Problem Sheet 2

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Problem 1. Let V be an n-dimensional real vector space.

(1) Show that $v_1, \cdots, v_k \in V$ are linearly independent if and only if

$$v_1 \wedge \cdots \wedge v_k \neq 0.$$

(2) Let $\{v_1, \cdots, v_k\}$ be k linearly independent vectors in V, and $w \in \Lambda^r(V)$ $(1 \leqslant r \leqslant n)$. Show that there exists $\psi_1, \cdots, \psi_k \in \Lambda^{r-1}(V)$ such that

$$w = v_1 \wedge \psi_1 + \cdots + v_k \wedge \psi_k$$

if and only if

$$v_1 \wedge \cdots \wedge v_k \wedge w = 0.$$

(3) Write down the change of basis formula for $\Lambda^r(V)$.

(4) Let $0 \leqslant r \leqslant n$. For $\alpha \in \Lambda^r(V)$, define a linear map

$$A_\alpha : \ \Lambda^{n-r}(V) \to \Lambda^n(V)$$

by $A_\alpha(\beta) = \alpha \wedge \beta$. Show that the map $\alpha \mapsto A_\alpha$ is a linear isomorphism from $\Lambda^r(V)$ to $\mathrm{End}(\Lambda^{n-r}(V), \Lambda^n(V))$, where $\mathrm{End}(\Lambda^{n-r}(V), \Lambda^n(V))$ denotes the space of linear maps from $\Lambda^{n-r}(V)$ to $\Lambda^n(V)$.

Problem 2. (1) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $F(x, y, z) = (xy, yz, zx)$. Calculate $F^*(xdy \wedge dz)$ and $F^*(xdy + ydz)$.

(2) Let M be an n-dimensional manifold, and let $\omega_1, \cdots, \omega_k$ be k one forms which are pointwisely linearly independent $(k \leqslant n)$. Let $\theta_1, \cdots, \theta_k$ be k one forms such that

$$\sum_{i=1}^{k} \theta_i \wedge \omega_i = 0.$$

Show that there exists $A_{ij} \in C^\infty(M)$ with $A_{ij} = A_{ji}$ such that

$$\theta_i = \sum_{j=1}^{k} A_{ij}\omega_j, \ \forall i = 1, \cdots, k.$$

This is called *Cartan's Lemma.*

(3) Let ω be an r-form on a manifold M, and $X_1, \cdots, X_{r+1}$ be $r+1$ smooth vector fields. Show that

$$\begin{aligned} d\omega(X_1, \cdots, X_{r+1}) &= \sum_{i=1}^{r+1}(-1)^{i+1}X_i(\langle X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_{r+1}, \omega\rangle) \\ &\quad + \sum_{1\leqslant i<j\leqslant r+1}(-1)^{i+j}\langle [X_i, X_j] \wedge X_1 \wedge \cdots \\ &\quad \wedge \widehat{X_i} \wedge \cdots \wedge \widehat{X_j} \wedge \cdots \wedge X_{r+1}, \omega\rangle, \end{aligned}$$

Here $\widehat{\cdot}$ means removing the term, and the evaluation on the L.H.S. and $\langle \cdot, \cdot \rangle$ are defined pointwisely, where $\langle \cdot, \cdot \rangle$ is the pairing defined in the notes (see (2.13) in Section 2.2).

Problem 3. (1) Show that every smooth vector field on a compact manifold is complete.

(2) Find the smooth vector fields X, Y, Z on $\mathbb{R}^2$ given by the following three one-parameter groups of diffeomorphisms.

(i) $\varphi_t(x^1, x^2) = (x^1 + t, x^2)$.

(ii) $\varphi_t(x^1, x^2) = (x^1, x^2 + t)$.

(iii) $\varphi_t(x^1, x^2) = (x^1 \cos t + x^2 \sin t, -x^1 \sin t + x^2 \cos t)$.

Show that the Lie bracket of any pair of X, Y, Z is a linear combination of X, Y, Z with constant coefficients.

(3) For an $n \times n$ matrix C, define the exponential of C by

$$\exp C = \sum_{i=0}^{\infty} \frac{C^n}{n!}.$$

Show that this is well-defined. Given an $n \times n$ matrix A, consider the smooth vector field X on $\mathbb{R}^n$ defined by

$$X = \sum_{i,j=1}^{n} A^i_j x^j \frac{\partial}{\partial x^i}.$$

Integrate this vector field to a one-parameter group of diffeomorphisms on $\mathbb{R}^n$.

(4) Let g be the symmetric $(0, 2)$-type tensor on $\mathbb{R}^2$ given by

$$g = dx \otimes dx + dy \otimes dy.$$

A smooth vector field X on $\mathbb{R}^2$ is called a *Killing vector field* if

$$L_X g = 0.$$

Characterize all Killing vector fields on $\mathbb{R}^2$. Moreover, for each Killing vector field X, integrate X to a one-parameter group of diffeomorphisms.

Problem 4. (1) Let (M, φ) be an immersion in $\mathbb{R}^{n+1}$, where M is an n-dimensional manifold. Show that M is orientable if and only if there exists a smooth map

$$N: \ M \to \mathbb{R}^{n+1}$$

such that $N(p)$ is orthogonal to $(d\varphi)_p(T_pM)$, where we identify $T_{\varphi(p)}\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$.

(2) Let $F: \ \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^∞ map and $c \in \mathbb{R}$. Assume that for any $a \in F^{-1}(c)$, $(dF)_a$ is surjective. Show that $F^{-1}(c)$ is orientable (for the manifold structure of $F^{-1}(c)$ see Theorem 1.8 in the notes).

(3) Show that the product of two orientable manifolds is orientable.

(4) Show that the n-sphere S^n is orientable, and the n-dimensional real projective space $\mathbb{R}P^n$ is orientable if and only if n is odd. Is $S^2 \times \mathbb{R}P^2$ orientable? How about $\mathbb{R}P^2 \times \mathbb{R}P^2$?

(5) Let M be the set of straight lines in $\mathbb{R}^2$. Let U_0 be the set of non-vertical lines, and for each $l \in U_0$, the equation of l is given uniquely by

$$y = ax + b.$$

Define $\varphi_0: \ U_0 \to \mathbb{R}^2$ by $\varphi_0(l) = (a, b)$. Similarly, let U_1 be the set of non-horizontal lines and define $\varphi_1: \ U_1 \to \mathbb{R}^2$ be $\varphi_1(l) = (c, d)$ where l is uniquely given by

$$x = cy + d.$$

Show that $\{(U_0, \varphi_0), \ (U_1, \varphi_1)\}$ defines an atlas on M which makes M a two dimensional manifold. Is M orientable?

Problem 5. Consider the embedding $i: \ S^n \to \mathbb{R}^{n+1}$. Define an n-form ω on $\mathbb{R}^{n+1}$ by

$$\omega = \sum_{i=0}^{n+1} (-1)^i x_i dx^0 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Show that the pull back $i^*\omega$ is nonvanishing on S^n and hence defines an orientation on S^n. Under this orientation, compute $\int_{S^n} i^*\omega$.