21-241: Matrix Algebra – Summer I, 2006

Homework 5 Solutions

Monday, June 19

5.2.1 (a) Denote the three vectors by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. By the Gram-Schmidt formula, we can construct an orthogonal basis for $\mathbb{R}^3$:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \|\mathbf{w}_1\| = \sqrt{2},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \|\mathbf{w}_2\| = 1,$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \frac{0}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \|\mathbf{w}_3\| = \sqrt{2}.$$

After normalization, we obtain an orthonormal basis as follows:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}. \qquad \square$$

(c) Denote the three vectors by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. By the Gram-Schmidt formula, we can construct an orthogonal basis for $\mathbb{R}^3$:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad \|\mathbf{w}_1\| = \sqrt{14},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} - \frac{14}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}, \qquad \|\mathbf{w}_2\| = 3\sqrt{3},$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} - \frac{5}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{18}{27}\begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{14} \\ \frac{2}{7} \\ -\frac{1}{14} \end{pmatrix}, \quad \|\mathbf{w}_3\| = \frac{\sqrt{30}}{14}.$$

After normalization, we obtain an orthonormal basis as follows:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}, \qquad \mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \begin{pmatrix} -\frac{5}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{30}} \end{pmatrix}. \qquad \square$$

5.2.4 (a) Denote the two vectors by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. By the Gram-Schmidt formula, we can construct an orthogonal basis:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \qquad \|\mathbf{w}_1\| = \sqrt{5},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} - \frac{-5}{5}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \|\mathbf{w}_2\| = 1.$$

After normalization, we obtain an orthonormal basis as follows:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{pmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \qquad \square$$

5.2.6 (b) Denote the matrix by A. We first find a basis for $\ker A$.

$$\begin{pmatrix} 2 & 1 & 0 & -1 \\ 3 & 2 & -1 & -1 \end{pmatrix} \xrightarrow{R_2 - \frac{3}{2}R_1} \begin{pmatrix} \boxed{2} & 1 & 0 & -1 \\ 0 & \boxed{\frac{1}{2}} & -1 & \frac{1}{2} \end{pmatrix}.$$

The general solution to the system $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 + x_4 \\ 2x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 \\ 2x_3 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} x_4 \\ -x_4 \\ 0 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Let $\mathbf{v}_1 = (-1, 2, 1, 0)^T$, $\mathbf{v}_2 = (1, -1, 0, 1)^T$. Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ form a basis for $\ker A$. By the Gram-Schmidt formula, we can construct an orthogonal basis starting with $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \|\mathbf{w}_1\| = \sqrt{6},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} - \frac{-3}{6} \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{pmatrix}, \qquad \|\mathbf{w}_2\| = \sqrt{6}/2.$$

After normalization, we obtain an orthonormal basis as follows:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ 0 \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}. \qquad \square$$

(d) Denote the matrix by A. We first find a basis for $\operatorname{Col} A$.

$$\begin{pmatrix} 1 & -2 & 2 \\ 2 & -4 & 1 \\ 0 & 0 & -1 \\ -2 & 4 & 5 \end{pmatrix} \xrightarrow[R_3 + 2R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 9 \end{pmatrix} \xrightarrow[R_4 + 3R_2]{R_3 - \frac{1}{3}R_2} \begin{pmatrix} \boxed{1} & -2 & 2 \\ 0 & 0 & \boxed{-3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so the first column $\mathbf{v}_1 = (1, 2, 0, -2)^T$ and the third column $\mathbf{v}_2 = (2, 1, -1, 5)^T$ form a basis for

Col A. By the Gram-Schmidt formula, we can construct an orthogonal basis starting with $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \end{pmatrix}, \qquad \|\mathbf{w}_1\| = 3,$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 5 \end{pmatrix} - \frac{-6}{9}\begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{7}{3} \\ -1 \\ \frac{11}{3} \end{pmatrix}, \qquad \|\mathbf{w}_2\| = 3\sqrt{3}.$$

After normalization, we obtain an orthonormal basis as follows:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \\ -\frac{2}{3} \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{pmatrix} \frac{8}{9\sqrt{3}} \\ \frac{7}{9\sqrt{3}} \\ -\frac{1}{3\sqrt{3}} \\ \frac{11}{9\sqrt{3}} \end{pmatrix}.$$

□

5.3.9 Suppose Q is an orthogonal matrix, then $Q^T = Q^{-1}$. Let $P = Q^{-1}$, then

$$P^T = (Q^{-1})^T = (Q^T)^{-1} = P^{-1},$$

which implies that P is also orthogonal. □

5.3.12 Suppose U is an orthogonal upper triangular matrix, then $U^T = U^{-1}$. Note that U^T is lower triangular and U^{-1} is upper triangular. Therefore U^T has to be diagonal, so is U. Now suppose further $U = \operatorname{diag}(d_1, d_2, \cdots, d_n)$. Then,

$$U^T = \operatorname{diag}(d_1, d_2, \cdots, d_n), \qquad U^{-1} = \operatorname{diag}(1/d_1, 1/d_2, \cdots, 1/d_n).$$

Hence, $d_i = 1/d_i$, namely $d_i = \pm 1$ for all $1 \leqslant i \leqslant n$. One the other hand, when U is a diagonal matrix with diagonal entries either 1 or -1, U is clearly an orthogonal matrix. □

Tuesday, June 20

5.2.8 (a) Denote the three vectors by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. By the Gram-Schmidt formula, we can construct an orthogonal basis for $\mathbb{R}^3$:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \|\mathbf{w}_1\| = 2,$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{4}{4}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \|\mathbf{w}_2\| = \sqrt{2},$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \frac{-2}{4}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{4}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix}, \quad \|\mathbf{w}_3\| = \sqrt{3}.$$

After normalization, we obtain an orthonormal basis as follows:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \qquad \mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \begin{pmatrix} -\frac{1}{2\sqrt{3}} \\ 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix}.$$

□

(c) Denote the three vectors by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. By the Gram-Schmidt formula, we can construct an orthogonal basis for $\mathbb{R}^3$:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad \|\mathbf{w}_1\| = 2\sqrt{5},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} - \frac{32}{20}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{3}{5}\begin{pmatrix} 4 \\ 3 \\ -8 \end{pmatrix}, \qquad \|\mathbf{w}_2\| = \frac{3\sqrt{130}}{5},$$

$$\begin{aligned} \mathbf{w}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 \\ &= \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} - \frac{15}{20}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{30}{1170/25}\begin{pmatrix} \frac{12}{5} \\ \frac{9}{5} \\ -\frac{24}{5} \end{pmatrix} = \frac{3}{52}\begin{pmatrix} -5 \\ 6 \\ -3 \end{pmatrix}, \qquad \|\mathbf{w}_3\| = \frac{3\sqrt{156}}{52}. \end{aligned}$$

After normalization, we obtain an orthonormal basis as follows:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{20}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{130}}\begin{pmatrix} 4 \\ 3 \\ -8 \end{pmatrix}, \quad \mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{\sqrt{156}}\begin{pmatrix} -5 \\ 6 \\ -3 \end{pmatrix}.$$ □

5.3.1 **(b)** proper orthogonal.

(e) not orthogonal.

5.3.6 According to the description, there exists an elementary permutation matrix P such that $R = PQ$. Since $P = P^T = P^{-1}$, $Q^T = Q^{-1}$, we have

$$R^T = (PQ)^T = Q^T P^T = Q^{-1}P^{-1} = (PQ)^{-1} = R^{-1}.$$

So R is orthogonal. More over, since $\det P = -1$, $\det Q = 1$, we have $\det R = \det P \det Q = -1$. Therefore, R is an improper orthogonal matrix. □

5.3.8 **(a)** Suppose Q is an orthogonal matrix, then $Q^T = Q^{-1}$. Let $P = Q^T$. Then

$$P^T = (Q^T)^T = Q, \qquad P^{-1} = (Q^T)-1 = (Q^{-1})^{-1} = Q.$$

Therefore $P^T = P^{-1}$, thus $P = Q^T$ is also orthogonal. □

5.3.27 **(b)** Denote the columns of the matrix (denoted by A) by $\mathbf{v}_1$, $\mathbf{v}_2$. Apply Gram-Schmidt formula to get an orthogonal basis:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \frac{18}{25}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

Then we normalize $\mathbf{w}_1$, $\mathbf{w}_2$ to obtain an orthonormal basis:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}.$$

Thus we obtain the orthogonal matrix Q by combining $\mathbf{u}_1$, $\mathbf{u}_2$ in a single matrix

$$Q = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{pmatrix}.$$

The upper triangular matrix R equals

$$R = Q^T A = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 5 & \frac{18}{5} \\ 0 & \frac{1}{5} \end{pmatrix}.$$

□

(d) Denote the columns of the matrix (denoted by A) by $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Apply Gram-Schmidt formula to get an orthogonal basis:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-2}{2}\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \frac{-4}{2}\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} - \frac{2}{1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Then we normalize $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ to obtain an orthonormal basis:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Thus we obtain the orthogonal matrix Q by combining $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ in a single matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The upper triangular matrix R equals

$$R = Q^T A = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & -2\sqrt{2} \\ 0 & 1 & 2 \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

□

Wednesday, June 21

5.3.27 (f) Denote the columns of the matrix (denoted by A) by $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$. Apply Gram-Schmidt formula to get an orthogonal basis:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix}1\\2\\1\\0\end{pmatrix} - \frac{4}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\-1\end{pmatrix},$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_3\cdot\mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = \begin{pmatrix}1\\1\\2\\1\end{pmatrix} - \frac{5}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{0}{2}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}-1\\-1\\3\\-1\end{pmatrix},$$

$$\begin{aligned}\mathbf{w}_4 &= \mathbf{v}_4 - \frac{\mathbf{v}_4\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_4\cdot\mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 - \frac{\mathbf{v}_4\cdot\mathbf{w}_3}{\|\mathbf{w}_3\|^2}\mathbf{w}_3\\ &= \begin{pmatrix}1\\0\\1\\1\end{pmatrix} - \frac{3}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{-1}{2}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} - \frac{1/4}{3/4}\frac{1}{4}\begin{pmatrix}-1\\-1\\3\\-1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}2\\-1\\0\\-1\end{pmatrix}.\end{aligned}$$

Then we normalize $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ to obtain an orthonormal basis:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{pmatrix}\frac12\\\frac12\\\frac12\\\frac12\end{pmatrix},\quad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{pmatrix}0\\\frac{1}{\sqrt2}\\0\\-\frac{1}{\sqrt2}\end{pmatrix},\quad \mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \begin{pmatrix}-\frac{1}{\sqrt{12}}\\-\frac{1}{\sqrt{12}}\\\frac{3}{\sqrt{12}}\\-\frac{1}{\sqrt{12}}\end{pmatrix},\quad \mathbf{u}_4 = \frac{\mathbf{w}_4}{\|\mathbf{w}_4\|} = \begin{pmatrix}\frac{2}{\sqrt6}\\-\frac{1}{\sqrt6}\\0\\-\frac{1}{\sqrt6}\end{pmatrix}.$$

Thus we obtain the orthogonal matrix Q by combining $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ in a single matrix

$$Q = \begin{pmatrix}\frac12 & 0 & -\frac{1}{\sqrt{12}} & \frac{2}{\sqrt6}\\ \frac12 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt6}\\ \frac12 & 0 & \frac{3}{\sqrt{12}} & 0\\ \frac12 & -\frac{1}{\sqrt2} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt6}\end{pmatrix}.$$

The upper triangular matrix R equals

$$R = Q^TA = \begin{pmatrix}\frac12 & \frac12 & \frac12 & \frac12\\ 0 & \frac{1}{\sqrt2} & 0 & -\frac{1}{\sqrt2}\\ -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & \frac{3}{\sqrt{12}} & -\frac{1}{\sqrt{12}}\\ \frac{2}{\sqrt6} & -\frac{1}{\sqrt6} & 0 & -\frac{1}{\sqrt6}\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&2&1&0\\1&1&2&1\\1&0&1&1\end{pmatrix} = \begin{pmatrix}2 & 2 & \frac52 & \frac32\\ 0 & \sqrt2 & 0 & -\frac{1}{\sqrt2}\\ 0 & 0 & \frac{3}{\sqrt{12}} & \frac{1}{\sqrt{12}}\\ 0&0&0&\frac{1}{\sqrt6}\end{pmatrix}.$$

□

5.5.3 Denote the columns of the matrix (denoted by A) by $\mathbf{w}_1$, $\mathbf{w}_2$. Since $\mathbf{w}_1\cdot\mathbf{w}_2 = 0$, they form an orthogonal basis for the range of A. Let $\mathbf{b} = (1,2,3)^T$, then

$$\mathrm{proj}_{\mathrm{Col}\,A}\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 + \frac{\mathbf{b}\cdot\mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = \frac{10}{14}\begin{pmatrix}3\\2\\1\end{pmatrix} + \frac{-8}{12}\begin{pmatrix}2\\-2\\-2\end{pmatrix} = \begin{pmatrix}17/21\\58/21\\43/21\end{pmatrix}.$$

□

5.5.11 (a) Denote the columns of the coefficient matrix (denoted by A) by $\mathbf{w}_1$, $\mathbf{w}_2$. Since $\mathbf{w}_1\cdot\mathbf{w}_2 = 0$, they form an orthogonal basis for the range of A. Let $\mathbf{b} = (1,0,-1)^T$, then the least squares solution is the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Col}\,A$:

$$\mathrm{proj}_{\mathrm{Col}\,A}\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 + \frac{\mathbf{b}\cdot\mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = \frac{-2}{1}4\begin{pmatrix}1\\2\\3\end{pmatrix} + \frac{0}{6}\begin{pmatrix}-1\\2\\-1\end{pmatrix} = \begin{pmatrix}-1/7\\-4/7\\-6/7\end{pmatrix}.$$

□

5.5.12 Denote the two vectors by $\mathbf{w}_1$, $\mathbf{w}_2$. Since $\mathbf{w}_1\cdot\mathbf{w}_2 = 0$, they form an orthogonal basis for $W = \mathrm{span}\,\{\mathbf{w}_1,\mathbf{w}_2\}$. Then the closest point to $\mathbf{b}$ in W is the orthogonal projection of $\mathbf{b}$ onto W:

$$\mathrm{proj}_W\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 + \frac{\mathbf{b}\cdot\mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = \frac{3}{2}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + \frac{2}{7}\begin{pmatrix}2\\1\\1\\-1\end{pmatrix} = \begin{pmatrix}4/7\\2/7\\25/14\\17/14\end{pmatrix}.$$

□