21-241: Matrix Algebra – Summer I, 2006

Homework 3 Solutions

Monday, June 5

2.3.2 We need to show the system $\left(\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ -3 & 6 & 4 & 7 \\ -2 & 3 & 6 & 6 \\ 0 & 4 & -7 & 1 \end{array}\right)$ is consistent. By Gaussian elimination,

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ -3 & 6 & 4 & 7 \\ -2 & 3 & 6 & 6 \\ 0 & 4 & -7 & 1 \end{array}\right) \xrightarrow[R_3+2R_1]{R_2+3R_1} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 0 & 0 & -2 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 4 & -7 & 1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 4 & -7 & 1 \end{array}\right)$$

$$\xrightarrow{R_4+4R_2} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_4+\frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

There is no pivot in the augmented column. So, the system is consistent. □

2.3.4 **(b)** $(2,-1)^T$ and $(1,3)^T$ are not multiples of each other, thus linearly independent. Therefore, they span a 2-dimensional subspace of $\mathbb{R}^2$, which has to be $\mathbb{R}^2$ itself.

(d) Since $(6,-9)^T = -\frac{3}{2}(-4,6)^T$, they are linearly dependent. Therefore, they span a 1-dimensional subspace of $\mathbb{R}^2$, which is the entire line in the direction of $(6,-9)^T$, not $\mathbb{R}^2$.

(f) Since $(0,0)^T$ and $(2,-2)^T$ are both multiples of $(1,-1)^T$, they span a 1-dimensional subspace of $\mathbb{R}^2$, which is the entire line in the direction of $(1,-1)^T$, not $\mathbb{R}^2$. □

2.3.17 This statement is **false**. Here is a counter-example. Let

$$\mathbf{z} = (1,1,0)^T, \quad \mathbf{u} = (1,0,0)^T, \quad \mathbf{v} = (0,1,0)^T, \quad \mathbf{w} = (0,0,1)^T.$$

Then $\mathbf{z} = \mathbf{u} + \mathbf{v} + 0\,\mathbf{w}$, a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$. But $\mathbf{w}$ can't be a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{z}$, because the third entry of $\mathbf{w}$ is 1, while those of $\mathbf{u}, \mathbf{v}, \mathbf{z}$ are all 0's. □

2.3.21 **(c)** There are at most 2 linearly independent vectors in 2-dimensional vector space. So any three vectors in $\mathbb{R}^2$ have to be linearly dependent.

(e) Combine the three vectors as columns in a single matrix A, and reduce it in the echelon form,

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & -1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

There is a free column, so the homogeneous system $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions. This means the three vectors are linearly dependent.

(g) It's obvious that $(4,2,0,-6)^T = -\frac{2}{3}(-6,-3,0,9)^T$. So they are linearly dependent. □

2.4.9 **(a)** It's easy to see that $(-6,-2,2)^T = -2(3,1,-1)^T$. So, $\{(3,1,-1)\}$ is a basis for the span with dimension 1.

(c) Combine the four vectors as columns in a single matrix A, and reduce A in the echelon form,

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ -1 & 1 & -3 & 1 \\ 2 & 3 & 1 & 1 \end{pmatrix} \xrightarrow[R_4-2R_1]{R_3+R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & -3 & -1 \end{pmatrix} \xrightarrow[R_4-3R_2]{R_3-R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$
$$\xrightarrow{R_4-\frac{5}{4}R_3} \begin{pmatrix} \boxed{1} & 0 & 2 & 1 \\ 0 & \boxed{1} & -1 & -2 \\ 0 & 0 & 0 & \boxed{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The first, second and fourth columns contain pivots. So $\{(1,0,-1,2)^T, (0,1,1,3)^T, (1,-2,1,1)^T\}$ is a basis of the span (namely $\operatorname{Col} A$), with dimension 3. □

Tuesday, June 6

2.5.5 (c) Apply Gauss-Jordan,

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 2 & 0 & -4 & -6 \\ 2 & -1 & -2 & -4 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 2 & -4 & -4 \\ 0 & 1 & -2 & -2 \end{array}\right) \xrightarrow{R_3-\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 2 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$\xrightarrow{R_2/2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|c} \boxed{1} & 0 & -2 & -3 \\ 0 & \boxed{1} & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore, the general solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z-3 \\ 2z-2 \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} + z\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \mathbf{x}^* + \mathbf{z},$$

where, $\mathbf{x}^* = (-3,-2,0)^T$ is a particular solution, $\mathbf{z} = z(2,2,1)^T$ is the general element of the kernel.

(e) Apply Gauss-Jordan,

$$\left(\begin{array}{cc|c} 1 & -2 & -1 \\ 2 & -4 & -2 \\ -3 & 6 & 3 \\ -1 & 2 & 1 \end{array}\right) \xrightarrow[R_4+R_1]{\substack{R_2-2R_1 \\ R_3+3R_1}} \left(\begin{array}{cc|c} \boxed{1} & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Therefore, the general solution is

$$\mathbf{x} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2v-1 \\ v \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + v\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{x}^* + \mathbf{z},$$

where, $\mathbf{x}^* = (-1,0)^T$ is a particular solution, $\mathbf{z} = v(2,1)^T$ is the general element of the kernel. □

2.5.22 Denote columns of the matrix by $\mathbf{v}_1, \cdots, \mathbf{v}_5$. Apply Gaussian,

$$\begin{pmatrix} -1 & 2 & 0 & -3 & 5 \\ 2 & -4 & 1 & 1 & -4 \\ -3 & 6 & 2 & 0 & 8 \end{pmatrix} \xrightarrow[R_3-3R_1]{R_2+2R_1} \begin{pmatrix} -1 & 2 & 0 & -3 & 5 \\ 0 & 0 & 1 & -5 & 6 \\ 0 & 0 & 2 & 9 & -7 \end{pmatrix} \xrightarrow{R_3-2R_2} \begin{pmatrix} \boxed{-1} & 2 & 0 & -3 & 5 \\ 0 & 0 & \boxed{1} & -5 & 6 \\ 0 & 0 & 0 & \boxed{19} & -19 \end{pmatrix}$$

Therefore, $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4$ form a basis for range of the matrix. Then

$$\mathbf{v}_1 = \mathbf{v}_1, \qquad \mathbf{v}_2 = -2\mathbf{v}_1, \qquad \mathbf{v}_3 = \mathbf{v}_3, \qquad \mathbf{v}_4 = \mathbf{v}_4, \qquad \mathbf{v}_5 = -2\mathbf{v}_1 + \mathbf{v}_3 - \mathbf{v}_4$$

□

2.5.26 Write $\mathbf{v}$ in linear combination form:

$$\mathbf{v} = \begin{pmatrix} a-3b \\ a+2c+4d \\ b+3c-d \\ c-d \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 2 \\ 3 \\ 1 \end{pmatrix} + d\begin{pmatrix} 0 \\ 4 \\ -1 \\ -1 \end{pmatrix} \doteq a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 + d\mathbf{v}_4.$$

Hence, the set of all vectors of the form of $\mathbf{v}$ is $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, thus a subspace of $\mathbb{R}^4$. Combine these four vectors as columns in a single matrix A, and reduce A in the echelon form:

$$\begin{pmatrix} 1 & -3 & 0 & 0 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 3 & 2 & 4 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_3-\frac{1}{3}R_2} \begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & \frac{7}{3} & -\frac{7}{3} \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_4-\frac{3}{7}R_3} \begin{pmatrix} \boxed{1} & -3 & 0 & 0 \\ 0 & \boxed{3} & 2 & 4 \\ 0 & 0 & \boxed{\frac{7}{3}} & -\frac{7}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are 3 pivot columns, the dimension of $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ (namely $\operatorname{Col} A$) is 3. □

2.5.27 (c) Reduce the coefficient matrix in the echelon form:

$$\begin{pmatrix} 1 & -1 & -2 & 4 \\ 2 & 1 & 0 & -1 \\ -2 & 0 & 2 & -2 \end{pmatrix} \xrightarrow[R_3+2R_1]{R_2-2R_1} \begin{pmatrix} 1 & -1 & -2 & 4 \\ 0 & 3 & 4 & -9 \\ 0 & -2 & -2 & 6 \end{pmatrix} \xrightarrow{R_3+\frac{2}{3}R_2} \begin{pmatrix} \boxed{1} & -1 & -2 & 4 \\ 0 & \boxed{3} & 4 & -9 \\ 0 & 0 & \boxed{\frac{2}{3}} & 0 \end{pmatrix}$$

So x_4 is free. By back substitution, we obtain the general solution in vector form:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_4 \\ 3x_4 \\ 0 \\ x_4 \end{pmatrix} = x_4\begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

Hence $(2,3,0,1)^T$ is a basis for the solution space (namely the kernel). □

2.5.38 Suppose $\mathbf{x} \in \ker A$, then $A\mathbf{x} = \mathbf{0}$. Therefore, $BA\mathbf{x} = B(A\mathbf{x}) = B\mathbf{0} = \mathbf{0}$, implying $\mathbf{x} \in \ker BA$. Thus we've proved that $\ker A \subseteq \ker BA$. Particularly, let $B = A$, we get $\ker A \subseteq \ker A^2$. □

Wednesday, June 7

2.5.29 Let $A = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3)$, $B = (\mathbf{w}_1\ \mathbf{w}_2\ \mathbf{w}_3)$. We are to characterize $\operatorname{Col} A$ and $\operatorname{Col} B$. First consider the system $A\mathbf{x} = \mathbf{b}$, where the right hand side will remain unspecified for the moment. Apply Gaussian to the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & b_1 \\ 2 & 1 & 0 & b_2 \\ 0 & 1 & -4 & b_3 \\ -1 & -1 & 3 & b_4 \end{array}\right) \xrightarrow[R_4+R_1]{R_2-2R-1} \left(\begin{array}{ccc|c} 1 & -3 & 2 & b_1 \\ 0 & 7 & -4 & b_2-2b_1 \\ 0 & 1 & -4 & b_3 \\ 0 & -4 & 5 & b_4+b_1 \end{array}\right) \xrightarrow{R_2\leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -3 & 2 & b_1 \\ 0 & 1 & -4 & b_3 \\ 0 & 7 & -4 & b_2-2b_1 \\ 0 & -4 & 5 & b_4+b_1 \end{array}\right)$$

$$\xrightarrow[R_4+4R_2]{R_3-7R_2} \left(\begin{array}{ccc|c} 1 & -3 & 2 & b_1 \\ 0 & 1 & -4 & b_3 \\ 0 & 0 & 24 & b_2-2b_1-7b_3 \\ 0 & 0 & -11 & b_4+b_1+4b_3 \end{array}\right) \xrightarrow{R_4+\frac{11}{24}R_3} \left(\begin{array}{ccc|c} 1 & -3 & 2 & b_1 \\ 0 & 1 & -4 & b_3 \\ 0 & 0 & 24 & b_2-2b_1-7b_3 \\ 0 & 0 & 0 & \frac{1}{12}b_1+\frac{11}{24}b_2+\frac{19}{24}b_3+b_4 \end{array}\right)$$

Hence, $\operatorname{Col} A = \{(b_1, b_2, b_3, b_4)^T \mid \frac{1}{12}b_1 + \frac{11}{24}b_2 + \frac{19}{24}b_3 + b_4 = 0\}$, a three-dimensional subspace of $\mathbb{R}^4$. Similarly,

$$\left(\begin{array}{ccc|c} 3 & 2 & 0 & b_1 \\ 2 & 3 & 3 & b_2 \\ -4 & -7 & -3 & b_3 \\ 2 & 4 & 1 & b_4 \end{array}\right) \xrightarrow{R_1\leftrightarrow R_4} \left(\begin{array}{ccc|c} 2 & 4 & 1 & b_4 \\ 2 & 3 & 3 & b_2 \\ -4 & -7 & -3 & b_3 \\ 3 & 2 & 0 & b_1 \end{array}\right) \xrightarrow[R_4-\frac{3}{2}R_1]{\substack{R_2-R_1 \\ R_3+2R_1}} \left(\begin{array}{ccc|c} 2 & 4 & 1 & b_4 \\ 0 & -1 & 2 & b_2-b_4 \\ 0 & 1 & -1 & b_3+2b_4 \\ 0 & -4 & -\frac{3}{2} & b_1-\frac{3}{2}b_4 \end{array}\right)$$

$$\xrightarrow[R_4-4R_2]{R_3+R_2} \left(\begin{array}{ccc|c} 2 & 4 & 1 & b_4 \\ 0 & -1 & 2 & b_2-b_4 \\ 0 & 0 & 1 & b_2+b_3+b_4 \\ 0 & 0 & -\frac{19}{2} & b_1-4b_2+\frac{5}{2}b_4 \end{array}\right) \xrightarrow{R_4+\frac{19}{2}R_3} \left(\begin{array}{ccc|c} 2 & 4 & 1 & b_4 \\ 0 & -1 & 2 & b_2-b_4 \\ 0 & 0 & 1 & b_2+b_3+b_4 \\ 0 & 0 & 0 & b_1+\frac{11}{2}b_2+\frac{19}{2}b_3+12b_4 \end{array}\right)$$

So, $\operatorname{Col} B = \{(b_1, b_2, b_3, b_4)^T \mid b_1 + \frac{11}{2}b_2 + \frac{19}{2}b_3 + 12b_4 = 0\}$. Clearly, $\operatorname{Col} A = \operatorname{Col} B$, denoted by V. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ are two bases for V, which is a 3-dimensional subspace of $\mathbb{R}^4$. □

3.1.1 It's easy to verify that bilinearity and symmetry hold for all b. Then whether $\langle \mathbf{v}, \mathbf{w}\rangle$ defines an inner product depends on whether positivity holds. When $b > 1$,

$$\langle \mathbf{v}, \mathbf{v}\rangle = v_1^2 - 2v_1v_2 + bv_2^2 = (v_1 - v_2)^2 + (b-1)v_2^2 \geqslant 0.$$

The equality holds only when $v_1 - v_2 = v_2 = 0$, or $v_1 = v_2 = 0$, namely $\mathbf{v} = \mathbf{0}$. So positivity holds. When $b \leqslant 1$, let $\mathbf{v} = (1,1)^T \neq \mathbf{0}$. Then $\langle \mathbf{v}, \mathbf{v}\rangle = 1 - b \leqslant 0$, implying positivity doesn't hold. Therefore, $\langle \mathbf{v}, \mathbf{w}\rangle$ defines an inner product if and only if $b > 1$. □

3.1.2 **(c)** No. Let $\mathbf{v} = (1,-1)^T \neq \mathbf{0}$, but $\langle \mathbf{v}, \mathbf{v}\rangle = 0$. So positivity doesn't hold.

(e) No. Let $\mathbf{u} = (1,0)^T$, $\mathbf{v} = \mathbf{w} = (0,1)^T$, then $\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle = \sqrt{2}$, $\langle \mathbf{u}, \mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle = 1$. So $\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle \neq \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$, bilinearity doesn't hold.

(g) Yes. Bilinearity and symmetry are straightforward. To verify positivity, note that

$$\langle \mathbf{v}, \mathbf{v}\rangle = 4v_1^2 - 4v_1v_2 + 4v_2^2 = 4(v_1 - \frac{1}{2}v_2)^2 + 3v_2^2 \geqslant 0,$$

and equality holds only if $v_1 - \frac{1}{2}v_2 = v_2 = 0$, or $v_1 = v_2 = 0$. So $\mathbf{v} = \mathbf{0}$, thus positivity holds. □

3.1.8 Using bilinearity we have

$$\begin{aligned}\langle a\mathbf{v}+b\mathbf{w}, c\mathbf{v}+d\mathbf{w}\rangle &= a\langle \mathbf{v}, c\mathbf{v}+d\mathbf{w}\rangle + b\langle \mathbf{w}, c\mathbf{v}+d\mathbf{w}\rangle \\ &= a(c\langle \mathbf{v},\mathbf{v}\rangle + d\langle \mathbf{v},\mathbf{w}\rangle) + b(c\langle \mathbf{w},\mathbf{v}\rangle + d\langle \mathbf{w},\mathbf{w}\rangle) \\ &= ac\langle \mathbf{v},\mathbf{v}\rangle + (ad+bc)\langle \mathbf{v},\mathbf{w}\rangle + bd\langle \mathbf{w},\mathbf{w}\rangle \\ &= ac\|\mathbf{v}\|^2 + (ad+bc)\langle \mathbf{v},\mathbf{w}\rangle + bd\|\mathbf{w}\|^2\end{aligned}$$

□

3.1.12 **(a)** We can apply problem 3.1.8 to $\|\mathbf{x}+\mathbf{y}\|^2$ and $\|\mathbf{x}-\mathbf{y}\|^2$, and get

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle = \|\mathbf{x}\|^2 + 2\langle \mathbf{x},\mathbf{y}\rangle + \|\mathbf{y}\|^2 \\ \|\mathbf{x}-\mathbf{y}\|^2 = \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle = \|\mathbf{x}\|^2 - 2\langle \mathbf{x},\mathbf{y}\rangle + \|\mathbf{y}\|^2\end{aligned}$$

Adding these two equations, we obtain the required equality

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2).$$

(b) Let $\mathbf{x}$ and $\mathbf{y}$ be two adjacent sides of a parallelogram. Then the two diagonals are $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$. So this equality tells us that the sum of squares of lengths of diagonals of a parallelogram equals the sum of squares of lengths of its four sides. □

Thursday, June 8

3.2.5 $\langle \mathbf{v},\mathbf{w}\rangle = 19$, $\|\mathbf{v}\| = \sqrt{38}$, $\|\mathbf{w}\| = \sqrt{10}$. Since $19 < \sqrt{38}\sqrt{10}$, the Cauchy-Schwarz inequality is true. □

3.2.6 Let $\mathbf{u} = (a,b)^T$, $\mathbf{v} = (\cos\theta, \sin\theta)$. Then $\mathbf{u}\cdot\mathbf{v} = a\cos\theta + b\sin\theta$, $\|\mathbf{u}\| = \sqrt{a^2+b^2}$, $\|\mathbf{v}\| = 1$. By Cauchy-Schwarz inequality, $(\mathbf{u}\cdot\mathbf{v})^2 \leqslant \|\mathbf{u}\|^2\|\mathbf{v}\|^2$, namely, $(a\cos\theta + b\sin\theta)^2 \leqslant a^2+b^2$. □

3.2.19 Suppose vector $(x,y,z,w)^T \in \mathbb{R}^4$ is orthogonal to the vector $(1,2,-1,3)^T$, then their dot product has to be 0, namely,

$$x + 2y - z + 3w = 0.$$

Look at this equation as a homogeneous linear system, then all vectors orthogonal to $(1,2,-1,3)^T$ are solutions to this system, and thus W is the kernel. We can write the general solution in vector form:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2y+z-3w \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2y \\ y \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \\ z \\ 0 \end{pmatrix} + \begin{pmatrix} -3w \\ 0 \\ 0 \\ w \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, $\{(-2,1,0,0)^T, (1,0,1,0)^T, (-3,0,0,1)^T\}$ is basis for W. □

3.2.39 By triangle inequality,

$$\|\mathbf{v}\| = \|(\mathbf{v}-\mathbf{w})+\mathbf{w}\| \leqslant \|\mathbf{v}-\mathbf{w}\| + \|\mathbf{w}\|,$$

or,

$$\|\mathbf{v}\| - \|\mathbf{w}\| \leqslant \|\mathbf{v}-\mathbf{w}\|.$$

Exchange $\mathbf{v}$ and $\mathbf{w}$, we get

$$\|\mathbf{w}\| - \|\mathbf{v}\| \leqslant \|\mathbf{w}-\mathbf{v}\| = \|\mathbf{v}-\mathbf{w}\|.$$

Therefore,

$$|\ \|\mathbf{v}\| - \|\mathbf{w}\|\ | \leqslant \|\mathbf{v}-\mathbf{w}\|.$$

This inequality says the length of a side of a triangle is at least equal to the difference of the lengths of the other two sides. □

3.3.9 Positivity and homogeneity are straightforward. Let's verify the triangle inequality. Let $\mathbf{x} = (x_1, y_1)^T$, $\mathbf{y} = (x_2, y_2)^T$. Then

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\| &= |x_1+x_2| + 2|(x_1+x_2)-(y_1+y_2)| = |x_1+x_2| + 2|(x_1-y_1)+(x_2-y_2)| \\ &\leqslant (|x_1|+|x+2|) + 2(|x_1-y_1|+|x_2-y_2|) = (|x_1|+2|x_1-y_1|) + (|x_2|+|x_2-y_2|) \\ &= \|\mathbf{x}\| + \|\mathbf{y}\|.\end{aligned}$$

Thus we complete the proof. □

Friday, June 9

3.4.1 First of all, (f) is not symmetric, thus not positive definite. We've learn that a symmetric 2×2 matrix $\left(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right)$ is positive definite if and only if $a > 0$ and $ac - b^2 > 0$. Therefore, both (b) and (d) are not positive definite, too. □

3.4.7 **(a)** First, since K and L are symmetric, $(K+L)^T = K^T + L^T = K + L$. So $K + L$ is also symmetric. For all $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x}^T(K+L)\mathbf{x} = \mathbf{x}^T K\mathbf{x} + \mathbf{x}^T L\mathbf{x} > 0$. Hence, $K + L$ is positive definite.

(b) Let $A = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & -2 \\ -2 & 1\end{smallmatrix}\right)$. Both matrices are not positive definite. However, $A+B = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2\end{smallmatrix}\right)$ is positive definite. □

3.4.20 Since $q(\mathbf{x})$ is a scalar, $q(\mathbf{x}) = q(\mathbf{x})^T = (\mathbf{x}^T A\mathbf{x})^T = \mathbf{x}^T A^T\mathbf{x}$. Therefore,

$$q(\mathbf{x}) = \frac{1}{2}(\mathbf{x}^T A\mathbf{x} + \mathbf{x}^T A^T\mathbf{x}) = \mathbf{x}^T K\mathbf{x},$$

where, $K = \frac{1}{2}(A + A^T)$. K is symmetric because $K^T = \frac{1}{2}(A+A^T)^T = \frac{1}{2}(A^T + A) = K$. □

3.5.5 **(c)** $x^2 - 2xy - y^2 = (x-y)^2 - 2y^2$, not positive definite.

(d) $x^2 + 6xy = (x+3y)^2 - 9y^2$, not positive definite. □

3.5.19 (a)

$$\begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix} \xrightarrow{R_2+\frac{2}{3}R_1} \begin{pmatrix} 3 & -2 \\ 0 & \frac{2}{3} \end{pmatrix} = U, \qquad D = \operatorname{diag}(3, 2/3),$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_2-\frac{2}{3}R_1} \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & 1 \end{pmatrix} = L, \qquad S = \operatorname{diag}(\sqrt{3}, \sqrt{6}/3).$$

Let $M = LS = \begin{pmatrix} \sqrt{3} & 0 \\ -\frac{2}{3}\sqrt{3} & \frac{\sqrt{6}}{3} \end{pmatrix}$, then the Cholesky factorization is

$$\begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix} = MM^T = \begin{pmatrix} \sqrt{3} & 0 \\ -\frac{2}{3}\sqrt{3} & \frac{\sqrt{6}}{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & -\frac{2}{3}\sqrt{3} \\ 0 & \frac{\sqrt{6}}{3} \end{pmatrix}.$$

(c)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & -2 & 14 \end{pmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & -3 & 13 \end{pmatrix} \xrightarrow{R_3+3R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{pmatrix} = U, \qquad D = \operatorname{diag}(1, 1, 4),$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2+R_1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_3-3R_2} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix} = L, \qquad S = \operatorname{diag}(1, 1, 2).$$

Let $M = LS = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -3 & 2 \end{pmatrix}$, then the Cholesky factorization is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & -2 & 14 \end{pmatrix} = MM^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{pmatrix}.$$

□