} b_J(x,t)dx^J \wedge dt$$

on $\mathbb{R}^n \times \mathbb{R}$ to the compactly supported $(k-1)$-form

$$\pi_*^k(\omega) = \sum_{J=(j_1<\cdots<j_{k-1})} (\int_{\mathbb{R}^1} b_J(x,t)dt)dx^J$$

on $\mathbb{R}^n$ (this map is called *integration along fiber*). Show that $\pi_* = (\pi_*^{\cdot})$ is a chain map from the cochain complex $\Omega_c^*(\mathbb{R}^n \times \mathbb{R})$ to the cochain complex $\Omega_c^{*-1}(\mathbb{R}^n) = (\Omega_c^{\cdot-1}(\mathbb{R}^n), d)$. Moreover, show that the induced homomorphism on cohomology is an isomorphism.

(4) Compute the de Rham cohomology groups of $\mathbb{R}^n$ with compact supports.

Problem 2. (1) Let $A^* = (A^{\cdot}, d_A^{\cdot})$ and $B^* = (B^{\cdot}, d_B^{\cdot})$ be cochain complexes. Show that for each r,

$$H^r(A^* \oplus B^*) \cong H^r(A^*) \oplus H^r(B^*),$$

where $A^* \oplus B^*$ is the cochain complex $(A^{\cdot} \oplus B^{\cdot}, d_A^{\cdot} \oplus d_B^{\cdot})$.

(2) Prove the Five Lemma.

(3) Let A^* be a cochain complex of the form

$$0 \longrightarrow A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A^n \longrightarrow 0$$

such that

$$\dim A^r < \infty, \ \forall r = 0, 1, \cdots, n.$$

Define the *Euler characteristic* of A^* by

$$\chi(A^*) = \sum_{r=0}^{n}(-1)^r \dim A^r.$$

Show that

$$\chi(A^*) = \sum_{r=0}^{n}(-1)^r \dim H^r(A^*).$$

Problem 3. (1) Compute the de Rham cohomology of the following manifolds:

(i) the n-dimensional tours $T^n = S^1 \times \cdots \times S^1$;

(ii) $\mathbb{R}^2 \backslash \{p, q\}$, where p, q are two distinct points in $\mathbb{R}^2$;

(iii) the n-dimensional real projective space $\mathbb{R}P^n$.

(2) What is a generator of $H^{n-1}(\mathbb{R}^n \backslash \{0\})$? Let A be an invertible matrix of order n. Define the diffeomorphism

$$f_A : \ \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$$

to be the multiplication by A. What is the induced homomorphism

$$f_A^* : \ H^{n-1}(\mathbb{R}^n \backslash \{0\}) \to H^{n-1}(\mathbb{R}^n \backslash \{0\})$$

on the $(n-1)$-th de Rham cohomology group of $\mathbb{R}^n \backslash \{0\}$?

(3) Let $i_k : \ S^1 \to T^n$ be the inclusion

$$i_k(z) = (1, \cdots, z, \cdots, 1), \ z \in S^1,$$

into the k-th component, where we regard S^1 as a subset in $\mathbb{C}$ and use complex numbers. Take the product orientation on T^n induced by the one on the circle S^1. Define a map

$$I : \ \Omega^1(T^n) \to \mathbb{R}^n$$

by

$$I(\omega) = (\int_{S^1} i_1^* \omega, \cdots \int_{S^1} i_n^* \omega).$$

Show that it induces a linear map

$$I : \ H^1(T^n) \to \mathbb{R}^n$$

which is an isomorphism. Construct a basis of $H^1(T^n)$ explicitly.

Problem 4. (1) Show that for any C^∞ map $F : \ S^n \to T^n$ $(n \geqslant 2)$, $\deg F = 0$.

(2) For each $n \in \mathbb{Z}$, construct a smooth map $F : \ S^1 \to S^1$ such that $\deg F = n$. Can you do the same thing on all spheres S^m?

(3) The *quaternions* consist of the 4-dimensional associative algebra $\mathbb{H}$ of expressions

$$q = x^0 + ix^1 + jx^2 + kx^3$$

where $(x^0, x^1, x^2, x^3) \in \mathbb{R}^4$ and i, j, k satisfy the relations

$$i^2 = j^2 = k^2 = -1; \; ij = -ji = k; \; jk = -kj = i; \; ki = -ik = j.$$

How many solutions are there to the equation $q^2 = 1$? How about $q^2 = -1$?

Define an atlas on $M = \mathbb{H} \cup \{\infty\}$ by taking $U_0 = \mathbb{H}$, $\varphi_0 : \; U_0 \to \mathbb{R}^4$ to be the identity map, and $U_1 = (\mathbb{H} \backslash \{0\}) \cup \{\infty\}$, $\varphi_1 : \; U_1 \to \mathbb{R}^4$ by

$$\varphi_1(x) = \begin{cases} \frac{q}{\|q\|^2}, & q \neq \infty; \\ 0, & q = \infty, \end{cases}$$

where $\| \cdot \|$ is the Euclidean norm. This defines a differential structure on M which is diffeomorphic to S^4 (similar to the case of the Riemann sphere).

Define a C^∞ map $F : \; M \to M$ by

$$F(q) = \begin{cases} q^2, & q \in \mathbb{H}; \\ \infty, & q = \infty. \end{cases}$$

What is the degree of F?

Problem 5. Let M be a two dimensional connected and orientable manifold embedded in $\mathbb{R}^3$ via the inclusion map. We assume that the orientation on M is taken by specifying a unit normal vector field N on M. We always think of the tangent space at each point of M as an embedded linear subspace in $\mathbb{R}^3$ via the embedding.

(1) Choose an oriented chart $\{U; (u, v)\}$ and define the 2-form

$$\nu = \sqrt{\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \rangle \langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \rangle - \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \rangle^2} du \wedge dv$$

on U, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^3$. Show that ν is invariant under change of oriented charts, and hence it defines a global 2-form on M.

(2) Consider the C^∞ map $S : \; M \to S^2$ defined by

$$S(p) = N(p), \; p \in M.$$

For each $p \in M$, by identifying the tangent space T_pM with $T_{N(p)}S^2$, the differential $(dS)_p$ defines linear map from T_pM to itself. Show that for any $v, w \in T_pM$,

$$\langle (dS)_p v, w \rangle = \langle v, (dS)_p w \rangle.$$

(3) From (2) and standard results for real symmetric matrices, we know that $(dS)_p$ has two real eigenvalues λ and Λ(may be equal). Let $K(p)$ be the product of λ and Λ. Show that K defines a C^∞ function on M. For the following cases, classify the points on M at which $K > 0$, $K < 0$ and $K = 0$: M is a plane in $\mathbb{R}^3$; M is S^2; M is the cylinder $S^1 \times \mathbb{R}$; M is the 2-dimensional torus embedded in $\mathbb{R}^3$.

(4) Assume further that M is compact. Show that

$$\deg S = \frac{1}{4\pi} \int_M K\nu,$$

where the orientation on S^2 is induced by the outer normal vector field.

(5) Let $(U; u, v)$ be an oriented chart. For notation convenience, we use $X : U \to M \subset \mathbb{R}^3$ to denote the inverse of the chart map, which maps (u, v) to the manifold M, and we use subscripts to denote partial derivatives. For example, X_u is $\frac{\partial X}{\partial u} \in \mathbb{R}^3$, and X_{uv} is $\frac{\partial^2 X}{\partial u \partial v} \in \mathbb{R}^3$. It is easy to see that X_u is just $\frac{\partial}{\partial u}$ and X_v is $\frac{\partial}{\partial v}$. Similar convention applies to the unit normal vector field $N(u, v)$ on U. It follows that

$$N_u = (dS)_{(u,v)} X_u, \; N_v = (dS)_{(u,v)} X_v, \; \forall (u, v) \in U.$$

Let

$$E = \langle X_u, X_u \rangle, \; F = \langle X_u, X_v \rangle, \; G = \langle X_v, X_v \rangle,$$

and

$$\begin{aligned} P &= \begin{pmatrix} \langle X_{uu}, X_{vv} \rangle & \langle X_{uu}, X_u \rangle & \langle X_{uu}, X_v \rangle \\ \langle X_u, X_{vv} \rangle & \langle X_u, X_u \rangle & \langle X_u, X_v \rangle \\ \langle X_v, X_{vv} \rangle & \langle X_v, X_u \rangle & \langle X_v, X_v \rangle \end{pmatrix}, \\ Q &= \begin{pmatrix} \langle X_{uv}, X_{uv} \rangle & \langle X_{uv}, X_u \rangle & \langle X_{uv}, X_v \rangle \\ \langle X_u, X_{uv} \rangle & \langle X_u, X_u \rangle & \langle X_u, X_v \rangle \\ \langle X_v, X_{uv} \rangle & \langle X_v, X_u \rangle & \langle X_v, X_v \rangle \end{pmatrix}. \end{aligned}$$

Show that

$$K = \frac{\det P - \det Q}{(EG - F^2)^2}.$$

Moreover, by showing that

$$\langle X_{uu}, X_{vv} \rangle - \langle X_{uv}, X_{uv} \rangle$$

can be computed in terms of the functions E, F, G and their derivatives, conclude that K can be computed in terms of the functions E, F, G and their derivatives on U. What does this fact tell you?

(6) Appreaciate the elegance and profoundness of the result of (5). It leads Riemann to study intrinsic metric geometry in higher dimensions which opens a new era of geometry from the end of 19th century, and also leads to Einstein's general theory of relativity which changes the whole physics around that time.