Chapter 2

Manifolds and Bundles

21. Charts, Atlases and Manifolds

Let a set $\mathcal{M}$ and $r \in \tilde{\ }$ be given. A **chart** χ for $\mathcal{M}$ is defined to be a bijection whose domain is included in $\mathcal{M}$ and whose codomain is an open subset of a specified flat space, denote by $\text{Pag}\,\chi$ and called the **page** of χ. The translation space of $\text{Pag}\,\chi$ is denoted by

$$\mathcal{V}_\chi := \text{Pag}\,\chi - \text{Pag}\,\chi. \tag{21.1}$$

Let f be a mapping whose domain is a subset of $\mathcal{M}$ and whose codomain is an open subset $\mathcal{D}$ of a specified flat space. We say that f is **C^r-related** to a given chart χ for $\mathcal{M}$ if

(R1) $\chi_>(\text{Dom}\,\chi \cap \text{Dom}\,f)$ is an open subset of $\text{Pag}\,\chi$,

(R2) $f \circ \chi^{\leftarrow} : \chi_>(\text{Dom}\,\chi \cap \text{Dom}\,f) \to \mathcal{D}$ is of class C^r.

We say that two charts χ and γ for $\mathcal{M}$ are **C^r-compatible** if γ is C^r-related to χ and χ is C^r-related to γ.

Pitfall: In general, C^r-compatibility is not an equivalence relation. ∎

A class $\mathfrak{A}$ of charts for $\mathcal{M}$ is called a **C^r-atlas** of $\mathcal{M}$ if

(A1) Any two charts in $\mathfrak{A}$ are C^r-compatible,

(A2) The domain of the charts in $\mathfrak{A}$ cover $\mathcal{M}$, i.e.

$$\mathcal{M} = \bigcup \{\text{Dom}\,\chi \mid \chi \in \mathfrak{A}\}. \tag{21.2}$$

It is clear that a C^r-atlas is also a C^s-atlas for every $s \in 0..r$.

Proposition 1: *Let $\mathfrak{A}$ be a C^r-atlas for $\mathcal{M}$ and let χ be a chart that is C^r-compatible with all charts in $\mathfrak{A}$. If f is a mapping that is C^r-related to every chart in $\mathfrak{A}$ then it is also C^r-related to χ .*

Proof: Let $x \in \text{Dom}\,\chi \cap \text{Dom}\,f$ be given. By (A2) we may may choose $\alpha \in \mathcal{A}$ such that $x \in \text{Dom}\,\alpha$. We put

$$\mathcal{G} := \text{Dom}\,\chi \cap \text{Dom}\,\alpha \cap \text{Dom}\,f. \tag{21.3}$$

Since α is injective we have

$$\alpha_>(\mathcal{G}) = \alpha_>(\text{Dom}\,\chi \cap \text{Dom}\,\alpha) \cap \alpha_>(\text{Dom}\,f \cap \text{Dom}\,\alpha).$$

Since χ and f are both C^r-related to α, it follows from (R1) that both $\alpha_>(\mathrm{Dom}\,\chi \cap \mathrm{Dom}\,\alpha)$ and $\alpha_>(\mathrm{Dom}\,f \cap \mathrm{Dom}\,\alpha)$ are open subsets of $\mathrm{Pag}\alpha$ and hence that $\alpha_>(\mathcal{G})$ is also open in $\mathrm{Pag}\alpha$. Since $\alpha \,\square\, \chi^{\leftarrow}$ is continuous by (R2), it follows that $\chi_>(\mathcal{G}) = (\alpha \,\square\, \chi^{\leftarrow})^<(\alpha_>(\mathcal{G}))$ is an open neighborhood of $\chi(x)$ in $\mathrm{Pag}\,\chi$. Using (0.1) and (0.2) it is easily seen that

$$(f \,\square\, \chi^{\leftarrow})\Big|_{\chi_>(\mathcal{G})} = (f \,\square\, \alpha^{\leftarrow})\Big|_{\alpha_>(\mathcal{G})} \circ (\alpha \,\square\, \chi^{\leftarrow})\Big|_{\chi_>(\mathcal{G})}^{\alpha_>(\mathcal{G})}.$$

Since both $f \,\square\, \alpha^{\leftarrow}$ and $\alpha \,\square\, \chi^{\leftarrow}$ are of class C^r by (R2), it follows from the chain rule that the restriction of $f \,\square\, \alpha^{\leftarrow}$ to a neighborhood $\chi_>(\mathcal{G})$ of $\chi(x)$ in $\mathrm{Pag}\,\chi$ is of class C^r. Since $x \in \mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f$ was arbitrary, it follows that the domain $\chi_>(\mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f)$ of $f \,\square\, \chi^{\leftarrow}$ is open in $\mathrm{Pag}\,\chi$ and that $f \,\square\, \chi^{\leftarrow}$ is of class C^r, *i.e.* that f is C^r-related to χ. ∎

We say that a C^r-atlas $\mathfrak{A}$ for $\mathcal{M}$ is **C^r-saturated** if every chart for $\mathcal{M}$ that is C^r-compatible with all charts in $\mathfrak{A}$ already belongs to $\mathfrak{A}$. The following is an immediate consequence of Prop. 1.

Proposition 2: *Let $\mathfrak{A}$ be a C^r-atlas for $\mathcal{M}$. Then there is exactly one saturated C^r-atlas $\overline{\mathfrak{A}}$ that includes $\mathfrak{A}$. In fact, $\overline{\mathfrak{A}}$ consists of all charts that are C^r-compatible with all charts in $\mathfrak{A}$.*

Definition: *Let $r \in {}^{\sim}$ be given. A C^r-**manifold** is a set $\mathcal{M}$ endowed with structure by the prescription of a saturated C^r-atlas for $\mathcal{M}$, which is called the* **chart-class** *of $\mathcal{M}$ and is denoted by $\mathrm{Ch}^r\mathcal{M}$, or if no confusion is likely, simply by $\mathrm{Ch}\mathcal{M}$.*

In view of Prop. 2, the structure of a C^r-manifold on $\mathcal{M}$ is uniquely determined by specifying a C^r-atlas included in $\mathrm{Ch}\mathcal{M}$. Of course, two different such atlases may determine one and the same C^r-structure.

Let $\mathcal{M}$ be a C^r-manifold with chart-class $\mathrm{Ch}^r\mathcal{M}$. Then, for every $s \in 0..r$, $\mathcal{M}$ has also the natural structure of a C^s-manifold, determined by $\mathrm{Ch}^r\mathcal{M}$ regarded as a C^s-atlas. Of course, the chart-class $\mathrm{Ch}^s\mathcal{M}$ of the C^s-manifold structure includes $\mathrm{Ch}^r\mathcal{M}$, but we have $\mathrm{Ch}^r\mathcal{M}\ \ \mathrm{Ch}^s\mathcal{M}$ if $s < r$.

Examples of manifold

Example 1: Let $\mathcal{D}$ be an open subset of a flat space. Then the singleton $\{\mathbf{1}_{\mathcal{D}}\}$ is a C^ω-atlas of $\mathcal{D}$. It determines on $\mathcal{D}$ a natural C^ω-structure and hence a natural C^r-structure for every $r \in$.

Example 2: (**Product manifold**) Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds of class C^r, then the product $\mathcal{M} \times \mathcal{N}$ has the natural structure of a C^r manifold. ∎

We now assume that a C^r-manifold $\mathcal{M}$ with chart-class $\mathrm{Ch}\mathcal{M}$ is given. We use the notation

$$\mathrm{Ch}_x\mathcal{M} := \left\{ \chi \in \mathrm{Ch}\mathcal{M} \;\middle|\; x \in \mathrm{Dom}\,\chi \right\} . \tag{21.4}$$

It is easily seen that the spaces $\mathrm{Pag}\,\chi$ and $\mathcal{V}_\chi$, $\chi \in \mathrm{Ch}_x\mathcal{M}$, all have the same dimension. This dimension is called the **dimension of $\mathcal{M}$ at x**, and is denoted by $\dim_x\mathcal{M}$.

The C^r-manifold $\mathcal{M}$ is endowed with a natural topology, namely the coarsest topology that renders all $\chi \in \mathrm{Ch}\mathcal{M}$ continuous. A subset $\mathcal{P}$ of $\mathcal{M}$ is open if and only if, for each $\chi \in \mathrm{Ch}\mathcal{M}$, the image $\chi_>(\mathcal{P} \cap \mathrm{Dom}\,\chi)$ is an open subset of $\mathrm{Pag}\,\chi$. Given $x \in \mathcal{M}$, one can construct a neighborhood-basis $\mathfrak{B}_x$ of x in $\mathcal{M}$ in the following manner: Choose a chart $\chi \in \mathrm{Ch}_x\mathcal{M}$ and a neighborhood-basis $\mathfrak{N}_{\chi(x)}$ of $\chi(x)$ in $\mathrm{Pag}\,\chi$. Then put

$$\mathfrak{B}_x := \left\{ \chi^<(\mathcal{N} \cap \mathrm{Cod}\,\chi) \;\middle|\; \mathcal{N} \in \mathfrak{N}_{\chi(x)} \right\} . \tag{21.5}$$

Pitfall: The natural topology of $\mathcal{M}$ need not be separating.

Let $\mathcal{P}$ be an open subset of $\mathcal{M}$. Then $\mathcal{P}$ has the natural structure of a C^r-manifold whose chart-class $\mathrm{Ch}\,\mathcal{P}$ is

$$\mathrm{Ch}\,\mathcal{P} := \left\{ \chi \in \mathrm{Ch}\mathcal{M} \;\middle|\; \mathrm{Dom}\,\chi \subset \mathcal{P} \right\}. \tag{21.6}$$

The natural topology of $\mathcal{P}$ as a C^r-manifold concides with the topology of $\mathcal{P}$ induced by the topology of $\mathcal{M}$.

Let f be a mapping whose domain is an open subset of $\mathcal{M}$ and whose codomain is an open subset $\mathcal{D}$ of a specified flat space $\mathcal{E}$ with translation space $\mathcal{V} := \mathcal{E} - \mathcal{E}$. We say that f is **of class** C^s, with $s \in 0..r$, if it is C^s-related to every chart $\chi \in \mathrm{Ch}\mathcal{M}$, i.e. if $f \circ \chi^\leftarrow$ is of class C^s for all charts $\chi \in \mathrm{Ch}\mathcal{M}$. (Since $\mathrm{Dom}\,f$ is open, $\mathrm{Dom}\,f \circ \chi^\leftarrow = \chi_>(\mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f)$ is automatically open in $\mathrm{Pag}\,\chi$ when $\chi \in \mathrm{Ch}\mathcal{M}$.) It follows from Prop. 1 that f is of class C^s if $f \circ \chi^\leftarrow$ is of class C^s for every chart χ in some C^r-atlas included in $\mathrm{Ch}\mathcal{M}$. If f is of class C^s with $s \geq 1$ and if $\chi \in \mathrm{Ch}\mathcal{M}$, we define the **gradient**

$$\nabla_\chi f : \mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f \to \mathrm{Lin}(\mathcal{V}_\chi, \mathcal{V})$$

of f in the chart χ by

$$(\nabla_\chi f)(x) := \nabla_{\chi(x)}(f \circ \chi^\leftarrow) \quad \text{for all} \quad x \in \mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f. \tag{21.7}$$

More generally, for every $s \in 1..r$, the **gradient of order** s

$$\nabla_\chi^{(s)} f : \mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f \to \mathrm{Sym}_s((\mathcal{V}_\chi)^s, \mathcal{V})$$

of f in the chart χ defined by

$$(\nabla_\chi^{(s)} f)(x) := \nabla_{\chi(x)}^{(s)}(f \circ \chi^{\leftarrow}) \quad \text{for all} \quad x \in \operatorname{Dom}\chi \cap \operatorname{Dom} f. \tag{21.8}$$

The following transformation rules are easy concequences of the rules of calculus.

Proposition 3: *Let f be a mapping of class C^1, $x \in \operatorname{Dom} f$ and $\chi, \gamma \in \operatorname{Ch}_x\mathcal{M}$. Then*

$$(\nabla_\gamma f)(x) = (\nabla_\chi f)(x)(\nabla_\gamma \chi)(x). \tag{21.9}$$

If f is also of class C^2, then

$$(\nabla_\gamma^{(2)} f)(x) = (\nabla_\chi^{(2)} f)(x) \circ (\nabla_\gamma \chi(x) \times \nabla_\gamma \chi(x)) + (\nabla_\chi f)(x)\nabla_\gamma^{(2)}\chi(x). \tag{21.10}$$

In the case when $f := \gamma$ the formulas (21.7) and (21.8) reduce to

$$(\nabla_\gamma \gamma)(x) = \mathbf{1}_{\mathcal{V}_\gamma} \qquad \text{and} \qquad (\nabla_\gamma^{(2)}\gamma)(x) = \mathbf{0}.$$

Hence Prop. 3 has the following consequence:

Proposition 4: *Let $x \in \mathcal{M}$ and $\chi, \gamma \in \operatorname{Ch}_x\mathcal{M}$ be given. If $r \geq 1$, then $(\nabla_\chi \gamma)(x) \in \operatorname{Lin}(\mathcal{V}_\chi, \mathcal{V}_\gamma)$ is invertible and*

$$(\nabla_\chi \gamma)(x)^{-1} = (\nabla_\gamma \chi)(x). \tag{21.11}$$

If $r \geq 2$, we also have

$$(\nabla_\gamma^{(2)}\chi)(x) = -(\nabla_\gamma\chi)(x)\big((\nabla_\chi^{(2)}\gamma)(x) \circ (\nabla_\gamma\chi(x) \times \nabla_\gamma\chi(x))\big). \tag{21.12}$$

If the manifold $\mathcal{M}$ is itself the underlying manifold of an open subset of a flat space (see Example 1 above), then a mapping f is of class C^s as described above if and only if it is of class C^s in the ordinary sence (see Notations).

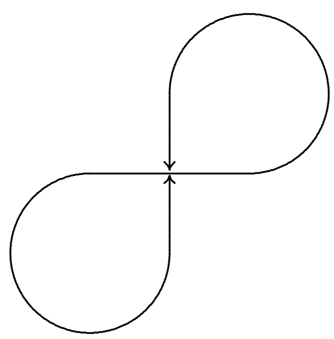
Let f be a mapping whose domain is a neighborhood of a given point $x \in \mathcal{M}$ and whose codomain is an open subset of a specified flat space. We say that f is **differentiable at** x if $f \circ \chi^{\leftarrow}$ is differentiable at $\chi(x)$ for some, and hence all, $\chi \in \operatorname{Ch}_x\mathcal{M}$. If this is the case, (21.7) remains meaningful for the given $x \in \mathcal{M}$ and the transformation formula (21.9) remains valid. The concept of "s times differentiable at x" when $s \in 0..r$ is defined in a similar way.

More generally, let C^r-manifolds $\mathcal{M}$ and $\mathcal{M}'$ be given. Let g be a mapping whose domain and codomain are open subsets of $\mathcal{M}$ and $\mathcal{M}'$, respectively. We say that g is **of class** C^s with $s \in 0..r$ if $\chi' \circ g \circ \chi^{\leftarrow}$ is of class C^s in the ordinary sense for all $\chi \in \operatorname{Ch}\mathcal{M}$ and all $\chi' \in \operatorname{Ch}\mathcal{M}'$.

Definition: *Let $\mathcal{M}$ be a C^r-manifold and let $\mathcal{P}$ be a subset of $\mathcal{M}$. We say that $\mathcal{P}$ is a* **submanifold** *of $\mathcal{M}$ if for each point $x \in \mathcal{P}$ there is a chart $\chi \in \mathrm{Ch}_x\mathcal{M}$ such that $\chi_>(\mathcal{P} \cap \mathrm{Dom}\,\chi)$ is an open subset of a flat $\mathcal{F}_\chi$ of $\mathrm{Pag}\,\chi$.*

Let $\mathcal{P}$ be a C^r submanifold of the manifold $\mathcal{M}$. We left it the readers to show that $\mathcal{P}$ has the natural structure of a C^r manifold. The natural topology of $\mathcal{P}$ as a C^r-manifold concides with the topology of $\mathcal{P}$ induced by the topology of $\mathcal{M}$, i.e. $\mathcal{P}$ a topological subspace of $\mathcal{M}$.

Let $f : \mathcal{S} \to \mathcal{M}$ be a C^s mapping from a manifold $\mathcal{S}$ to another manifold $\mathcal{M}$. The mapping f is called a C^s **immersion at** $x \in \mathcal{S}$ if there exists an open neighborhood $\mathcal{N}_x$ of x (in $\mathcal{S}$) such that the restriction $f|_{\mathcal{N}_x}$ is injective and $f_>(\mathcal{N}_x)$ is a submanifold of $\mathcal{M}$. We say that f is an **immersion** if it is an immersion at every $y \in \mathcal{S}$. If f is an immersion, the domain $\mathcal{S}$ called an **immersed manifold** of $\mathcal{M}$. However, being an immersion is a “local property” and hence the range $\mathrm{Rng}\, f := f_>(\mathcal{S})$ of f may not be a submanifold of $\mathcal{M}$. For example (see [L]):



Figue11.1

An injective immersion f from manifold $\mathcal{A}$ to manifold $\mathcal{B}$ is an **imbedding** if the range $\mathrm{Rng}\, f := f_>(\mathcal{A})$ is a submanifold of $\mathcal{B}$. The domain of an imbedding is called an **imbedded manifold** of its codomain manifold. It is clear that for every submanifold $\mathcal{P}$ of a given manifold $\mathcal{M}$ the inclusion $\mathbf{1}_{\mathcal{P}\subset\mathcal{M}}$ is an imbedding.

Remark: Let $\mathcal{A}$ and $\mathcal{B}$ be topological spaces and $f : \mathcal{A} \to \mathcal{B}$ be an injection. We say that f is an imbedding if the topology of $\mathcal{A}$ is induced by f from the topology of $\mathcal{B}$. ▮

More details on submanifolds

22. Bundles

We assume that $r \in {}^{\sim}$ with $r \geq 2$ and a C^r-manifold $\mathcal{M}$ are given. Let a number $s \in 0..r$ be given and let $\tau : \mathcal{B} \to \mathcal{M}$ be a surjective mapping from a given set $\mathcal{B}$ to the manifold $\mathcal{M}$.

Let a concrete isocategory ISO with object class *OBJ* be given with the following properties:

(i) Each set in *OBJ* has the natural structure of a C^s-manifold.

(ii) Every isomorphism in ISO is a C^s-diffeomorphism.

The most inportant special cases are (1) the isocategory of LIS consisting of all linear isomorphisms, whose object class *LS* consist of all (finite dimensional) linear spaces and (2) the isocategory of FIS consisting of all flat isomorphisms, whose object class *FS* consist of all flat spaces. The object sets in *LS* and *FS* have the natural structure of C^ω-manifolds and the isomorphisms in LIS and FIS are C^ω-diffeomorphisms.

<u>Definition</u>: *An* ISO-**bundle chart** *for $\mathcal{B}$ (for τ) is a bijection*

$$\phi : \tau^<(\mathcal{O}_\phi) \to \mathcal{O}_\phi \times \mathcal{V}_\phi,$$

where $\mathcal{O}_\phi$ is an open subset of $\mathcal{M}$ and $\mathcal{V}_\phi$ is a set in OBJ such that the diagram

$$\begin{array}{ccc} \tau^<(\mathcal{O}_\phi) & \xrightarrow{\phi} & \mathcal{O}_\phi \times \mathcal{V}_\phi \\ & {\scriptstyle \tau|^{\mathcal{O}_\phi}_{\tau^<(\mathcal{O}_\phi)}} \searrow & \downarrow {\scriptstyle \mathrm{ev}_1} \\ & & \mathcal{O}_\phi \end{array} \quad . \tag{22.1}$$

is commutative, i.e. $\mathrm{ev}_1 \circ \phi = \tau|^{\mathcal{O}_\phi}_{\tau^<(\mathcal{O}_\phi)}$.

<u>Notation</u>: *For every $y \in \mathcal{M}$, we denote $\mathcal{B}_y := \tau^<(\{y\})$ and for every* ISO-*bundle chart ϕ we use the following notations*

$$\phi\rfloor_y := \mathrm{ev}_2 \circ \phi \circ \left(\mathbf{1}_{\mathcal{B}_y \subset \tau^<(\mathcal{O}_\phi)}\right) : \mathcal{B}_y \to \mathcal{V}_\phi \tag{22.2}$$

for all $y \in \mathcal{O}_\phi$, i.e. we have the following commutative diagram

$$\begin{array}{ccccc} & & & & \mathcal{V}_\phi \\ & & {\scriptstyle \phi\rfloor_y} \nearrow & & \uparrow {\scriptstyle \mathrm{ev}_2} \\ \mathcal{B}_y & \hookrightarrow & \tau^<(\mathcal{O}_\phi) & \xrightarrow{\phi} & \mathcal{O}_\phi \times \mathcal{V}_\phi \end{array} \quad .$$

Put (22.1) and (22.2) together, we have the following commutative diagram

$$\begin{array}{ccccc} & & & & \mathcal{V}_\phi \\ & \overset{\phi\rfloor_y}{\nearrow} & & & \uparrow \mathrm{ev}_2 \\ \mathcal{B}_y & \hookrightarrow & \tau^{<}(\mathcal{O}_\phi) & \xrightarrow{\phi} & \mathcal{O}_\phi \times \mathcal{V}_\phi \\ & & & \underset{\tau\big|^{\mathcal{O}_\phi}_{\tau^{<}(\mathcal{O}_\phi)}}{\searrow} & \downarrow \mathrm{ev}_1 \\ & & & & \mathcal{O}_\phi \end{array} .$$

Let ϕ and ψ be ISO-bundle charts for $\mathcal{B}$. We say that ϕ and ψ are C^s-**compatible** if

$$\psi \sqcdot \phi^{\leftarrow} : (\mathcal{O}_\phi \cap \mathcal{O}_\psi) \times \mathcal{V}_\phi \to (\mathcal{O}_\phi \cap \mathcal{O}_\psi) \times \mathcal{V}_\psi \tag{22.3}$$

is a C^s-diffeomophism such that, for every $y \in \mathcal{O}_\phi \cap \mathcal{O}_\psi$, the mapping

$$\psi\rfloor_y \circ \phi\rfloor_y^{\leftarrow} : \mathcal{V}_\phi \to \mathcal{V}_\psi \tag{22.4}$$

belongs to ISO.

A class $\mathfrak{A}$ of ISO-bundle charts for $\mathcal{B}$ is called a C^s ISO-**bundle atlas** for $\mathcal{B}$ if

(BA1) every two ISO-bundle charts in $\mathfrak{A}$ are C^s-compatiable,

(BA2) for every $x \in \mathcal{M}$ there is a bundle chart $\phi \in \mathfrak{A}$ with $x \in \mathcal{O}_\phi$; i.e. we have
$$\mathcal{M} = \bigcup_{\phi \in \mathfrak{A}} \mathcal{O}_\phi .$$

Proposition 1: *Let $\mathfrak{A}$ be a* ISO*-bundle atlas for $\mathcal{B}$ and let ϕ be a* ISO*-bundle chart that is C^s-compatible with all* ISO*-bundle charts in $\mathfrak{A}$. If ψ is a* ISO*-bundle chart that is C^s-compatible with every* ISO*-bundle chart in $\mathfrak{A}$ then it is also C^s-compatible with ϕ.*

Proof: Let $x \in \mathcal{O}_\phi \cap \mathcal{O}_\psi$ be given. By (BA2), we may choose a ISO-bundle chart $\theta \in \mathfrak{A}$ such that $x \in \mathcal{O}_\theta$. Put $\mathcal{O} := \mathcal{O}_\phi \cap \mathcal{O}_\psi \cap \mathcal{O}_\theta$. Since both ϕ and ψ are C^s-compatible with θ, we see that the restriction

$$\psi \sqcdot \phi^{\leftarrow}\Big|_{\phi(\tau^{<}\{\mathcal{O}\})} = (\psi \sqcdot \theta^{\leftarrow})\Big|_{\theta(\tau^{<}\{\mathcal{O}\})} \circ (\theta \sqcdot \phi^{\leftarrow})\Big|^{\theta(\tau^{<}\{\mathcal{O}\})}_{\phi(\tau^{<}\{\mathcal{O}\})}$$

on $\phi(\tau^{<}\{\mathcal{O}\})$ is a C^s-diffeomorphism and the induced mapping

$$\psi\rfloor_x \circ \phi\rfloor_x^{\leftarrow} = (\psi\rfloor_x \circ \theta\rfloor_x^{\leftarrow}) \circ (\theta\rfloor_x \circ \phi\rfloor_x^{\leftarrow})$$

is a ISO-isomorphism. Since $x \in \mathcal{O}_\phi \cap \mathcal{O}_\psi$ was arbitrary, we conclude that ψ and ϕ are C^s-compatible. ∎

We say that a ISO-bundle atlas $\mathfrak{A}$ of $\mathcal{B}$ is C^s**-saturated** if every ISO-bundle chart for $\mathcal{B}$ that is C^s-compatible with all ISO-bundle charts in $\mathfrak{A}$ already belongs to $\mathfrak{A}$. The following is an immediate consequence of Prop. 1.

Proposition 2: *Let $\mathfrak{A}$ be a C^s* ISO-*bundle atlas for $\mathcal{B}$. Then there is exactly one C^s-saturated* ISO-*bundle atlas $\overline{\mathfrak{A}}$ that includes $\mathfrak{A}$. In fact, $\overline{\mathfrak{A}}$ consists of all* ISO-*bundle charts that are C^s-compatible with all* ISO-*bundle charts in $\mathcal{B}$.*

Let $\mathfrak{A}$ be a saturated ISO-atlas for $\mathcal{B}$ and let ϕ be a ISO-bundle chart in $\mathfrak{A}$. On each fibre $\mathcal{B}_x$, $x \in \mathcal{O}_\phi$, we can transport the ISO-structure of $\mathcal{V}_\phi$ by means of $\phi\rfloor_x : \mathcal{B}_x \to \mathcal{V}_\phi$. The result is independent of the choice of ϕ, since every pair of bundle charts ϕ and ψ in $\mathfrak{A}$ are compatible and hence $\psi\rfloor_x \circ \phi\rfloor_x^{\leftarrow} : \mathcal{V}_\phi \to \mathcal{V}_\psi$ is a ISO-isomorphism.

<u>Definition</u>: *A C^s* ISO-**bundle** *over $\mathcal{M}$ is a set $\mathcal{B}$ and a mapping $\tau : \mathcal{B} \to \mathcal{M}$ endowed with structure by the prescription of a saturated C^s* ISO-*bundle atlas for $\mathcal{B}$, which is called the* **bundle structure** *for $\mathcal{B}$ and is denoted by* $\mathrm{Ch}^s(\mathcal{B}, \mathcal{M})$, *or if no confusion is likely, simply by* $\mathrm{Ch}(\mathcal{B}, \mathcal{M})$. *We denote the* ISO-*bundle by* $(\mathcal{B}, \tau, \mathcal{M})$ *or simply by $\mathcal{B}$.*

The mapping τ is called the **bundle-projection**. For every $x \in \mathcal{M}$, $\mathcal{B}_x := \tau^<(\{x\})$ is called the **fiber over** x and the inclusion mapping of $\mathcal{B}_x$ in $\mathcal{B}$ is called the **bundle inclusion at** x. Right inverses of τ are called **cross sections of** $\mathcal{B}$. We also use the following notation

$$\mathrm{Ch}_x(\mathcal{B}, \mathcal{M}) := \left\{ \phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M}) \;\middle|\; x \in \mathcal{O}_\phi \right\}. \tag{22.5}$$

As explained above, for every $x \in \mathcal{M}$, the fiber $\mathcal{B}_x$ is naturally endowed with the structure of a ISO-set in such a way that $\phi\rfloor_x : \mathcal{B}_x \to \mathcal{V}_\phi$ is in ISO (is an isomorphism) for all $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$. Thus the dimension of $\mathcal{B}_x$ can be obtained from all $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$.

Locally (relative to $\mathcal{M}$), the manifold structure of the **bundle manifold** $\mathcal{B}$ is completely determined by the manifold structure of the **base manifold** $\mathcal{M}$ and the manifold structures of $\mathcal{V}_\phi$ for a single $\phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$. Every bundle chart ϕ in $\mathrm{Ch}(\mathcal{B}, \mathcal{M})$ transports the manifold structure from $\mathcal{O}_\phi \times \mathcal{V}_\phi$ to $\tau^<(\mathcal{O}_\phi)$, and hence a manifold chart can be easily obtained from ϕ.

Let $\mathbf{b} \in \mathcal{B}$ be given and put $x := \tau(\mathbf{b})$. The dimension of $\mathcal{B}$ at $\mathbf{b}$ can be obtained from the codomain of each bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$. We have

$$\dim_{\mathbf{b}} \mathcal{B} = m + n,$$

where $\dim_x \mathcal{M} = m$ and $\dim_{\mathbf{b}} \mathcal{B}_x = n$.

Let ISO-bundles $(\mathcal{B}', \tau', \mathcal{M}')$ and $(\mathcal{B}, \tau, \mathcal{M})$ be given. We say that $(\mathcal{B}', \tau', \mathcal{M}')$ is a ISO-**subbundle** of $(\mathcal{B}, \tau, \mathcal{M})$ provided $\mathcal{B}'$ is a submanifold of $\mathcal{B}$, $\mathcal{M}'$ is a submanifold of $\mathcal{M}$ and $\tau' = \tau\big|_{\mathcal{B}'}^{\mathcal{M}'}$ such that, for each bundle chart $\varphi \in \mathrm{Ch}(\mathcal{B}', \mathcal{M}')$, we have $\varphi = \phi\big|_{\mathrm{Dom}\,\varphi}^{\mathrm{Cod}\,\varphi}$ for some bundle chart $\phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$.

It is easily seen that for every open subset $\mathcal{P}$ of $\mathcal{M}$, $\left(\tau^{<}(\mathcal{P}), \tau\big|_{\tau^{<}(\mathcal{P})}^{\mathcal{P}}, \mathcal{P}\right)$ is an open subbundle of $(\mathcal{B}, \tau, \mathcal{M})$.

<u>**Definition**</u>**:** *A* **cross section on** $\mathcal{O}$ *of* $\mathcal{B}$*, where* $\mathcal{O}$ *is an open submanifold of* $\mathcal{M}$*, is a mapping* $\mathbf{s} : \mathcal{O} \to \mathcal{B}$ *such that* $\tau \circ \mathbf{s} = \mathbf{1}_{\mathcal{O}\subset\mathcal{M}}$*. For every* $p \in 0..s$*, we denote the collection of all* C^p *cross sections of* $\mathcal{B}$ *by* $\mathrm{Sec}^p\mathcal{B}$*.*

If ISO is the category DIF_s that consists of all C^s-diffeomorphisms between C^s manifolds, we call $\mathcal{B}$ a C^s-**bundle**. If ISO = FIS, we call $\mathcal{B}$ a **flat-space bundle**. If ISO = LIS, we call $\mathcal{B}$ a **linear-space bundle**.

Proposition 3: *Let* $\mathcal{D}$ *be an open subset of a flat space* $\mathcal{E}$ *and let* $\mathcal{V}$*,* $\mathcal{W}$ *be linear spaces. Let* $F : \mathcal{D} \to \mathrm{Lin}(\mathcal{V}, \mathcal{W})$ *be given. If* $f : \mathcal{D} \times \mathcal{V} \to \mathcal{W}$ *is defined by*

$$f(x, \mathbf{v}) := F(x)\mathbf{v} \qquad \text{for all} \quad (x, \mathbf{v}) \in \mathcal{D} \times \mathcal{V} \tag{22.6}$$

then f *is of class* C^p*,* $p \in$ *, if and only if* F *is of class* C^p*.*

Proof: The assertion follows from the Partial Gradient Theorem [FDS]. ∎

If $\mathcal{B}$ is a linear-space bundle, then it follows from (22.3), (22.4) and Prop. 3 that for every pair of bundle charts $\phi, \psi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$, the mapping

$$\psi \diamond \phi : \mathcal{O}_\phi \cap \mathcal{O}_\psi \to \mathrm{Lin}(\mathcal{V}_\phi, \mathcal{V}_\psi)$$

defined by

$$(\psi \diamond \phi)(x) := \psi\big\rfloor_x \circ \phi\big\rfloor_x^{-1} \quad \text{for all} \quad x \in \mathcal{O}_\phi \cap \mathcal{O}_\psi \tag{22.7}$$

is of class C^s.

Before closing this section, we give two examples of constructing a new bundle from given ones. We omit the details.

Examples :

(1) **Trivial bundles :** $\mathcal{M} \times \mathcal{G}$, where $\mathcal{G} \in OBJ$. The fiber $\mathcal{B}_x = \{x\} \times \mathcal{G}$ at $x \in \mathcal{M}$ is $\mathcal{G}$ tagged with x. ∎

(2) **Fiber-product bundles :** Let two bundles $(\mathcal{A},\alpha,\mathcal{M})$ and $(\mathcal{B},\beta,\mathcal{M})$ over the same base manifold $\mathcal{M}$ be given. Put

$$\begin{aligned}\mathcal{A}\times_{\mathcal{M}}\mathcal{B} &:= \bigcup_{x\in\mathcal{M}} \mathcal{A}_x\times\mathcal{B}_x \\ \alpha\times_{\mathcal{M}}\beta &:= \alpha\circ \mathrm{ev}_1 = \beta\circ\mathrm{ev}_2\end{aligned} \quad ; \quad \begin{array}{ccc} \mathcal{A}\times_{\mathcal{M}}\mathcal{B} & \xrightarrow{\mathrm{ev}_2} & \mathcal{B} \\ {\scriptstyle \mathrm{ev}_1}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ \mathcal{A} & \xrightarrow[\alpha]{} & \mathcal{M}\end{array} \; . \tag{22.8}$$

The bundle $(\,\mathcal{A}\times_{\mathcal{M}}\mathcal{B}\,,\,\alpha\times_{\mathcal{M}}\beta\,,\,\mathcal{M}\,)$ is called the **fiber-product bundle** of $(\mathcal{A},\alpha,\mathcal{M})$ and $(\mathcal{B},\beta,\mathcal{M})$. The bundle projection $\alpha\times_{\mathcal{M}}\beta:\mathcal{A}\times_{\mathcal{M}}\mathcal{B}\to\mathcal{M}$ is given by

$$\alpha\times_{\mathcal{M}}\beta(\mathbf{v}) :\in \big\{ y \,\big|\, \mathbf{v}\in\mathcal{A}_y\times\mathcal{B}_y\big\}. \tag{22.9}$$

Let bundle charts $\phi\in\mathrm{Ch}(\mathcal{A},\mathcal{M})$ and $\psi\in\mathrm{Ch}(\mathcal{B},\mathcal{M})$ be given. The mapping

$$\phi\times_{\mathcal{M}}\psi : (\tau_1\times_{\mathcal{M}}\tau_2)^{<}(\mathcal{O}_\phi\cap\mathcal{O}_\psi)\to(\mathcal{O}_\phi\cap\mathcal{O}_\psi)\times(\mathcal{V}_\phi\times\mathcal{V}_\psi) \tag{22.10}$$

given by

$$\phi\times_{\mathcal{M}}\psi(\mathbf{v}) = \big(\, y\;,\;(\phi\rfloor_y\times\psi\rfloor_y)\mathbf{v}\,\big) \quad \text{for all} \quad \mathbf{v}\in\mathcal{A}\times_{\mathcal{M}}\mathcal{B} \tag{22.11}$$

is a bundle chart for $(\,\mathcal{A}\times_{\mathcal{M}}\mathcal{B}\,,\,\phi\times_{\mathcal{M}}\psi\,,\,\mathcal{M}\,)$. ▮

23. The tangent bundle

Let $r \in {}^{\infty\times}$, a C^r-manifold $\mathcal{M}$, and a point $x \in \mathcal{M}$ be given.

<u>Definition</u>: *The* **tangent space** *of $\mathcal{M}$ at x is defined to be*

$$\mathrm{T}_x\mathcal{M} := \left\{ \mathbf{t} \in \bigtimes_{\alpha \in \mathrm{Ch}_x\mathcal{M}} \mathcal{V}_\alpha \;\middle|\; (23.2) \textit{ holds} \right\}, \tag{23.1}$$

where the condition (23.2) *is given by*

$$\mathbf{t}_\gamma = \nabla_\chi \gamma(x)\, \mathbf{t}_\chi \quad \text{for all} \quad \chi, \gamma \in \mathrm{Ch}_x\mathcal{M}. \tag{23.2}$$

$\mathrm{T}_x\mathcal{M}$ *is endowed with the natural structure of a linear space as shown below and* $\dim \mathrm{T}_x\mathcal{M} = \dim_x\mathcal{M}$.

For every $\chi \in \mathrm{Ch}_x\mathcal{M}$, define the evaluation mapping $\mathrm{ev}_\chi : \mathrm{T}_x\mathcal{M} \to \mathcal{V}_\chi$ by

$$\mathrm{ev}_\chi(\mathbf{t}) := \mathbf{t}_\chi \qquad \text{for all} \qquad \mathbf{t} \in \mathrm{T}_x\mathcal{M}.$$

It follows from (21.10) that the evaluation mapping ev_χ is invertible and that its inverse $\mathrm{ev}_\chi^{\leftarrow} : \mathcal{V}_\chi \to \mathrm{T}_x\mathcal{M}$ is given by

$$(\mathrm{ev}_\chi^{\leftarrow})(\mathbf{u}) = \left(\nabla_\chi \alpha(x)\mathbf{u} \;\middle|\; \alpha \in \mathrm{Ch}_x\mathcal{M} \right) \quad \text{for all} \quad \mathbf{u} \in \mathcal{V}_\chi.$$

Hence we have

$$\mathrm{ev}_\chi \circ \mathrm{ev}_\gamma^{\leftarrow} = \nabla_\gamma \chi(x) \;\in\; \mathrm{Lis}\,(\mathcal{V}_\gamma, \mathcal{V}_\chi) \tag{23.3}$$

for all $\gamma, \chi \in \mathrm{Ch}_x\mathcal{M}$. It follows from that the linear-space structure on $\mathrm{T}_x\mathcal{M}$ obtained from that of $\mathcal{V}_\chi$ by ev_χ does not depend on the choice of $\chi \in \mathrm{Ch}_x\mathcal{M}$ and hence is intrinsic to $\mathrm{T}_x\mathcal{M}$. We consider $\mathrm{T}_x\mathcal{M}$ to be endowed with this structure.

Let f be a mapping whose domain $\mathcal{D}$ is a neighborhood of x in $\mathcal{M}$ and whose codomain is an open subset of a flat space with translation space $\mathcal{V}$. It follows from (23.3) and (21.7) that

$$\nabla_\chi f(x) \circ \mathrm{ev}_\chi \;\in\; \mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathcal{V})$$

is the same for all $\chi \in \mathrm{Ch}_x\mathcal{M}$. Hence we may define the **gradient of** f **at** x by

$$\nabla_x f := \nabla_\chi f(x) \circ \mathrm{ev}_\chi \;\in\; \mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathcal{V}) \tag{23.4}$$

for all $\chi \in \mathrm{Ch}_x\mathcal{M}$. In particular, if we put $f := \chi$ we get $\nabla_x \chi = \mathrm{ev}_\chi$ and hence

$$(\nabla_x \chi)\, \mathbf{t} = \mathbf{t}_\chi \qquad \text{for all} \qquad \chi \in \mathrm{Ch}_x\mathcal{M}. \tag{23.5}$$

Also, if f is given as above, we have

$$\nabla_x f = \nabla_\chi f(x) \nabla_x \chi \qquad \text{for all} \qquad \chi \in \mathrm{Ch}_x\mathcal{M}. \tag{23.6}$$

Let $\mathcal{P}$ be an open neighborhood of x in $\mathcal{M}$. By (21.6) we have $\mathrm{Ch}_x\mathcal{P} \subset \mathrm{Ch}_x\mathcal{M}$ and the mapping

$$(\mathbf{t} \mapsto \mathbf{t}|_{\mathrm{Ch}_x\mathcal{P}}) : \mathrm{T}_x\mathcal{M} \to \mathrm{T}_x\mathcal{P}$$

is a natural bijection; we use it to identify

$$\mathrm{T}_x\mathcal{P} \cong \mathrm{T}_x\mathcal{M}. \tag{23.7}$$

<u>**Definition:**</u> *The* **tangent bundle** $\mathrm{T}\mathcal{M}$ *of* $\mathcal{M}$ *is defined to be the union of all the tangent spaces of* $\mathcal{M}$:

$$\mathrm{T}\mathcal{M} := \bigcup_{x\in\mathcal{M}} \mathrm{T}_x\mathcal{M}. \tag{23.8}$$

It is endowed with the natural structure of a C^{r-1}-linear-space bundle as shown below.

In view of the identifications (23.7) we may regard $\mathrm{T}\mathcal{P}$ as a subset of $\mathrm{T}\mathcal{M}$ when $\mathcal{P}$ is an open subset of $\mathcal{M}$.

Let $\mathcal{D}$ be an open subset of a flat space $\mathcal{E}$ with translation space $\mathcal{V} := \mathcal{E} - \mathcal{E}$. Then the singleton $\{\mathbf{1}_{\mathcal{D}}\}$ is a C^ω-atlas of $\mathcal{D}$. It determines on $\mathcal{D}$ a natural C^ω-manifold structure and hence a natural C^r-manifold structure for every $r \in$. Given $x \in \mathcal{D}$, the linear isomorphism $\mathrm{ev}_{\mathbf{1}_{\mathcal{D}}} : \mathrm{T}_x\mathcal{D} \to \mathcal{V}$ will be used for the identification

$$\mathrm{T}_x\mathcal{D} \cong \{x\} \times \mathcal{V}. \tag{23.9}$$

Let f be a mapping whose domain is an open neighborhood of x and whose codomain is an open subset of a flat space $\mathcal{E}'$ with translation space $\mathcal{V}'$. If f is differentiable at $x \in \mathcal{D}$ then the gradient $\nabla_x f$ in the ordinary sense of (23.4) belongs to $\mathrm{Lin}(\{x\} \times \mathcal{V}, \mathcal{V}')$ when the identification (23.9) is used. No confusion is likely because we have

$$\nabla_x f(x, \mathbf{v}) = \nabla_x f\, \mathbf{v} \qquad \text{for all} \qquad \mathbf{v} \in \mathcal{V} \tag{23.10}$$

when $\nabla_x f$ is used with both meanings.

If $\mathcal{D}$ is the underlying manifold of an open subset of a flat space, then (23.9) gives rise to the idetification

$$\mathrm{T}\mathcal{D} \cong \mathcal{D} \times \mathcal{V}. \tag{23.11}$$

Note that the family $(\, \mathrm{T}_x\mathcal{M} \,|\, x \in \mathcal{M}\,)$ is disjoint. The **bundle projection** $\mathrm{pt} : \mathrm{T}\mathcal{M} \to \mathcal{M}$ of the tangent bundle is given by

$$\mathrm{pt}(\mathbf{t}) :\in \left\{\, x \in \mathcal{M} \;\middle|\; \mathbf{t} \in \mathrm{T}_x\mathcal{M} \,\right\}. \tag{23.12}$$

Every manifold chart $\chi \in \mathrm{Ch}\mathcal{M}$ induces a bundle chart for $\mathrm{T}\mathcal{M}$ as shown in the following. We define the **tangent-bundle chart**

$$\mathrm{tgt}_\chi : \mathrm{pt}^<(\mathrm{Dom}\ \chi) \to \mathrm{Dom}\ \chi \times \mathcal{V}_\chi \tag{23.13}$$

by

$$\mathrm{tgt}_\chi(\mathbf{t}) = (\, z\, ,\ (\nabla_z \chi)\, \mathbf{t}\,) \qquad \text{where} \quad z := \mathrm{pt}(\mathbf{t}). \tag{23.14}$$

It is easily seen that tgt_χ is invertible and that

$$\mathrm{tgt}_\chi^{\leftarrow}(z, \mathbf{u}) = (\nabla_z \chi)^{-1} \mathbf{u} \tag{23.15}$$

for all $z \in \mathrm{Dom}\chi$ and all $\mathbf{u} \in \mathcal{V}_\chi$. Let $\chi, \gamma \in \mathrm{Ch}\mathcal{M}$ be given. It follows from (21.7) and (23.6) that

$$\nabla_{\chi(z)}(\gamma \,\square\, \chi^{\leftarrow}) = (\nabla_\chi \gamma)(z) = (\nabla_z \gamma)(\nabla_z \chi)^{-1} \tag{23.16}$$

for all $z \in \mathrm{Dom}\ \gamma \cap \mathrm{Dom}\ \chi$. Hence, by (23.14) and (23.15) with χ replaced by γ, we have

$$(\mathrm{tgt}_\gamma \,\square\, \mathrm{tgt}_\chi^{\leftarrow})(z, \mathbf{u}) = \big(\ z\ ,\ \nabla_{\chi(z)}(\gamma \,\square\, \chi^{\leftarrow})\, \mathbf{u}\ \big) \tag{23.17}$$

for all $z \in \mathrm{Dom}\,\gamma \cap \mathrm{Dom}\,\chi$ and all $\mathbf{u} \in \mathcal{V}_\chi$. It is clear that $\mathrm{tgt}_\gamma \,\square\, \mathrm{tgt}_\chi^{\leftarrow}$ is of class C^{r-1}. Since $\chi, \gamma \in \mathrm{Ch}\mathcal{M}$ were arbitrary, it follows from (23.17) that

$$\{\ \mathrm{tgt}_\alpha\ \big|\ \alpha \in \mathrm{Ch}\mathcal{M}\ \}$$

is a C^{r-1} bundle-atlas of $\mathrm{T}\mathcal{M}$. We consider $\mathrm{T}\mathcal{M}$ has being endowed with the C^{r-1} linear space bundle structure determined by this atlas.

It is also easily seen that $\{\,(\alpha \times \mathbf{1}_{\mathcal{V}_\alpha}) \circ \mathrm{tgt}_\alpha \,|\, \alpha \in \mathrm{Ch}\mathcal{M}\,\}$ is a C^{r-1} manifold-atlas of $\mathrm{T}\mathcal{M}$. If $\chi \in \mathrm{Ch}\mathcal{M}$ then the page of the manifold chart $(\chi \times \mathbf{1}_{\mathcal{V}_\chi}) \circ \mathrm{tgt}_\chi$ is

$$\mathrm{Pag}\ ((\chi \times \mathbf{1}_{\mathcal{V}_\chi}) \circ \mathrm{tgt}_\chi) = \mathrm{Pag}\ \chi \times \mathcal{V}_\chi \tag{23.18}$$

and we have

$$\mathcal{V}_{(\chi \times \mathbf{1}_{\mathcal{V}_\chi}) \circ \mathrm{tgt}_\chi} = (\mathcal{V}_\chi)^2 \tag{23.19}$$

and hence

$$\dim_{\mathbf{t}} \mathrm{T}\mathcal{M} = 2 \dim_{\mathrm{pt}(\mathbf{t})} \mathcal{M} \quad \text{for all} \quad \mathbf{t} \in \mathrm{T}\mathcal{M}. \tag{23.20}$$

It is easily seen that the bundle projection $\mathrm{pt} : \mathrm{T}\mathcal{M} \to \mathcal{M}$ defined by (23.12) is of class C^{r-1}.

Let $r \in$ and C^r-manifolds $\mathcal{M}$ and $\mathcal{M}'$ be given. Let g be a mapping whose domain and codomain are open subsets of $\mathcal{M}$ and $\mathcal{M}'$, respectively. We say that g is **of class** C^s with $s \in 0..r$ if $\chi' \,\square\, g \,\square\, \chi^{\leftarrow}$ is of class C^s in the ordinary sense for all $\chi \in \mathrm{Ch}\mathcal{M}$ and all $\chi' \in \mathrm{Ch}\mathcal{M}'$. This is the case if and only if $\chi' \,\square\, g$ is of class C^s in the sense of Sect.21 for all $\chi' \in \mathrm{Ch}\mathcal{M}'$. Also, g is of class C^s if $\chi' \,\square\, g \,\square\, \chi^{\leftarrow}$ is of class C^s for all χ in some atlas included in $\mathrm{Ch}\mathcal{M}$ and for all

χ' in some atlas included in Ch$\mathcal{M}'$. The notion of differentiablity of g is defined in a similar way.

Assume that g is differentiable at $x \in \mathcal{M}$. It follows from (23.16) that

$$\nabla_x g := (\nabla_{g(x)}\chi')^{-1}\nabla_{\chi(x)}(\chi' \circ g \circ \chi^{\leftarrow})\nabla_x\chi \tag{23.21}$$

does not depend on the choice of $\chi \in \mathrm{Ch}_x\mathcal{M}$ and $\chi' \in \mathrm{Ch}_{g(x)}\mathcal{M}'$. We call

$$\nabla_x g \in \mathrm{Lin}\,(\,\mathrm{T}_x\mathcal{M}\,,\,\mathrm{T}_{g(x)}\mathcal{M}'\,) \tag{23.22}$$

the **gradient of** g **at** x. Appropriate versions of the chain rule apply to gradients in this sense. If $\mathcal{M}'$ is an open subset of a flat space $\mathcal{E}'$ with translation space $\mathcal{V}'$, then the gradient $\nabla_x g$ in the sense of (23.22) is related to the gradient $\nabla_x g$ in the sense of (23.4) by

$$(\nabla_x g)\mathbf{t} = (\,g(x)\,,\,(\nabla_x g)\mathbf{t}\,) \qquad \text{for all} \qquad \mathbf{t} \in \mathrm{T}_x\mathcal{M} \tag{23.23}$$

when the identification $\mathrm{T}_{g(x)}\mathcal{M}' \cong \{g(x)\} \times \mathcal{V}'$ is used.

Definition: *A mapping* $\mathbf{h} : \mathcal{M} \to \mathrm{T}\mathcal{M}$ *is called a* **vector-field** *on* $\mathcal{M}$ *if it is a right-inverse of* pt*, i.e. if*

$$\mathbf{h}(x) \in \mathrm{T}_x\mathcal{M} \qquad \text{for all} \qquad x \in \mathcal{M}. \tag{23.24}$$

If $\mathbf{h}$ and $\mathbf{k}$ are vector-fields, then $\mathbf{h}+\mathbf{k}$ is the vector-field defined by value-wise addition, i.e. by $(\mathbf{h}+\mathbf{k})(x) := \mathbf{h}(x)+\mathbf{k}(x)$ for all $x \in \mathcal{M}$. If $\mathbf{h}$ is a vector-field and f a real-valued function on $\mathcal{M}$ (often called a “scalar-field”), then $f\,\mathbf{h}$ is defined by value-wise sacalar multiplication, *i.e.* by $(f\,\mathbf{h})(x) := f(x)\mathbf{h}(x)$ for all $x \in \mathcal{M}$.

The set of all real-valued functions of class C^s, $s \in 0..(r-1)$, on $\mathcal{M}$ will be denoted by $\mathrm{C}^s(\mathcal{M})$. The set of all vector-fields of class C^s, $s \in 0..(r-1)$, on $\mathcal{M}$ will be denoted by $\mathfrak{X}^s(\mathrm{T}\mathcal{M})$. Using value-wise addition and mutiplication, $\mathrm{C}^s(\mathcal{M})$ acquires the natural structure of a commutative algebra over . The constants form a subalgebra of $\mathrm{C}^s(\mathcal{M})$ that is isomorphic to . Using value-wise addition and mutiplication, $\mathfrak{X}^s(\mathrm{T}\mathcal{M})$ acquires the natural structure of a $\mathrm{C}^s(\mathcal{M})$-module.

Let $\mathbf{h} : \mathcal{M} \to \mathrm{T}\mathcal{M}$ be a vector-field and $\chi \in \mathrm{Ch}\mathcal{M}$. Define $\mathbf{h}^\chi : \mathrm{Dom}\chi \to \mathcal{V}_\chi$ by

$$\mathbf{h}^\chi(y) := (\nabla_y\chi)\mathbf{h}(y) \quad \text{foa all} \quad y \in \mathrm{Dom}\chi. \tag{23.25}$$

Given $x \in \mathrm{Dom}\,\chi$, we define

$$\nabla_x^\chi\mathbf{h} := (\nabla_x\chi)^{-1}\nabla_x\mathbf{h}^\chi \in \mathrm{Lin}\,\mathrm{T}_x\mathcal{M}. \tag{23.26}$$

It is easily seen from $(\nabla_x \chi)^{-1} \nabla_x \mathbf{h}^\chi = (\nabla_x \chi)^{-1} \big(\nabla_\chi \mathbf{h}^\chi(x)\big) \nabla_x \chi$ that $\nabla_x^\chi \mathbf{h}$ is simply the ordinary gradient of $\mathbf{h}^\chi$ in the chart χ, transported from $\operatorname{Lin} \mathcal{V}_\chi$ to $\operatorname{Lin} \mathrm{T}_x \mathcal{M}$ by $\nabla_x \chi$.

A continuous mapping $p : J \to \mathcal{M}$ from some genuine interval $J \in$ into the manifold $\mathcal{M}$ will be called a **process**. If p is differentiable at a given $t \in J$, then

$$\partial_t p := (\nabla_{p(x)} \chi)^{-1} \partial_t (\chi \circ p) \tag{23.27}$$

does not depend on the choice of $\chi \in \mathrm{Ch}_{p(t)} \mathcal{M}$. We call $\partial_t p \in \mathrm{T}_{p(t)} \mathcal{M}$ the **derivative of** p **at** t. If p is differentiable, we define the **derivative** (-process) $p^{\cdot} : J \to \mathrm{T}\mathcal{M}$ by

$$p^{\cdot}(t) := \partial_t \, p \qquad \text{for all} \qquad t \in J. \tag{23.28}$$

24. Tensor Bundles

We now assume that a number $s \in^{\sim}$and a C^s linear-space bundle $(\mathcal{B}, \tau, \mathcal{M})$ are given.

With each analytic tensor functor $\mathbf{\Phi}$ one can construct what is called the associated **$\mathbf{\Phi}$-bundle** of $\mathcal{B}$

$$\mathbf{\Phi}(\mathcal{B}) := \bigcup_{y \in \mathcal{M}} \mathbf{\Phi}(\mathcal{B}_y). \tag{24.1}$$

It has the natural structure of a C^s linear-space bundle over $\mathcal{M}$. For every open subset $\mathcal{P}$ of $\mathcal{M}$, we also use the following notation

$$\mathbf{\Phi}(\tau^{<}(\mathcal{P})) := \bigcup_{y \in \mathcal{P}} \mathbf{\Phi}(\mathcal{B}_y). \tag{24.2}$$

We define the **bundle projection** $\tau^{\mathbf{\Phi}} : \mathbf{\Phi}(\mathcal{B}) \to \mathcal{M}$ of the bundle $\mathbf{\Phi}(\mathcal{B})$ by

$$\tau^{\mathbf{\Phi}}(\mathbf{v}) :\in \left\{ \, y \in \mathcal{M} \;\middle|\; \mathbf{v} \in \mathbf{\Phi}(\mathcal{B}_y) \, \right\}. \tag{24.3}$$

For every bundle chart $\phi : \tau^{<}(\mathcal{O}_\phi) \to \mathcal{O}_\phi \times \mathcal{V}_\phi$, we have

$$\phi(\mathbf{v}) = \big(\, y \, , \, \phi\rfloor_y(\mathbf{t}) \, \big) \qquad \text{where} \quad y := \tau(\mathbf{t})$$

We define the mapping

$$\mathbf{\Phi}(\phi) : \mathbf{\Phi}(\pi^{<}(\mathcal{O}_\phi)) \to \mathcal{O}_\phi \times \mathbf{\Phi}(\mathcal{V}_\phi) \tag{24.4}$$

by

$$(\mathbf{\Phi}(\phi))(\mathbf{v}) := (\ y\ ,\ \mathbf{\Phi}(\phi\rfloor_y)\mathbf{v}\ ) \qquad \text{when} \quad y := \tau^{\mathbf{\Phi}}(\mathbf{v}). \tag{24.5}$$

It follows from the analyticity of the mapping $(\mathbf{L} \mapsto \mathbf{\Phi}(\mathbf{L}))$ that

$$\{\ \mathbf{\Phi}(\phi)\ \big|\ \phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})\ \}$$

is a C^s-bundle-atlas of $\mathbf{\Phi}(\mathcal{B})$. It determines the C^s linear-space bundle structure of $(\mathbf{\Phi}(\mathcal{B}), \tau^{\mathbf{\Phi}}, \mathcal{M})$.

The bundle projection $\tau^{\mathbf{\Phi}} : \mathbf{\Phi}(\mathcal{B}) \to \mathcal{M}$ defined by (24.3) is easily seen to be of class C^s.

<u>Notation:</u> *For every $p \in 0..s$, we denote the collection of all C^p cross sections of $\mathbf{\Phi}(\mathcal{B})$ by $\mathfrak{X}^p(\mathbf{\Phi}(\mathcal{B}))$. The collection of all differentiable cross sections of $\mathbf{\Phi}(\mathcal{B})$ is denoted by $\mathfrak{X}(\mathbf{\Phi}(\mathcal{B}))$.*

In the special case $\mathcal{B} = \mathrm{T}\mathcal{M}$, we call $\mathbf{\Phi}(\mathrm{T}\mathcal{M})$ the **tansor bundle** of $\mathcal{M}$ of type $\mathbf{\Phi}$. A cross section of the tensor bundle $\mathbf{\Phi}(\mathrm{T}\mathcal{M})$ is called a **tensor-field** of type $\mathbf{\Phi}$. When $\mathbf{\Phi} := \mathrm{Dl}$ is the duality functor (see Sect.13), we call $\mathrm{Dl}\,(\mathrm{T}\mathcal{M})$ the **cotangent bundle** of $\mathcal{M}$ which will be denoted by $\mathrm{T}^*\mathcal{M}$.

Remark: Let $\mathcal{M}$ be a C^∞-manifold. With every $\mathbf{h} \in \mathfrak{X}^\infty(\mathrm{T}\mathcal{M})$ we can then associate a mapping $\overset{\triangledown}{\mathbf{h}} : \mathrm{C}^\infty(\mathcal{M}) \to \mathrm{C}^\infty(\mathcal{M})$ defined by

$$\overset{\triangledown}{\mathbf{h}}(f) := (\nabla f)\mathbf{h} \qquad \text{for all} \qquad f \in \mathrm{C}^\infty(\mathcal{M}) \tag{24.6}$$

where the gradient ∇f of f is the covector field of class C^∞ given by $\nabla f(x) := \nabla_x f$ for all $x \in \mathrm{Dom}\, f$. It is clear that $\overset{\triangledown}{\mathbf{h}}$ is -linear. By using the product rule $\nabla fg = f\nabla g + g\nabla f$, we have

$$\overset{\triangledown}{\mathbf{h}}(fg) = f\overset{\triangledown}{\mathbf{h}}(g) + g\overset{\triangledown}{\mathbf{h}}(f) \quad \text{for all} \quad f, g \in \mathbf{C}^\infty(\mathcal{M}). \tag{24.7}$$

This shows that $\overset{\triangledown}{\mathbf{h}}$ is a derivation of the module $\mathrm{C}^\infty(\mathcal{M})$. One can prove that every derivation of $\mathrm{C}^\infty(\mathcal{M})$ can be obtained in this manner. (The proof is fairly difficult.) ∎

Let a cross section section $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$ be given. For every bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ we define the mapping

$$\mathbf{H}^\phi : \mathcal{O}_\phi \to \mathbf{\Phi}(\mathcal{V}_\phi)$$

by

$$\mathbf{H}^\phi(y) := \mathbf{\Phi}(\phi\rfloor_y)\mathbf{H}(y), \quad \text{for all} \quad y \in \mathcal{O}_\phi. \tag{24.8}$$

Given $x \in \mathcal{O}_\phi$, we define

$$\nabla_x^\phi \mathbf{H} := \mathbf{\Phi}(\phi\rfloor_x^{-1})\nabla_x \mathbf{H}^\phi \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathbf{\Phi}(\mathcal{B}_x)). \tag{24.9}$$

When $\mathbf{\Phi} = \mathrm{Id}$ and $\mathcal{B} = \mathrm{T}\mathcal{M}$, we have $\nabla_x^{\mathrm{tgt}_\chi}\mathbf{h} = \nabla_x^{\chi}\mathbf{h}$ for all $\chi \in \mathrm{Ch}\mathcal{M}$ and all $x \in \mathrm{Dom}\,\chi$.

One defines value-wise addition of cross sections of $\mathbf{\Phi}(\mathcal{B})$ and value-wise scalar multiplication of a real function on $\mathcal{M}$ and a cross section of $\mathbf{\Phi}(\mathcal{B})$ in the obvious manner. $\mathfrak{X}^p\mathbf{\Phi}(\mathcal{B})$ has the natural structure of a $\mathrm{C}^p(\mathcal{M})$-module, where $\mathrm{C}^p(\mathcal{M})$ is the ring of all real-valued functions of class C^p on $\mathcal{M}$.

Let $(\mathcal{L}_1, \tau_1, \mathcal{M})$ and $(\mathcal{L}_2, \tau_2, \mathcal{M})$ be linear-space bundles over $\mathcal{M}$ and let $\mathcal{L}_1 \times_{\mathcal{M}} \mathcal{L}_2$ be the fiber product bundle of $\mathcal{L}_1$ and $\mathcal{L}_2$. For every tensor bifunctor $\mathbf{\Upsilon}$, it follows form (24.5) that for each bundle chart $\phi_1 \in \mathrm{Ch}(\mathcal{L}_1, \mathcal{M})$ and each buhdle chart $\phi_2 \in \mathrm{Ch}(\mathcal{L}_2, \mathcal{M})$

$$\mathbf{\Upsilon}(\phi_1 \times_{\mathcal{M}} \phi_2)(\mathbf{v}) = \big(\, y\ ,\ \mathbf{\Upsilon}(\varphi\rfloor_y \times \phi\rfloor_y)\mathbf{v}\,\big) \tag{24.10}$$

where $y := (\tau_1 \times_{\mathcal{M}} \tau_2)^{\mathbf{\Upsilon}}(\mathbf{v})$ (see 24.3).

Let a cross section $\mathbf{H} : \mathcal{M} \to \mathbf{\Upsilon}(\mathcal{L}_1 \times_{\mathcal{M}} \mathcal{L}_2)$ be given. For each bundle chart $\phi_1 \in \mathrm{Ch}(\mathcal{L}_1, \mathcal{M})$ and each buhdle chart $\phi_2 \in \mathrm{Ch}(\mathcal{L}_2, \mathcal{M})$, we define the mapping

$$\mathbf{H}^{\phi_1,\phi_2} : \mathcal{O}_\phi \to \mathbf{\Upsilon}(\mathcal{V}_{\phi_1} \times \mathcal{V}_{\phi_2})$$

by

$$\mathbf{H}^{\phi_1,\phi_2}(y) := \mathbf{\Phi}(\phi\rfloor_y)\mathbf{H}(y), \quad \text{for all} \quad y \in \mathcal{O}_{\phi_1} \cap \mathcal{O}_{\phi_2}. \tag{24.11}$$

Given $x \in \mathcal{O}_{\phi_1} \cap \mathcal{O}_{\phi_2}$, we define

$$\nabla_x^{\phi_1,\phi_2}\mathbf{H} := \mathbf{\Upsilon}(\phi_1\rfloor_x^{-1} \times \phi_2\rfloor_x^{-1})\nabla_x\mathbf{H}^{\phi_1,\phi_2} \tag{24.12}$$

which is in $\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathbf{\Upsilon}(\mathcal{L}_{1x} \times \mathcal{L}_{2x}))$.