Problem Sheet 1

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Problem 1. (1) Consider $\mathbb{R}$ as an additive group, and $\mathbb{Z}$ to be the subgroup of integers. Let $M = \mathbb{R}/\mathbb{Z}$ be the quotient group, and

$$\begin{array}{rcl} p: \mathbb{R} & \to & \mathbb{R}/\mathbb{Z}, \\ x & \mapsto & x + \mathbb{Z} = \{x + z : \ z \in \mathbb{Z}\}, \end{array}$$

be the quotient homomorphism. Set $U_1 = p(0,1)$ and $U_2 = p(-\frac{1}{2}, \frac{1}{2})$, where $(0,1), (-\frac{1}{2}, \frac{1}{2})$ are the open intervals in $\mathbb{R}$. By definition it is obvious that p is injective when restricted to $(0,1)$ or $(-\frac{1}{2}, \frac{1}{2})$. Let

$$\varphi_1 = p^{-1} : \ U_1 \to (0,1), \ \varphi_2 = p^{-1} : \ U_2 \to (-\frac{1}{2}, \frac{1}{2}).$$

Show that $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ defines an atlas on M, which makes M into a one dimensional manifold. Moreover, show that p is a local diffeomorphism and M is diffeomorphic to S^1.

(2) Let M be the extended complex plane $M = \mathbb{C} \cup \{\infty\}$ (one can just think of "∞" as some abstract element not in $\mathbb{C}$). Let $U_1 = \mathbb{C}$ with

$$\varphi_1 : \ U_1 \to \mathbb{C} \cong \mathbb{R}^2, \ \varphi_1(z) = z,$$

and $U_2 = \mathbb{C} \backslash \{0\} \cup \{\infty\}$ with

$$\varphi_2 : \ U_2 \to \mathbb{C} \cong \mathbb{R}^2$$

defined by

$$\varphi_2(z) = \begin{cases} z^{-1}, & z \neq \infty; \\ 0, & z = \infty. \end{cases}$$

Show that $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ defines an atlas on M, which makes M into a two dimensional manifold. Moreover, show that M is diffeomorphic to S^2. M is usually called the *Riemann Sphere*.

(3) Let $S^n \subset \mathbb{R}^{n+1}$ be the n-sphere, and let $U_1 = S^n \backslash \{(0, \cdots, 0, 1)\}$ and $U_2 = S^n \backslash \{(0, \cdots, 0, -1)\}$. For $x = (x^0, \cdots, x^n) \in U_1$, define $\varphi_1(x)$ to be the point in $\mathbb{R}^n$ such that the straight line in $\mathbb{R}^{n+1}$ joining $(0, \cdots, 0, 1)$ to x meets the plane $x^n = -1$ at the point $(\varphi_1(x), -1) \in \mathbb{R}^{n+1}$. Similarly, for $x \in U_2$, define $\varphi_2(x)$ to be the point in $\mathbb{R}^n$ such that the straight line in $\mathbb{R}^{n+1}$ joining

$(0, \cdots, 0, -1)$ to x meets the plane $x^n = 1$ at the point $(\varphi_2(x), 1) \in \mathbb{R}^{n+1}$. Show that for $i = 1, 2$, $\varphi_i : U_i \to \mathbb{R}^n$ is a bijection, and $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ defines an atlas on S^n which is compatible with the one introduced in notes. Such construction is called the *stereographic projection*.

Problem 2. (1) Show that the notion of open sets introduced in the notes defines a topology on the manifold, and it is independent of the choice of atlases in the differential structure.

(2) Under the manifold topology, for any open set U of M and $p \in U$, show that there exists some open set V such that $\overline{V}$ is compact and

$$p \in V \subset \overline{V} \subset U.$$

Problem 3. (1) Let

$$f(t) = \begin{cases} \mathrm{e}^{-\frac{1}{t}}, & t > 0; \\ 0, & t \leqslant 0. \end{cases}$$

Show that $f \in C^\infty(\mathbb{R}^1)$. Let

$$g(t) = \frac{f(t)}{f(t) + f(1-t)},$$

sketch the graph of g.

(2) Construct $h \in C^\infty(\mathbb{R}^1)$, such that

$$h(t) = \begin{cases} 1, & |t| \leqslant 1 \\ 0, & |t| \geqslant 2. \end{cases}$$

Starting from this, construct a function $k(x^1, \cdots, x^n) \in C^\infty(\mathbb{R}^n)$, such that k is identically 1 in a ball of radius r and identically 0 outside the ball of radius $2r$ with the same center, where $r > 0$.

(3) Let M be a manifold, and U, V be open sets of M such that $\overline{U}$ is compact and

$$U \subset \overline{U} \subset V,$$

prove that there exists $f \in C^\infty(M)$, such that

$$f(p) = \begin{cases} 1, & p \in \overline{U}; \\ 0, & p \notin V. \end{cases}$$

This f is called a *bump function*.

Problem 4. (1) Let $f : S^n \to \mathbb{R}P^n$ be the map which associates to each unit vector in $\mathbb{R}^{n+1}$ the one dimensional vector space it spans. Show that f is a smooth surjective map, and it is a local diffeomorphism. Moreover, show that $\mathbb{R}P^n$ is Hausdorff.

(2) Let $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an invertible linear transformation. Show that A maps the set $\mathbb{R}P^n$ bijectively to itself. Show further that this defines a diffeomorphism F from $\mathbb{R}P^n$ to itself.

Problem 5. Let M be a manifold and let $p \in M$. Let Γ_p be the space of smooth curves γ through p, that is, $\gamma : (-\delta, \delta) \to M$ with $\gamma(0) = p$, for some $\delta > 0$.

(1) For each $\gamma \in \Gamma_p$, define a functional on the space $\mathcal{F}_p$ of C^∞ germs at p by

$$\ll \gamma, [f] \gg = \frac{d}{dt}|_{t=0} f \circ \gamma(t), \ [f] \in \mathcal{F}_p.$$

Show that $\ll \gamma, \cdot \gg$ is well-defined and linear.

(2) Show that

$$\mathcal{H}_p = \{[f] \in \mathcal{F}_p : \ll \gamma, [f] \gg = 0 \text{ for all } \gamma \in \Gamma_p\},$$

where $\mathcal{H}_p$ is the linear subspace of $\mathcal{F}_p$ introduced in the notes. It follows that each $\gamma \in \Gamma_p$ defines a linear functional $\ll \gamma, \cdot \gg$ on the cotangent space $T_p^*M = \mathcal{F}_p / \mathcal{H}_p$ by

$$\ll \gamma, (df)_p \gg = \ll \gamma, [f] \gg,$$

where $[f]$ is any representative in $(df)_p$.

(3) Introduce an equivalence relation "$\sim$" on Γ_p by

$$\gamma \sim \gamma' \iff \ll \gamma, (df)_p \gg = \ll \gamma', (df)_p \gg, \ \forall (df)_p \in T_p^*M.$$

Define a mapping $\Phi : \Gamma_p/_\sim \to T_pM = (T_p^*M)^*$ by

$$\langle \Phi([\gamma]), (df)_p \rangle = \ll \gamma, (df)_p \gg, \ (df)_p \in T_p^*M,$$

where $\Gamma_p / \sim$ is the space of $\sim$-equivalence classes and γ is any representative in $[\gamma]$. Show that Φ is well-defined, and it is a bijection. This shows that each $[\gamma]$ is canonically identified with a tangent vector representating the "direction" of $[\gamma]$. In the case when M is a submanifold of $\mathbb{R}^N$, $\gamma \sim \gamma'$ is equivalent to saying that γ and γ' are tangent to each other at p (or have the same tangent vector at p in the classical sense).

Problem 6. (1) Show that the manifold topology of the tangent bundle TM induced by the differential structure constructed in the notes, is Hausdorff and has a second countable base of open sets.

(2) Consider the n-sphere $S^n \subset \mathbb{R}^{n+1}$. For any $x \in S^n$, show that T_xS^n is naturally isomorphic to the vector space

$$\{y \in \mathbb{R}^{n+1} : \ y \cdot x = 0\},$$

where "$\cdot$" is the inner product in $\mathbb{R}^{n+1}$. If n is an odd number, construct a non-vanishing smooth vector field on S^n.