21-241: Matrix Algebra – Summer I, 2006

Practice Exam 4 Solutions

1. Let $A = \begin{pmatrix} 1 & -3 & -7 \\ 0 & -2 & -6 \\ 0 & 2 & 5 \end{pmatrix}$.

   (a) Find the characteristic polynomial of A.

   SOLUTION.

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -3 & -7 \\ 0 & -2-\lambda & -6 \\ 0 & 2 & 5-\lambda \end{pmatrix} = (1-\lambda)\det\begin{pmatrix} -2-\lambda & -6 \\ 2 & 5-\lambda \end{pmatrix}$$
$$= (1-\lambda)[(-2-\lambda)(5-\lambda) - (-6)2]$$
$$= -\lambda^3 + 4\lambda^2 - 5\lambda + 2.$$

   (b) Find all eigenvalues and their multiplicities of A.

   SOLUTION. Since $\det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda-1)^2(\lambda-2)$, A has two distinct eigenvalues: $\lambda_1 = 1$ with multiplicity 2, $\lambda_2 = 2$ with multiplicity 1.

   (c) For each eigenvalue, find a basis for the corresponding eigenspace. Determine which eigenvalues are complete.

   SOLUTION. To find a basis for V_{λ_1}, we are to solve the system $(A - \lambda_1 I)\mathbf{x} = \mathbf{0}$.

$$A - \lambda_1 I = \begin{pmatrix} 0 & -3 & -7 \\ 0 & -3 & -6 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow[R_3 + \frac{2}{3}R_1]{R_2 - R_1} \begin{pmatrix} 0 & -3 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \xrightarrow{R_3 + \frac{2}{3}R_2} \begin{pmatrix} 0 & \boxed{-3} & -7 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix}$$

   Thus the general solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

   So $(1,0,0)^T$ is a basis for V_{λ_1}. Since $\dim V_{\lambda_1} = 1$, less than the multiplicity, λ_1 is not complete. Similarly, we solve the system $(A - \lambda_2 I)\mathbf{x} = \mathbf{0}$.

$$A - \lambda_2 I = \begin{pmatrix} -1 & -3 & -7 \\ 0 & -4 & -6 \\ 0 & 2 & 3 \end{pmatrix} \xrightarrow{R_3 + \frac{1}{2}R_2} \begin{pmatrix} \boxed{-1} & -3 & -7 \\ 0 & \boxed{-4} & -6 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Thus the general solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2}x_3 \\ -\frac{3}{2}x_3 \\ x_3 \end{pmatrix} = \frac{x_1}{2} \begin{pmatrix} -5 \\ -3 \\ 2 \end{pmatrix}.$$

   So $(-5,-3,2)^T$ is a basis for V_{λ_2}. Since $\dim V_{\lambda_2} = 1$, equal to the multiplicity, λ_2 is complete.

   (d) Is A complete? diagonalizable?

   SOLUTION. Since A has an incomplete eigenvalue, it's not complete, thus not diagonalizable. □

2. Write down a real matrix that has

(a) eigenvalues -1, 3 and corresponding eigenvectors $\begin{pmatrix}-1\\2\end{pmatrix}$, $\begin{pmatrix}1\\1\end{pmatrix}$.

SOLUTION. Since $\begin{pmatrix}-1\\2\end{pmatrix}$, $\begin{pmatrix}1\\1\end{pmatrix}$ are linearly independent (why?), they form an eigenvector basis for $\mathbb{R}^2$. Thus the matrix, denoted by A, is diagonalizable. Let $S = \begin{pmatrix}-1 & 1\\2 & 1\end{pmatrix}$, $\Lambda = \begin{pmatrix}-1 & 0\\0 & 3\end{pmatrix}$. Then,

$$A = S\Lambda S^{-1} = \begin{pmatrix}-1 & 1\\2 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}-\frac{1}{3} & \frac{1}{3}\\\frac{2}{3} & \frac{1}{3}\end{pmatrix} = \begin{pmatrix}\frac{5}{3} & \frac{4}{3}\\\frac{8}{3} & \frac{1}{3}\end{pmatrix}.$$

(b) an eigenvalue $-1+2\mathrm{i}$ and corresponding eigenvector $\begin{pmatrix}1+\mathrm{i}\\3\mathrm{i}\end{pmatrix}$.

SOLUTION. Since the wanted matrix, denoted by A, is real, the complex conjugate of $-1+2\mathrm{i}$, namely $-1-2\mathrm{i}$, must also be an eigenvalue of A. Moreover, the complex conjugate of $\begin{pmatrix}1+\mathrm{i}\\3\mathrm{i}\end{pmatrix}$, namely $\begin{pmatrix}1-\mathrm{i}\\-3\mathrm{i}\end{pmatrix}$, must be an eigenvector corresponding to $-1-2\mathrm{i}$. Since $\begin{pmatrix}1+\mathrm{i}\\3\mathrm{i}\end{pmatrix}$, $\begin{pmatrix}1-\mathrm{i}\\-3\mathrm{i}\end{pmatrix}$ are linearly independent, A allows an eigenvector basis for $\mathbb{C}^2$.

Let $S = \begin{pmatrix}1+\mathrm{i} & 1-\mathrm{i}\\3\mathrm{i} & -3\mathrm{i}\end{pmatrix}$, $\Lambda = \begin{pmatrix}-1+2\mathrm{i} & 0\\0 & -1-2\mathrm{i}\end{pmatrix}$. Then,

$$A = S\Lambda S^{-1} = \begin{pmatrix}1+\mathrm{i} & 1-\mathrm{i}\\3\mathrm{i} & -3\mathrm{i}\end{pmatrix}\begin{pmatrix}-1+2\mathrm{i} & 0\\0 & -1-2\mathrm{i}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{1}{6}-\frac{1}{6}\mathrm{i}\\\frac{1}{2} & -\frac{1}{6}+\frac{1}{6}\mathrm{i}\end{pmatrix} = \begin{pmatrix}-3 & \frac{4}{3}\\-6 & 1\end{pmatrix}.$$ □

3. Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T.

PROOF. If λ is an eigenvalue of A, then $\det(A-\lambda I)=0$. Thus,

$$\det(A^T-\lambda I) = \det(A^T-\lambda I^T) = \det(A-\lambda I)^T = \det(A-\lambda I) = 0.$$

This implies that λ is an eigenvalue of A^T. □

4. Find the minimum and maximum value of the rational function $\dfrac{2x^2+xy+3xz+2y^2+2z^2}{x^2+y^2+z^2}$.

SOLUTION. Let $\mathbf{v} = (x,y,z)^T$, then

$$\frac{2x^2+xy+3xz+2y^2+2z^2}{x^2+y^2+z^2} = \frac{\mathbf{v}^T K\mathbf{v}}{\|\mathbf{v}\|^2},$$

where $K = \begin{pmatrix}2 & \frac{1}{2} & \frac{3}{2}\\\frac{1}{2} & 2 & 0\\\frac{3}{2} & 0 & 2\end{pmatrix}$. By the optimization principles for eigenvalues,

$$\max\left\{\frac{\mathbf{v}^T K\mathbf{v}}{\|\mathbf{v}\|^2}\,\middle|\,\mathbf{v}\neq\mathbf{0}\right\} = \lambda_1, \qquad \min\left\{\frac{\mathbf{v}^T K\mathbf{v}}{\|\mathbf{v}\|^2}\,\middle|\,\mathbf{v}\neq\mathbf{0}\right\} = \lambda_3,$$

where λ_1 and λ_3 are respectively the largest and smallest eigenvalues of K. The characteristic equation of K is

$$\begin{aligned}
\det(K-\lambda I) &= \det\begin{pmatrix} 2-\lambda & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 2-\lambda & 0 \\ \frac{3}{2} & 0 & 2-\lambda \end{pmatrix} \\
&= \frac{3}{2}\det\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ 2-\lambda & 0 \end{pmatrix} + (2-\lambda)\det\begin{pmatrix} 2-\lambda & \frac{1}{2} \\ \frac{1}{2} & 2-\lambda \end{pmatrix} \\
&= \frac{3}{2}\left[0-\frac{3}{2}(2-\lambda)\right] + (2-\lambda)\left[(2-\lambda)^2-\left(\frac{1}{2}\right)^2\right] \\
&= -(\lambda-2)\left[(\lambda-2)^2-\frac{5}{2}\right] \\
&= -(\lambda-2)\left(\lambda-2-\sqrt{\frac{5}{2}}\right)\left(\lambda-2+\sqrt{\frac{5}{2}}\right) = 0.
\end{aligned}$$

Therefore, $\lambda_1 = 2+\sqrt{\frac{5}{2}}$, $\lambda_3 = 2-\sqrt{\frac{5}{2}}$. So, the maximum value of the function is $2+\sqrt{\frac{5}{2}}$ and the minimum value of the function is $2-\sqrt{\frac{5}{2}}$. □