21-241: Matrix Algebra – Summer I, 2006

Homework 4 Solutions

Wednesday, June 14

4.1.2 Since f is a sum of squares, $f \geqslant 0$. The equality holds if and only if both summands are zeros, i.e.,

$$3x - 2y + 1 = 0,$$
$$2x + y + 2 = 0.$$

This system has a unique solution $(x^*, y^*) = (-\frac{5}{7}, -\frac{4}{7})$, which is the only minimizer of f. $\square$

4.2.3 **(a)** $K = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}$, $\mathbf{f} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$, $c = -1$. By Gaussian,

$$\left(\begin{array}{cc|c} 1 & -1 & -\frac{1}{2} \\ -1 & 4 & 0 \end{array}\right) \xrightarrow{R_2+R_1} \left(\begin{array}{cc|c} \boxed{1} & -1 & -\frac{1}{2} \\ 0 & \boxed{3} & -\frac{1}{2} \end{array}\right)$$

Therefore, K is positive definite, the quadratic function has a minimum. By back substitution, the minimizer is $(-\frac{2}{3}, -\frac{1}{6})^T$, the minimum value is $-\frac{4}{3}$.

(c) $K = \begin{pmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 3 \end{pmatrix}$, $\mathbf{f} = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}$, $c = 0$. By Gaussian,

$$\left(\begin{array}{cc|c} 1 & \frac{5}{2} & -1 \\ \frac{5}{2} & 3 & \frac{1}{2} \end{array}\right) \xrightarrow{R_2-\frac{5}{2}R_1} \left(\begin{array}{cc|c} \boxed{1} & \frac{5}{2} & -1 \\ 0 & \boxed{-\frac{13}{4}} & 3 \end{array}\right)$$

Therefore, K is indefinite, the quadratic function has no minimum.

(e) $K = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$, $\mathbf{f} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $c = -3$. By Gaussian,

$$\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \xrightarrow{R_2-\frac{1}{2}R_1} \begin{pmatrix} \boxed{1} & \frac{1}{2} & 0 \\ 0 & \boxed{-\frac{5}{4}} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

Therefore, K is indefinite, the quadratic function has no minimum. $\square$

4.2.5 **(b)** $p(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \mathbf{f} + c = 3x_1^2 + 4x_1x_2 + x_2^2 - 8x_1 - 2x_2$. By Gaussian,

$$K = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \xrightarrow{R_2-\frac{2}{3}R_1} \begin{pmatrix} \boxed{3} & 2 \\ 0 & \boxed{-\frac{1}{3}} \end{pmatrix}$$

Therefore, K is indefinite, q has no minimum.

(c) $p(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \mathbf{f} + c = 3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_3x_1 - 2x_1 + 4x_3 - 3$. By Gaussian,

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ -1 & 2 & -1 & 0 \\ 1 & -1 & 3 & -2 \end{array}\right) \xrightarrow[R_3-\frac{1}{3}R_1]{R_2+\frac{1}{3}R_1} \left(\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & \frac{5}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{2}{3} & \frac{8}{3} & -\frac{7}{3} \end{array}\right) \xrightarrow{R_3+\frac{2}{5}R_2} \left(\begin{array}{ccc|c} \boxed{3} & -1 & 1 & 1 \\ 0 & \boxed{\frac{5}{3}} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \boxed{\frac{12}{5}} & -\frac{11}{5} \end{array}\right)$$

Therefore, K is positive definite, p has a minimum. By back substitution, the minimizer is $\mathbf{x}^* = (\frac{13}{60}, \frac{17}{30}, \frac{11}{12})^T$. The minimum value $p(\mathbf{x}^*) = \frac{83}{60}$. $\square$

4.2.7 (b) We only need to find out the minimum of the negative of the quadratic function. Let

$$p(\mathbf{x}) = 2x^2 - 6xy + 3y^2 - 4x + 3y = \mathbf{x}^T K\mathbf{x} - 2\mathbf{x}^T\mathbf{f} + c,$$

where $K = \begin{pmatrix} 2 & -3 \\ -3 & 3 \end{pmatrix}$, $\mathbf{f} = \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix}$, $c = 0$. By Gaussian,

$$\begin{pmatrix} 2 & -3 \\ -3 & 3 \end{pmatrix} \xrightarrow{R_2+\frac{3}{2}R_1} \begin{pmatrix} \boxed{2} & -3 \\ 0 & \boxed{-\frac{3}{2}} \end{pmatrix}$$

Therefore, K is indefinite, p has no minimum. Thus $-p$ has no maximum. □

4.2.9 When K is positive definite, p has a unique minimizer $\mathbf{x}^* = K^{-1}\mathbf{f}$, and $p(\mathbf{x}^*) = -\mathbf{x}^{*T}K\mathbf{x}^* \leqslant 0$. The equality holds only if $\mathbf{x}^* = \mathbf{0}$. Therefore, $\mathbf{f} = K\mathbf{x}^* = \mathbf{0}$. Vice verse, when $\mathbf{f} = \mathbf{0}$, $p(\mathbf{x}) = \mathbf{x}^T K\mathbf{x}$. It's obvious that the minimum value equal to zero. □

Thursday, June 15

3.5.1 (b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix} \boxed{1} & 1 \\ 0 & \boxed{0} \end{pmatrix}$, positive semi-definite.

(c) $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \xrightarrow[R_3-2R_1]{R_2-R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -3 \end{pmatrix} \xrightarrow{R_3+R_2} \begin{pmatrix} \boxed{1} & 1 & 2 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & \boxed{-4} \end{pmatrix}$, indefinite.

(d) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & -2 & 4 \end{pmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & -3 & 3 \end{pmatrix} \xrightarrow{R_3+3R_2} \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & \boxed{-6} \end{pmatrix}$, indefinite.

3.5.2 (b) $\begin{pmatrix} 5 & -1 \\ -1 & 3 \end{pmatrix} \xrightarrow{R_2+\frac{1}{5}R_1} \begin{pmatrix} \boxed{5} & -1 \\ 0 & \boxed{\frac{14}{5}} \end{pmatrix}$, positive definite.

(d) $\begin{pmatrix} -2 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix} \xrightarrow[R_3-\frac{1}{2}R_1]{R_2+\frac{1}{2}R_1} \begin{pmatrix} -2 & 1 & -1 \\ 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{3}{2} \end{pmatrix} \xrightarrow{R_3+\frac{1}{3}R_2} \begin{pmatrix} \boxed{-2} & 1 & -1 \\ 0 & \boxed{-\frac{3}{2}} & \frac{1}{2} \\ 0 & 0 & \boxed{-\frac{4}{3}} \end{pmatrix}$, negative definite.

(h) $\begin{pmatrix} 2 & 1 & -2 & 0 \\ 1 & 1 & -3 & 2 \\ -2 & -3 & 10 & -1 \\ 0 & 2 & -1 & 7 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2-\frac{1}{2}R_1} \begin{pmatrix} 2 & 1 & -2 & 0 \\ 0 & \frac{1}{2} & -2 & 2 \\ 0 & -2 & 8 & -1 \\ 0 & 2 & -1 & 7 \end{pmatrix} \xrightarrow[R_4-4R_2]{R_3+4R_2} \begin{pmatrix} 2 & 1 & -2 & 0 \\ 0 & \frac{1}{2} & -2 & 2 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 7 & -1 \end{pmatrix}$,
can't be reduced to an upper triangular matrix, so it's indefinite.

3.5.8 The associated matrix of the quadratic form is $\begin{pmatrix} 1 & \frac{a}{2} & \frac{b}{2} \\ \frac{a}{2} & 1 & \frac{c}{2} \\ \frac{b}{2} & \frac{c}{2} & 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & \frac{a}{2} & \frac{b}{2} \\ \frac{a}{2} & 1 & \frac{c}{2} \\ \frac{b}{2} & \frac{c}{2} & 1 \end{pmatrix} \xrightarrow[R_3-\frac{b}{2}R_1]{R_2-\frac{a}{2}R_1} \begin{pmatrix} 1 & \frac{a}{2} & \frac{b}{2} \\ 0 & 1-\frac{a^2}{4} & \frac{2c-ab}{4} \\ 0 & \frac{2c-ab}{4} & 1-\frac{b^2}{4} \end{pmatrix} \xrightarrow{R_3-\frac{2c-ab}{4-a^2}R_2} \begin{pmatrix} 1 & \frac{a}{2} & \frac{b}{2} \\ 0 & 1-\frac{a^2}{4} & \frac{2c-ab}{4} \\ 0 & 0 & \frac{16-4a^2-4b^2-4c^2+4abc}{4-a^2} \end{pmatrix}$$

Hence, when $a^2 < 4$ and $a^2 + b^2 + c^2 - abc < 4$, the quadratic form is positive definite. $\square$

4.2.6 When $n = 4$,

$$p(\mathbf{x}) = 4(x_1^2 + x_2^2 + x_3^2 + x_4^2) - 2(x_1x_2 + x_2x_3 + x_3x_4) + (x_1 + x_2 + x_3 + x_4) = \mathbf{x}^T K\mathbf{x} - 2\mathbf{x}^T\mathbf{f} + c,$$

where $K = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}$, $\mathbf{f} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$, $c = 0$. Reduce the augmented matrix $(K \,|\, \mathbf{f})$:

$$\left(\begin{array}{cccc|c} 4 & -1 & 0 & 0 & -\frac{1}{2} \\ -1 & 4 & -1 & 0 & -\frac{1}{2} \\ 0 & -1 & 4 & -1 & -\frac{1}{2} \\ 0 & 0 & -1 & 4 & -\frac{1}{2} \end{array}\right) \xrightarrow{R_2 + \frac{1}{4}R_1} \left(\begin{array}{cccc|c} 4 & -1 & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{15}{4} & -1 & 0 & -\frac{5}{8} \\ 0 & -1 & 4 & -1 & -\frac{1}{2} \\ 0 & 0 & -1 & 4 & -\frac{1}{2} \end{array}\right)$$

$$\xrightarrow{R_3 + \frac{4}{15}R_2} \left(\begin{array}{cccc|c} 4 & -1 & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{15}{4} & -1 & 0 & -\frac{5}{8} \\ 0 & 0 & \frac{56}{15} & -1 & -\frac{2}{3} \\ 0 & 0 & -1 & 4 & -\frac{1}{2} \end{array}\right) \xrightarrow{R_4 + \frac{15}{56}R_3} \left(\begin{array}{cccc|c} 4 & -1 & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{15}{4} & -1 & 0 & -\frac{5}{8} \\ 0 & 0 & \frac{56}{15} & -1 & -\frac{2}{3} \\ 0 & 0 & 0 & \frac{209}{56} & -\frac{19}{28} \end{array}\right)$$

Now that K has all positive pivots, it's positive definite, and p has a minimum. The minimizer can be obtained by back substitution $\mathbf{x}^* = (-\frac{2}{11}, -\frac{5}{22}, -\frac{5}{22}, -\frac{2}{11})^T$. The minimum value equals $p(\mathbf{x}^*) = c - \mathbf{x}^{*T}\mathbf{f} = -\frac{9}{22}$. $\square$

4.2.10 When A is positive semi-definite, $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} \geqslant 0$, and the zero vector is a minimizer. So the minimum value of $q(\mathbf{x})$ is 0.

When A is not positive semi-definite, there exists a vector $\mathbf{y}$ such that $q(\mathbf{y}) = \mathbf{y}^T A\mathbf{y} < 0$. Then $q(t\,\mathbf{y}) = (\mathbf{y}^T A\mathbf{y})\,t^2$. Let $t \to \infty$, $q(t\,\mathbf{y}) \to \infty$. So the minimum value of q is $-\infty$. $\square$

Friday, June 16

5.1.1 **(b)** Orthonormal basis.

(d) Basis, not orthogonal.

(f) Orthonormal basis.

5.1.2 **(a)** Basis, not orthogonal. The first and the third vectors are not orthogonal.

(c) Not a basis. The three vectors are linearly dependent, since their sum is zero.

5.1.4 Since

$$\begin{aligned} \langle \mathbf{e}_1, \mathbf{e}_2 \rangle &= 1 \cdot 0 + 2(0 \cdot 1) + 3(0 \cdot 0) = 0, \\ \langle \mathbf{e}_1, \mathbf{e}_3 \rangle &= 1 \cdot 0 + 2(0 \cdot 0) + 3(0 \cdot 1) = 0, \\ \langle \mathbf{e}_2, \mathbf{e}_3 \rangle &= 0 \cdot 0 + 2(1 \cdot 0) + 3(0 \cdot 1) = 0, \end{aligned}$$

they form an orthogonal basis with respect to the weighted inner product. An orthonormal basis is obtained by normalization:

$$\mathbf{u}_1 = \frac{\mathbf{e}_1}{\|\mathbf{e}_1\|} = (1,0,0)^T,$$
$$\mathbf{u}_2 = \frac{\mathbf{e}_2}{\|\mathbf{e}_2\|} = (0,1/\sqrt{2},0)^T,$$
$$\mathbf{u}_3 = \frac{\mathbf{e}_3}{\|\mathbf{e}_3\|} = (0,0,1/\sqrt{3})^T.$$

□

5.1.6 Since $\langle (1,2)^T, (-1,1)^T \rangle = -a+2b$, any pair of (a,b) that satisfies $-a+2b=0$ makes the two vectors an orthogonal basis in $\mathbb{R}^2$. The general form is $(2b,b)$, where b is arbitrary. □

5.1.16 Since $\mathbf{v}_1$, $\mathbf{v}_2$ are linearly independent, one of the two can't be a multiple of the other. Therefore, both $\mathbf{v}_1+\mathbf{v}_2$ and $\mathbf{v}_1-\mathbf{v}_2$ are not the zero vector. Now that

$$\langle \mathbf{v}_1+\mathbf{v}_2, \mathbf{v}_1-\mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle - \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \|\mathbf{v}_1\|^2 - \|\mathbf{v}_2\|^2 = 0,$$

we may conclude that $\mathbf{v}_1+\mathbf{v}_2$ and $\mathbf{v}_1-\mathbf{v}_2$ form an orthogonal basis. □