21-241: Matrix Algebra – Summer I, 2006

Homework 2 Solutions

Tuesday, May 30

1.8.1 (b) Applying Gaussian to reduce the augmented matrix in the echelon form, we get

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & 4 & -2 & -3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 4 & -2 & -3 \\ 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} \boxed{1} & 4 & -2 & -3 \\ 0 & \boxed{-7} & 7 & 7 \end{array}\right).$$

Now we don't have a zero row with nonzero right hand side, or, you can say the augmented column contains no pivot, so the system is consistent. We also have a free column (the third one), so the system has infinitely many solutions.

(d) Reducing the augmented matrix in the echelon form, we get

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 6 \\ 2 & 1 & -3 & -3 \\ 1 & -3 & 3 & 10 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 6 \\ 0 & 5 & -5 & -15 \\ 0 & -1 & 2 & 4 \end{array}\right) \xrightarrow{R_3 + (1/5)R_2} \left(\begin{array}{ccc|c} \boxed{1} & -2 & 1 & 6 \\ 0 & \boxed{5} & -5 & -15 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right)$$

It's clear that there is no pivot in the augmented column and no free variable. So the system has a unique solution.

(f) Reducing the augmented matrix as follows

$$\left(\begin{array}{ccc|c} 3 & -2 & 1 & 4 \\ 1 & 3 & -4 & -3 \\ 2 & -3 & 5 & 7 \\ 1 & -8 & 9 & 10 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 3 & -4 & -3 \\ 3 & -2 & 1 & 4 \\ 2 & -3 & 5 & 7 \\ 1 & -8 & 9 & 10 \end{array}\right) \xrightarrow[R_4 - R_1]{\substack{R_2 - 3R_1 \\ R_3 - 2R_1}} \left(\begin{array}{ccc|c} 1 & 3 & -4 & -3 \\ 0 & -11 & 13 & 13 \\ 0 & -9 & 13 & 13 \\ 0 & -11 & 13 & 13 \end{array}\right)$$

$$\xrightarrow[R_2 - R_3]{R_4 - R_2} \left(\begin{array}{ccc|c} 1 & 3 & -4 & -3 \\ 0 & -2 & 0 & 0 \\ 0 & -9 & 13 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_3 - (9/2)R_2} \left(\begin{array}{ccc|c} \boxed{1} & 3 & -4 & -3 \\ 0 & \boxed{-2} & 0 & 0 \\ 0 & 0 & \boxed{13} & 13 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Again, we have no pivot in the augmented column and no free variable. So the system has a unique solution. □

1.8.5 Reducing the augmented matrix in the echelon form, we have

$$\left(\begin{array}{ccc|c} 1 & 1 & b & 1 \\ b & 3 & -1 & -2 \\ 3 & 4 & 1 & c \end{array}\right) \xrightarrow[R_3 - 3R_1]{R_2 - bR_1} \left(\begin{array}{ccc|c} 1 & 1 & b & 1 \\ 0 & 3-b & -1-b^2 & -2-b \\ 0 & 1 & 1-3b & c-3 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & b & 1 \\ 0 & 1 & 1-3b & c-3 \\ 0 & 3-b & -1-b^2 & -2-b \end{array}\right)$$

$$\xrightarrow{R_3 - (3-b)R_2} \left(\begin{array}{ccc|c} \boxed{1} & 1 & b & 1 \\ 0 & \boxed{1} & 1-3b & c-3 \\ 0 & 0 & -4+10b-4b^2 & 7-4b-3c+bc \end{array}\right).$$

When $-4+10b-4b^2 \neq 0$, i.e., $b \neq 2$ or $1/2$, the system is consistent and has no free variable, thus has a unique solution. When $-4+10b-4b^2 = 0$ and $7-4b-3c+bc \neq 0$, i.e., $b = 2, c \neq -1$ or $b = 1/2, c \neq 2$, the augmented column has a pivot, thus the system is inconsistent, admitting no solution. When $-4+10b-4b^2 = 0$ and $7-4b-3c+bc = 0$, i.e., $b = 2, c = -1$ or $b = 1/2, c = 2$, the bottom row is of all zeros, making the system consistent again. But now the third column is free, so the system has infinitely many solutions. In summary, the system has

no solution, if $b = 2, c \neq -1$ or $b = 1/2, c \neq 2$;
exactly one solution, if $b \neq 2$ or $1/2$;
infinitely many solutions, if $b = 2, c = -1$ or $b = 1/2, c = 2$. □

1.8.6 (c) Use Gauss-Jordan.

$$\left(\begin{array}{ccc|c} 1 & \mathrm{i} & 1 & 1+4\mathrm{i} \\ -1 & 1 & -\mathrm{i} & -1 \\ \mathrm{i} & -1 & -1 & -1-2\mathrm{i} \end{array}\right) \xrightarrow[R_3-\mathrm{i}R_1]{R_2+R+1} \left(\begin{array}{ccc|c} 1 & \mathrm{i} & 1 & 1+4\mathrm{i} \\ 0 & 1+\mathrm{i} & 1-\mathrm{i} & 4\mathrm{i} \\ 0 & 0 & -1-\mathrm{i} & 3-3\mathrm{i} \end{array}\right)$$

$$\xrightarrow[R_3/(-1-\mathrm{i})]{R_2/(1+\mathrm{i})} \left(\begin{array}{ccc|c} 1 & \mathrm{i} & 1 & 1+4\mathrm{i} \\ 0 & 1 & -\mathrm{i} & 2+2\mathrm{i} \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow[R_2+\mathrm{i}R_3]{R_1-R_3} \left(\begin{array}{ccc|c} 1 & \mathrm{i} & 0 & -2+4\mathrm{i} \\ 0 & 1 & 0 & 2+5\mathrm{i} \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{R_1-\mathrm{i}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3+2\mathrm{i} \\ 0 & 1 & 0 & 2+5\mathrm{i} \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Thus, the solution is $(x_1, x_2, x_3)^T = (3+2\mathrm{i}, 2+5\mathrm{i}, 3)^T$. □

1.8.7 (c) Reduce the matrix in the echelon form.

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} \boxed{1} & -1 & 1 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix}$$

There are two pivots (boxed entries), so the rank is 2.

(e) Even this is a column vector, you can treat it as a common matrix.

$$\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{R_3+(2/3)R_1} \begin{pmatrix} \boxed{3} \\ 0 \\ 0 \end{pmatrix}$$

There is one pivot, so the rank is 1.

(g) Reduce to the echelon form.

$$\begin{pmatrix} 0 & -3 \\ 4 & -1 \\ 1 & 2 \\ -1 & -5 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 \\ 4 & -1 \\ 0 & -3 \\ -1 & -5 \end{pmatrix} \xrightarrow[R_4+R_1]{R_2-4R_1} \begin{pmatrix} 1 & 2 \\ 0 & -9 \\ 0 & -3 \\ 0 & -3 \end{pmatrix} \xrightarrow[R_4-(1/3)R_2]{R_3-(1/3)R_2} \begin{pmatrix} \boxed{1} & 2 \\ 0 & \boxed{\text{-9}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

There are two pivots, the rank is 2. □

1.8.22 (d) Reduce the coefficient matrix in the echelon form. (Here we don't deal with augmented matrix because the right hand sides are always zeros for homogeneous systems.)

$$\begin{pmatrix} 1 & 2 & -2 & 1 \\ -3 & 0 & 1 & -2 \end{pmatrix} \xrightarrow{R_2+3R_1} \begin{pmatrix} \boxed{1} & 2 & -2 & 1 \\ 0 & \boxed{6} & -5 & 1 \end{pmatrix}$$

x and y are basic variables, while z and w are free. By the second equation we get $y = \frac{5}{6}z - \frac{1}{6}w$. Substituting in the first equation we have $x = \frac{1}{3}z - \frac{2}{3}w$. The general solution is

$$\begin{cases} x = \frac{1}{3}z - \frac{2}{3}w \\ y = \frac{5}{6}z - \frac{1}{6}w \\ z, w \text{ free.} \end{cases}$$

(f) Reduce the coefficient matrix in the *reduced* echelon form.

$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 2 & 0 & 0 & -3 \\ 1 & 1 & 0 & -2 \\ 0 & 1 & -3 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 0 & -3 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} \xrightarrow[R_4 + 2R_2]{R_3 + R_2} \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -6 & 3 \end{pmatrix} \xrightarrow[R_3/(-2)]{R_4 - 3R_3} \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 + 3R_3} \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, w is free, and we can directly write out the general solution as

$$\begin{cases} x = \frac{3}{2}w \\ y = \frac{1}{2}w \\ z = \frac{1}{2}w \\ w \text{ free.} \end{cases}$$

□

Friday, June 2

2.1.1 Verify all 9 axioms one by one, all quite straightforward. You can also map the complex number $x + \mathrm{i}y$ to a 2×1 vector $(x, y)^T$. Then you'll find the operations of addition and scalar multiplication for complex numbers are exactly the same as those for vectors (matrices). Since we've showed $\mathbb{R}^2$ is a vector space, so is the set of complex numbers (equipped with the given operations).

2.1.3 Although it's kind of tedious, I'm here to justify the 9 axioms for the function space $\mathcal{F}(S)$. In the following, f, g, h are real-valued functions defined on set S, c, d are real scalars, x is an arbitrary element in S.

1. By definition of function addition, $(f + g)(x) = f(x) + g(x) \in \mathbb{R}$, $\forall\, x \in S$. So $f + g$ is also a real-valued function defined on S, i.e., $f + g \in \mathcal{F}(S)$.
2. Since $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$, $\forall\, x \in S$, we have $f + g = g + f$.
3. Since

$$\begin{aligned} ((f + g) + h)(x) &= (f + g)(x) + h(x) = (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)) = f(x) + (g + h)(x) = (f + (g + h))(x), \ \forall\, x \in S, \end{aligned}$$

   we have $(f + g) + h = f + (g + h)$.
4. Let $\mathbf{0}$ be the constant function taking value 0. Then $(f + \mathbf{0})(x) = f(x) + \mathbf{0}(x) = f(x) + 0 = f(x)$, $\forall\, x \in S$. So $f + \mathbf{0} = f$.

5. Given $f \in \mathcal{F}(S)$, define function $(-f)(x) = -f(x) \in \mathbb{R}$, $\forall\ x \in S$. Thus $-f$ is a real-valued function defined on S, i.e., $-f \in \mathcal{F}(S)$. Now that

$$(f + (-f))(x) = f(x) + (-f)(x) = f(x) + (-f(x)) = 0 = \mathbf{0}(x),\ \forall\ x \in S,$$

we get $f + (-f) = \mathbf{0}$.

6. By definition of function scalar multiplication, $(cf)(x) = c \cdot f(x) \in \mathbb{R}$, $\forall\ x \in S$. So cf is a real-valued function defined on S, i.e., $cf \in \mathcal{F}(S)$.
7. Since $\forall\ x \in S$,

$$(c(f+g))(x) = c\cdot(f+g)(x) = c\cdot(f(x)+g(x)) = c\cdot f(x)+c\cdot g(x) = (cf)(x)+(cg)(x) = (cf+cg)(x),$$

$$((c+d)f)(x) = (c+d)\cdot f(x) = c\cdot f(x) + d\cdot f(x) = (cf)(x) + (df)(x) = (cf+df)(x),$$

we have $c(f+g) = cf + cg$, $(c+d)f = cf + df$.

8. Since $\forall\ x \in S$,

$$(c(df))(x) = c\cdot(df)(x) = c\cdot(d\cdot f(x)) = (cd)\cdot f(x) = ((cd)f)(x),$$

we have $c(df) = (cd)f$.

9. Since $(1f)(x) = 1\cdot f(x) = f(x)$, $\forall\ x \in S$, we have $1f = f$.

In conclusion, $\mathcal{F}(S)$, the set of all real-valued functions defined on the set S, is a vector space. □

2.1.10 For two infinite real sequences $\mathbf{a} = (a_1, a_2, \cdots)$ and $\mathbf{b} = (b_1, b_2, \cdots)$, define

addition: $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \cdots)$

scalar multiplication: $c\,\mathbf{a} = (ca_1, ca_2, \cdots)$, $\forall\ c \in \mathbb{R}$

You need to justify all the axioms, although it's straightforward. □

2.2.1 (a) Denote the set by $W = \{(x,y,z)^T | x - y + 4z = 0\}$. We are to prove W is closed under addition and scalar multiplication. For any two vectors $\mathbf{u} = (x_1, y_1, z_1)^T, \mathbf{v} = (x_2, y_2, z_2)^T$ in W, we have

$$x_1 - y_1 + 4z_1 = 0, \qquad x_2 - y_2 + 4z_2 = 0.$$

Adding these two equations up, we get

$$(x_1 + x_2) - (y_1 + y_2) + 4(z_1 + z_2) = 0,$$

which implies the vector $\mathbf{u} + \mathbf{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)^T$ belongs to W. Multiplying the first equation by a scalar c, we get

$$(cx_1) - (cy_1) + 4(cz_1) = 0,$$

which shows the vector $c\mathbf{u} = (cx_1, cy_1, cz_1)^T$ belongs to W. Hence we complete the proof.

(b) The origin $(0,0,0)^T$ (the zero vector) does not satisfy $x - y + 4z = 1$, i.e., the set does not contain the zero vector, thus not form a subspace. □

2.2.5 False. For example, any interval not containing 0 can't be a vector space. □