21-241: Matrix Algebra – Summer I, 2006

Practice Exam 1

1. If $A = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$ and $AB = \begin{pmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{pmatrix}$, determine the first and second columns of B.

SOLUTION. Since A has size 2×2 and AB has size 2×3, B has size 2×3. Suppose $B = \begin{pmatrix} a & c & * \\ b & d & * \end{pmatrix}$. By the rule of matrix multiplication,

$$AB = \begin{pmatrix} a - 2b & c - 2d & * \\ -2a + 5b & -2c + 5d & * \end{pmatrix}.$$

Therefore, we have the following linear system:

$$\begin{aligned} a - 2b &= -1 \\ -2a + 5b &= 6 \\ c - 2d &= 2 \\ -2c + 5d &= -9 \end{aligned}$$

Solving the system, we get $a = 7$, $b = 4$, $c = -8$, $d = -5$. So the first and second columns of B are $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -8 \\ -5 \end{pmatrix}$. □

2. Two matrices A and B are said to be *similar*, denoted $A \sim B$, if there exists an invertible matrix P such that $B = P^{-1}AP$. Prove:
(a) $A \sim A$.
(b) If $A \sim B$, then $B \sim A$.
(c) If $A \sim B$ and $B \sim C$, then $A \sim C$.

PROOF. (a) Taking P as the identity matrix I, we have $A = I^{-1}AI$. So $A \sim A$.

(b) Since $A \sim B$, there exists an invertible matrix P such that $B = P^{-1}AP$. Notice that $Q = P^{-1}$ is also invertible, and $A = Q^{-1}BQ$. So $B \sim A$.

(c) Since $A \sim B$ and $B \sim C$, there exist invertible matrices P and Q such that $B = P^{-1}AP$, $C = Q^{-1}BQ$. Notice that PQ is also invertible, and $C = Q^{-1}BQ = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ)$. So $A \sim C$. □

3. (a) Explain why the inverse of a permutation matrix equals its transpose: $P^{-1} = P^T$.
(b) If $A^{-1} = A^T$, is A necessarily a permutation matrix? Give a proof or a counterexample to support your conclusion.

SOLUTION. (a) A permutation matrix is the product of a sequence of interchange elementary matrices. Suppose $P = E_1E_2 \cdots E_n$, each E_i interchanges some two rows of the identity matrix. It's obvious that E_i is symmetric, so $E_i^T = E_i$. Also, we have $E_i^2 = I$ because applying the same interchange trice returns to the identity. Therefore,

$$\begin{aligned} PP^T &= (E_1E_2 \cdots E_n)(E_1E_2 \cdots E_n)^T = (E_1E_2 \cdots E_n)(E_n^T E_{n-1}^T \cdots E_1^T) \\ &= (E_1E_2 \cdots E_n)(E_nE_{n-1} \cdots E_1) = E_1 \cdots E_{n-1}E_n^2E_{n-1} \cdots E_1 \\ &= E_1 \cdots E_{n-1}E_{n-1} \cdots E_1 = \cdots = I, \end{aligned}$$

which implies $P^{-1} = P^T$.

(b) No. Let $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Then $A^{-1} = A^T = A$. But A is not a permutation matrix, because it can't be obtained by interchanging rows of the identity matrix. (If we look at -1 as a 1×1 matrix, it's just an even simpler counterexample.) □

4. Suppose A, B, and X are $n \times n$ matrices with A, X, and $A - AX$ invertible, and suppose

$$(A - AX)^{-1} = X^{-1}B. \tag{1}$$

(a) Is B invertible? Explain why.
(b) Solve (1) for X. If you need to invert a matrix, explain why that matrix is invertible.

Solution. (a) Yes. From (1) we get $B = X(A - AX)^{-1}$, the product of two invertible matrices X and $(A - AX)^{-1}$. So B is invertible.

(b) Since $(A - AX)^{-1}$, X^{-1} and B are invertible, from (1) we have

$$A - AX = ((A - AX)^{-1})^{-1} = (X^{-1}B)^{-1} = B^{-1}X,$$

or,

$$A = (A + B^{-1})X. \tag{2}$$

Since X is invertible, $A + B^{-1} = AX^{-1}$, which is the product of two invertible matrices A and X^{-1}. Therefore, $A + B^{-1}$ is invertible, and thus, from (2), we have $X = (A + B^{-1})^{-1}A$. □

5. Find the determinant of the following Vandermonde matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}$$

Solution. We reduce A^T to an upper triangular matrix by elementary row operations.

$$A^T = \begin{pmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{pmatrix} \xrightarrow[R_4-R_1]{\substack{R_2-R_1 \\ R_3-R_1}} \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & c-a & c^2-a^2 & c^3-a^3 \\ 0 & d-a & d^2-a^2 & d^3-a^3 \end{pmatrix}$$

$$\xrightarrow[R_4/(d-a)]{\substack{R_2/(b-a) \\ R_3/(c-a)}} \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & b^2+ba+a^2 \\ 0 & 1 & c+a & c^2+ca+a^2 \\ 0 & 1 & d+a & d^2+da+a^2 \end{pmatrix} \xrightarrow[R_4-R_2]{R_3-R_2} \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & b^2+ba+a^2 \\ 0 & 0 & c-b & c^2-b^2+ca-ba \\ 0 & 0 & d-b & d^2-b^2+da-ba \end{pmatrix}$$

$$\xrightarrow[R_4/(d-b)]{R_3/(c-b)} \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & b^2+ba+a^2 \\ 0 & 0 & 1 & c+b+a \\ 0 & 0 & 1 & d+b+a \end{pmatrix} \xrightarrow{R_4-R_3} \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & b^2+ba+a^2 \\ 0 & 0 & 1 & c+b+a \\ 0 & 0 & 0 & d-c \end{pmatrix}$$

Therefore, $\det A = \det A^T = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$. □

6. When does the follow system have (i) a unique solution? (ii) no solution? (iii) infinitely many solutions?

$$\begin{aligned} x + 3y - 2z &= 2 \\ y + \ \ z &= -5 \\ x + 2y - 3z &= a \\ -2x - 8y + 4z &= b \end{aligned}$$

SOLUTION. We reduce the augmented matrix to echelon form.

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & a \\ -2 & -8 & 4 & b \end{array}\right) \xrightarrow[R_4+2R_1]{R_3-R_1} \left(\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & -1 & -1 & a-2 \\ 0 & -2 & 0 & b+4 \end{array}\right) \xrightarrow[R_4+2R_2]{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & a-7 \\ 0 & 0 & 2 & b-6 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 2 & b-6 \\ 0 & 0 & 0 & a-7 \end{array}\right)$$

Now we see each column contains a pivot, so the system can't have infinitely many solutions. When $a - 7 = 0$, or $a = 7$, the system is consistent and has a unique solution. When $a \neq 7$, the system is inconsistent and has no solution. □

7. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, show that $K = AA^T$ is well-defined, symmetric matrix. Find the LDL^T factorization of K.

SOLUTION. A has size 2×3, and A^T has size 3×2. So $K = AA^T$ is well-defined. K is symmetric because $K^T = (AA^T)^T = (A^T)^T A^T = AA^T = K$. Symmetry can also be seen by direct computation:

$$K = AA^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 32 \\ 32 & 77 \end{pmatrix}$$

To find the LDL^T factorization, we apply Gaussian method to K:

$$\begin{aligned} \begin{pmatrix} 14 & 32 \\ 32 & 77 \end{pmatrix} &\xrightarrow{R_2-(16/7)R_1} \begin{pmatrix} 14 & 32 \\ 0 & 27/7 \end{pmatrix} = U, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &\longrightarrow \begin{pmatrix} 1 & 0 \\ 16/7 & 1 \end{pmatrix} = L, \end{aligned}$$

and $D = \begin{pmatrix} 14 & 0 \\ 0 & 27/7 \end{pmatrix}$, the diagonal part of U. Thus the LDL^T factorization of K is

$$\begin{pmatrix} 14 & 32 \\ 32 & 77 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 16/7 & 1 \end{pmatrix} \begin{pmatrix} 14 & 0 \\ 0 & 27/7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 16/7 & 1 \end{pmatrix}^T.$$

□

8. Use the Gauss-Jordan method to find the inverse of the following complex matrix:

$$\begin{pmatrix} 0 & 1 & -\mathrm{i} \\ \mathrm{i} & 0 & -1 \\ -1 & \mathrm{i} & 1 \end{pmatrix}$$

SOLUTION.

$$\left(\begin{array}{ccc|ccc} 0 & 1 & -\mathrm{i} & 1 & 0 & 0 \\ \mathrm{i} & 0 & -1 & 0 & 1 & 0 \\ -1 & \mathrm{i} & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} -1 & \mathrm{i} & 1 & 0 & 0 & 1 \\ \mathrm{i} & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -\mathrm{i} & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \times (-1)} \left(\begin{array}{ccc|ccc} 1 & -\mathrm{i} & -1 & 0 & 0 & -1 \\ \mathrm{i} & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -\mathrm{i} & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 - \mathrm{i}R_1} \left(\begin{array}{ccc|ccc} 1 & -\mathrm{i} & -1 & 0 & 0 & -1 \\ 0 & -1 & \mathrm{i}-1 & 0 & 1 & \mathrm{i} \\ 0 & 1 & -\mathrm{i} & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \times (-1)} \left(\begin{array}{ccc|ccc} 1 & -\mathrm{i} & -1 & 0 & 0 & -1 \\ 0 & 1 & 1-\mathrm{i} & 0 & -1 & -\mathrm{i} \\ 0 & 1 & -\mathrm{i} & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 - R_2]{R_1 + \mathrm{i}R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & \mathrm{i} & 0 & -\mathrm{i} & 0 \\ 0 & 1 & 1-\mathrm{i} & 0 & -1 & -\mathrm{i} \\ 0 & 0 & -1 & 1 & 1 & \mathrm{i} \end{array}\right) \xrightarrow{R_3 \times (-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & \mathrm{i} & 0 & -\mathrm{i} & 0 \\ 0 & 1 & 1-\mathrm{i} & 0 & -1 & -\mathrm{i} \\ 0 & 0 & 1 & -1 & -1 & -\mathrm{i} \end{array}\right)$$

$$\xrightarrow[R_2 - (1-\mathrm{i})R_3]{R_1 - \mathrm{i}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \mathrm{i} & 0 & -1 \\ 0 & 1 & 0 & 1-\mathrm{i} & -\mathrm{i} & 1 \\ 0 & 0 & 1 & -1 & -1 & -\mathrm{i} \end{array}\right)$$

So,

$$\begin{pmatrix} 0 & 1 & -\mathrm{i} \\ \mathrm{i} & 0 & -1 \\ -1 & \mathrm{i} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \mathrm{i} & 0 & -1 \\ 1-\mathrm{i} & -\mathrm{i} & 1 \\ -1 & -1 & -\mathrm{i} \end{pmatrix}.$$

□