Chapter 1

Preliminaries

11. Multilinearity

Let $(\,\mathcal{V}_i \mid i \in I\,)$ be a family of linear spaces, we define (see (04.24) of [FDS]), for each $j \in I$ and each $\mathbf{v} \in \times_{i\in I} \mathcal{V}_i$, the mapping $(\mathbf{v}.j) : \mathcal{V}_j \to \times_{i\in I} \mathcal{V}_i$ by the rule

$$((\mathbf{v}.j)(\mathbf{u}))_i := \left\{ \begin{array}{lll} \mathbf{v}_i & \text{if} & i \in I\backslash\{j\} \\ \\ \mathbf{u} & \text{if} & i = j \end{array} \right\} \qquad \text{for all} \quad \mathbf{u} \in \mathcal{V}_j. \tag{11.1}$$

<u>Definition</u> : *Let the family $(\,\mathcal{V}_i \mid i \in I\,)$ and $\mathcal{W}$ be linear spaces. We say that the mapping $\mathbf{M} : \times_{i\in I} \mathcal{V}_i \to \mathcal{W}$ is* **multilinear** *if, for every $\mathbf{v} \in \times_{i\in I} \mathcal{V}_i$ and every $j \in I$ the mapping $\mathbf{M} \circ (\mathbf{v}.j) : \mathcal{V}_j \to \mathcal{W}$ is linear, so that $\mathbf{M} \circ (\mathbf{v}.j) \in \mathrm{Lin}(\mathcal{V}_j, \mathcal{W})$. The set of all multilinear mappings from $\times_{i\in I} \mathcal{V}_i$ to $\mathcal{W}$ is denoted by*

$$\mathrm{Lin}_I(\,\times_{i\in I} \mathcal{V}_i\,,\,\mathcal{W}\,). \tag{11.2}$$

Let linear spaces $\mathcal{V}$ and $\mathcal{W}$ and a set I be given.

Let $\mathrm{Perm}\, I$ be the permutation group, which consists of all invertible mappings from I to itself. For every permutation $\sigma \in \mathrm{Perm}\, I$ we define a mapping $\mathrm{T}_\sigma : \mathcal{V}^I \to \mathcal{V}^I$ by

$$\mathrm{T}_\sigma(\mathbf{v}) = \mathbf{v} \circ \sigma \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}^I, \tag{11.3}$$

that is $(\mathrm{T}_\sigma(\mathbf{v}))_i := \mathbf{v}_{\sigma(i)}$ for all $i \in I$. In view of $\mathbf{v} \circ (\sigma \circ \rho) = (\mathbf{v} \circ \sigma) \circ \rho$, we have $\mathrm{T}_{\sigma\circ\rho} = \mathrm{T}_\rho \circ \mathrm{T}_\sigma$ for all $\sigma, \rho \in \mathrm{Perm}\, I$. It is not hard to see that, for every multilinear mapping $\mathbf{M} : \mathcal{V}^I \to \mathcal{W}$ and every permutation σ, the composition $\mathbf{M}\circ\mathrm{T}_\sigma$ is again a multilinear mapping from $\mathcal{V}^I$ to $\mathcal{W}$, i.e. $\mathbf{M}\circ\mathrm{T}_\sigma \in \mathrm{Lin}_I(\,\mathcal{V}^I, \mathcal{W}\,)$.

<u>Definition</u> : *A multilinear mapping $\mathbf{M} : \mathcal{V}^I \to \mathcal{W}$ is said to be (completely)* **symmetric** *if*

$$\mathbf{M} \circ \mathrm{T}_\sigma = \mathbf{M} \qquad \text{for all} \quad \sigma \in \mathrm{Perm}\, I\,,$$

and is said to be (completely) **skew** *if*

$$\mathbf{M} \circ \mathrm{T}_\sigma = \mathrm{sgn}\,(\sigma)\, \mathbf{M} \qquad \text{for all} \quad \sigma \in \mathrm{Perm}\, I\,.$$

The set of all (completely) symmetric multilinear mappings and the set of all (completely) skew multilinear mappings from $\mathcal{V}^I$ to $\mathcal{W}$ will be denoted by $\mathrm{Sym}_I(\,\mathcal{V}^I, \mathcal{W}\,)$ and by $\mathrm{Skew}_I(\,\mathcal{V}^I, \mathcal{W}\,)$; respectively.

Both $\mathrm{Sym}_I(\mathcal{V}^I,\ \mathcal{W})$ and $\mathrm{Skew}_I(\mathcal{V}^I,\ \mathcal{W})$ are subspaces of the linear space $\mathrm{Lin}_I(\mathcal{V}^I,\ \mathcal{W})$ with dimensions

$$\dim \mathrm{Sym}_I(\mathcal{V}^I,\ \mathcal{W}) = \binom{\dim \mathcal{V} + \#I - 1}{\#I} \dim \mathcal{W} \tag{11.4}$$

and

$$\dim \mathrm{Skew}_I(\mathcal{V}^I,\ \mathcal{W}) = \binom{\dim \mathcal{V}}{\#I} \dim \mathcal{W}. \tag{11.5}$$

For every $k \in$, we write $\mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W})$, $\mathrm{Sym}_k(\mathcal{V}^k, \mathcal{W})$ and $\mathrm{Skew}_k(\mathcal{V}^k, \mathcal{W})$ for $\mathrm{Lin}_{k^]}(\mathcal{V}^{k^]}, \mathcal{W})$, $\mathrm{Sym}_{k^]}(\mathcal{V}^{k^]}, \mathcal{W})$ and $\mathrm{Skew}_{k^]}(\mathcal{V}^{k^]}, \mathcal{W})$; respectively.

In applicatins, we often use the following identifications

$$\begin{aligned}\mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W}) &\cong \mathrm{Lin}_{k-1}(\mathcal{V}^{k-1},\ \mathrm{Lin}\,(\mathcal{V}, \mathcal{W})) \\ &\cong \mathrm{Lin}(\mathcal{V},\ \mathrm{Lin}_{k-1}(\mathcal{V}^{k-1}, \mathcal{W}))\end{aligned}$$

and inclusions

$$\begin{aligned}\mathrm{Sym}_k(\mathcal{V}^k, \mathcal{W}) &\subset \mathrm{Sym}_{k-1}(\mathcal{V}^{k-1}, \mathrm{Lin}\,(\mathcal{V}, \mathcal{W})), \\ \mathrm{Skew}_k(\mathcal{V}^k, \mathcal{W}) &\subset \mathrm{Skew}_{k-1}(\mathcal{V}^{k-1}, \mathrm{Lin}\,(\mathcal{V}, \mathcal{W})).\end{aligned}$$

In particular, we shall use $\mathrm{Sym}_2(\mathcal{V}^2,) \cong \mathrm{Sym}\,(\mathcal{V}, \mathcal{V}^*) := \mathrm{Sym}\,(\mathcal{V}, \mathrm{Lin}\,(\mathcal{V},))$ and $\mathrm{Skew}_2(\mathcal{V}^2,) \cong \mathrm{Skew}\,(\mathcal{V}, \mathcal{V}^*) := \mathrm{Skew}\,(\mathcal{V}, \mathrm{Lin}\,(\mathcal{V},))$. It can be shown that $\mathrm{Skew}\,(\mathcal{V}, \mathcal{V}^*)$ has invertiable mapping if and only if $\dim \mathcal{V}$ is even. (See Prop.3 of Sect.87, [FDS].)

Given a number $k \in$ and a multilinear mapping $\mathbf{A} \in \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W})$, the mapping $\sum_{\sigma \in \mathrm{Perm}\, k^]} (\mathrm{sgn}\,\sigma)\, \mathbf{A} \circ \mathrm{T}_\sigma : \mathcal{V}^k \to \mathcal{W}$ is a completely skew multilinear mapping. Moreover, it can be easily shown that

$$\frac{1}{k!} \sum_{\sigma \in \mathrm{Perm}\, k^]} (\mathrm{sgn}\,\sigma)\ \mathbf{W} \circ \mathrm{T}_\sigma = \mathbf{W}$$

for all skew multilinear mapping $\mathbf{W} \in \mathrm{Skew}_k(\mathcal{V}^k, \mathcal{W})$.

<u>Definition</u> : *Given a number $k \in$, we define the* **alternating assignment** $\mathrm{Alt} : \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W}) \to \mathrm{Skew}_k(\mathcal{V}^k, \mathcal{W})$ *by*

$$\mathrm{Alt}\ \mathbf{A} := \frac{1}{k!} \sum_{\sigma \in \mathrm{Perm}\, k^]} (\mathrm{sgn}\,\sigma)\, \mathbf{A} \circ \mathrm{T}_\sigma \tag{11.6}$$

for all linear spaces $\mathcal{V}$ and $\mathcal{W}$ and all $\mathbf{A} \in \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W})$.

Given $p \in$. We define, for each $i \in (p+1)^]$, a mapping $\mathrm{del}_i : \mathcal{V}^{p+1} \to \mathcal{V}^p$ by

$$(\mathrm{del}_i(\mathbf{v}))_j := \left\{ \begin{array}{ll} \mathbf{v}_j & \text{if} \quad 1 \le i \le j-1 \\ \\ \mathbf{v}_{i+1} & \text{if} \quad j \le i \le p \end{array} \right\} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}^{p+1}. \tag{11.7}$$

Intuitively, $\mathrm{del}_i(\mathbf{v})$ is obtained from $\mathbf{v}$ by deleting the i-th term.

When the alternating assignment Alt restricted to the subspace $\mathrm{Lin}\,(\mathcal{V},\, \mathrm{Skew}_p(\mathcal{V}^p, \mathcal{W}))$ of $\mathrm{Lin}\,(\mathcal{V},\, \mathrm{Lin}_p(\mathcal{V}^p, \mathcal{W})) \cong \mathrm{Lin}_{p+1}(\,\mathcal{V}^{p+1}, \mathcal{W})$, we have

$$(p+1)\,(\mathrm{Alt}\,\mathbf{A})\mathbf{v} = \sum_{i\in(p+1)^]} (-1)^{i-1}\mathbf{A}(\mathbf{v}_i, \mathrm{del}_i\mathbf{v}) \tag{11.8}$$

for all $\mathbf{v} \in \mathcal{V}^{p+1}$ and all $\mathbf{A} \in \mathrm{Lin}\,(\mathcal{V},\, \mathrm{Skew}_p(\,\mathcal{V}^p, \mathcal{W}))$. Similarly, when the alternating assignment Alt restricted to the subspace $\mathrm{Skew}_p(\,\mathcal{V}^p, \mathrm{Lin}(\mathcal{V}, \mathcal{W}))$ of $\mathrm{Lin}\,(\,\mathcal{V},\, \mathrm{Lin}_p(\,\mathcal{V}^p, \mathcal{W})) \cong \mathrm{Lin}_{p+1}(\,\mathcal{V}^{p+1}, \mathcal{W})$, we have

$$(p+1)\,(\mathrm{Alt}\,\mathbf{B})\mathbf{v} = \sum_{i\in(p+1)^]} (-1)^{p+1-i}\mathbf{B}(\mathrm{del}_i\mathbf{v}, \mathbf{v}_i) \tag{11.9}$$

for all $\mathbf{v} \in \mathcal{V}^{p+1}$ and all $\mathbf{B} \in \mathrm{Skew}_p(\mathcal{V}^p, \mathrm{Lin}(\mathcal{V}, \mathcal{W}))$.

Definition: *An* **algebra** *is a linear space $\mathcal{V}$ together with a bilinear mapping $\mathbf{B} \in \mathrm{Lin}_2(\mathcal{V}^2, \mathcal{V})$. An algebra $\mathcal{V}$ is called a* **Lie Alegebra** *if the bilinear mapping $\mathbf{B}$ is skew-symmetric, i.e. $\mathbf{B} \in \mathrm{Skew}_2(\mathcal{V}^2, \mathcal{V})$, and satisfies* **Jacobi indetity**

$$\mathbf{B}(\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2), \mathbf{v}_3) + \mathbf{B}(\mathbf{B}(\mathbf{v}_2, \mathbf{v}_3), \mathbf{v}_1) + \mathbf{B}(\mathbf{B}(\mathbf{v}_3, \mathbf{v}_1), \mathbf{v}_2) = \mathbf{0} \tag{11.10}$$

for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathcal{V}$.

By using the inclusion $\mathrm{Skew}_2(\mathcal{V}^2, \mathcal{V}) \subset \mathrm{Lin}(\mathcal{V},\, \mathrm{Lin}(\mathcal{V}, \mathcal{V}))$ and (11.9), we see taht (11.10) can rewriten as

$$\mathrm{Alt}\,(\mathbf{B} \circ \mathbf{B}) = \mathbf{0} \tag{11.11}$$

where $(\mathbf{B} \circ \mathbf{B})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) := \mathbf{B}(\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2), \mathbf{v}_3)$ for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathcal{V}$.

Remark 1: In the literature the **alternating assignment** given in (11.6) is often called "*skew-symmetric operator*" ([B-W]), "*complete antisymmetrization*" ([F-C]). The **symmetric assignment**, "*symmetric operator*" or "*complete symmetrization*" $\mathrm{Sym} : \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W}) \to \mathrm{Sym}_k(\mathcal{V}^k, \mathcal{W})$ is given by

$$\mathrm{Sym}\,\mathbf{M} := \frac{1}{k!} \sum_{\sigma\in\mathrm{Perm}\,k^]} \mathbf{M} \circ \mathrm{T}_\sigma \tag{11.12}$$

for all linear spaces $\mathcal{V}$ and $\mathcal{W}$ and all $\mathbf{M} \in \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W})$. ∎

Remark 2: Both assignments given in (11.6) and (11.12) are "*natural linear assignments*" from a functor to another functor (see (13.16) of Sect.13). More precisely, the alternating assignment is a natural linear assgnment from the functor Ln_k to the functor Sk_k and the symmetric assignment is a natural linear assgnment from the functor Ln_k to the functor Sm_k (see Sect. 13). ∎

12. Isocategories, isofunctors and Natural Assignments

An **isocategory*** ‡ is given by the specification of a class OBJ whose members are called **objects**, a class ISO whose members are called ISO**morphisms**,

(i) a rule that associates with each $\phi \in \mathrm{ISO}$ a pair $(\mathrm{Dom}\,\phi, \mathrm{Cod}\,\phi)$ of objects, called the **domain** and **codomain** of ϕ,
(ii) a rule that associates with each $\mathcal{A} \in OBJ$ a member of ISO denoted by $1_{\mathcal{A}}$ and called the **identity** of $\mathcal{A}$,
(iii) a rule that associates with each pair (ϕ, ψ) in ISO such that $\mathrm{Cod}\,\phi = \mathrm{Dom}\,\psi$ a member of ISO denoted by $\psi \circ \phi$ and called the **composite** of ϕ and ψ, with $\mathrm{Dom}\,(\psi \circ \phi) = \mathrm{Dom}\,\phi$ and $\mathrm{Cod}\,(\psi \circ \phi) = \mathrm{Cod}\,\psi$.
(iv) a rule that associates with each $\phi \in \mathrm{ISO}$ a member of ISO denoted by $\phi^{\leftarrow}$ and called the **inverse** of ϕ.

subject to the following three axioms:

(I1) $\phi \circ 1_{\mathrm{Dom}\,\phi} = \phi = 1_{\mathrm{Cod}\,\phi} \circ \phi$ for all $\phi \in \mathrm{ISO}$,
(I2) $\chi \circ (\psi \circ \phi) = (\chi \circ \psi) \circ \phi$ for all $\phi, \psi, \chi \in \mathrm{ISO}$ such that $\mathrm{Cod}\,\phi = \mathrm{Dom}\,\psi$ and $\mathrm{Cod}\,\psi = \mathrm{Dom}\,\chi$.
(I3) $\phi^{\leftarrow} \circ \phi = 1_{\mathrm{Dom}\,\phi}$ and $\phi \circ \phi^{\leftarrow} = 1_{\mathrm{Cod}\,\phi}$ for all $\phi \in \mathrm{ISO}$.

Given $\phi \in \mathrm{ISO}$, one writes $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ or $\mathcal{A} \xrightarrow{\phi} \mathcal{B}$ to indicate that $\mathrm{Dom}\,\phi = \mathcal{A}$ and $\mathrm{Cod}\,\phi = \mathcal{B}$.

There is one to one correspondence between an object $\mathcal{A} \in OBJ$ and the corresponding identity $1_{\mathcal{A}} \in \mathrm{ISO}$. For this reason, we will usually name an isocategory by giving the name of its class of ISOmorphisms.

Let isocategories ISO and ISO$'$ with object-classes OBJ and OBJ' be given. We can then form the **product-isocategory** $\mathrm{ISO} \times \mathrm{ISO}'$ whose object-class $OBJ \times OBJ'$ consists of pairs $(\mathcal{A}, \mathcal{A}')$ with $\mathcal{A} \in OBJ$, $\mathcal{A}' \in OBJ'$ and ISOmorphism-class $\mathrm{ISO} \times \mathrm{ISO}'$ consists of pairs (ϕ, ϕ') with $\phi \in \mathrm{ISO}$, $\phi' \in \mathrm{ISO}'$ and the following

(a) For every $(\phi, \phi') \in \mathrm{ISO} \times \mathrm{ISO}'$, $\mathrm{Dom}\,(\phi, \phi') := (\mathrm{Dom}\,\phi, \mathrm{Dom}\,\phi')$ and $\mathrm{Cod}\,(\phi, \phi') := (\mathrm{Cod}\,\phi, \mathrm{Cod}\,\phi')$.

* A category, introduced by Eilenberg and MacLane, is defined by (i), (ii) and (iii) with the axioms (I1) and (I2). Roughly speaking, an isocategory is a special category whose "morphisms" are called ISO-morphisms.

‡ Since isocategories are widely used in differential geometry, we introduced them directly instead of making them as a special category.

(b) Composition in $\mathrm{ISO} \times \mathrm{ISO}'$ is defined by termwise composition, i.e. by $(\psi, \psi') \circ (\phi, \phi') := (\, \psi \circ \phi \,,\, \psi' \circ \phi' \,)$ for all $\phi, \psi \in \mathrm{ISO}$ and $\phi', \psi' \in \mathrm{ISO}'$ such that $\mathrm{Dom}\,(\psi, \psi') = \mathrm{Cod}\,(\phi, \phi')$.

(c) The identity of a given pair $(\mathcal{A}, \mathcal{A}') \in \mathit{OBJ} \times \mathit{OBJ}'$ is defined to be $1_{(\mathcal{A}, \mathcal{A}')} = (1_{\mathcal{A}}, 1_{\mathcal{A}'})$.

The product of an arbitary family of isocategories can be defined in a similar manner. In particular, if a isocategory ISO and an index set I are given, one can form the I-**power-isocategory** ISO^I of ISO; its ISOmorphism-class consists of all families in ISO indexed on I. In the case when I is of the form $I := n^{]}$, we write $\mathrm{ISO}^n := \mathrm{ISO}^{n^{]}}$ for short. For example, we write $\mathrm{ISO}^2 := \mathrm{ISO} \times \mathrm{ISO}$. We identify ISO^1 with ISO and ISO^{m+n} with $\mathrm{ISO}^m \times \mathrm{ISO}^n$ for all $m, n \in$ in the obvious manner. The isocategory ISO^0 is the trival one whose only object is $\emptyset$ and whose only ISOmorphism is $1_{\emptyset}$.

A **functor** Φ is given by the specification of:

(i) a pair $(\mathrm{Dom}\,\Phi, \mathrm{Cod}\,\Phi)$ of categories, called the **domain-category** and **codomain-category** of Φ,

(ii) a rule that associates with every $\phi \in \mathrm{Dom}\,\Phi$ a member of $\mathrm{Cod}\,\Phi$ denoted by $\Phi(\phi)$,

subject to the following conditions:

(F1) We have $\mathrm{Cod}\,\Phi(\phi) = \mathrm{Dom}\,\Phi(\psi)$ and $\Phi(\psi \circ \phi) = \Phi(\psi) \circ \Phi(\phi)$ for all $\phi, \psi \in \mathrm{Dom}\,\Phi$ such that $\mathrm{Cod}\,\phi = \mathrm{Dom}\,\psi$.

(F2) For every identity $1_{\mathcal{A}}$ in $\mathrm{Dom}\,\Phi$, where $\mathcal{A}$ belongs to the object-class of $\mathrm{Dom}\,\Phi$, $\Phi(1_{\mathcal{A}})$ is an identity in $\mathrm{Cod}\,\Phi$.

An **isofunctor** is a functor whose domain-category and codomain-category are isocategories. In this book we only deal with isofunctors.

Let isocategories ISO and ISO' with object-classes OBJ and OBJ' be given. We say that Φ is an **isofunctor from** ISO **to** ISO' and we write $\mathrm{ISO} \xrightarrow{\Phi} \mathrm{ISO}'$ or $\Phi : \mathrm{ISO} \longrightarrow \mathrm{ISO}'$ to indicate that $\mathrm{ISO} = \mathrm{Dom}\,\Phi$ and $\mathrm{ISO}' = \mathrm{Cod}\,\Phi$. By (F2), we can associate with each $\mathcal{A} \in \mathit{OBJ}$ exactly one object in OBJ', denoted by $\Phi(\mathcal{A})$, such that

$$\Phi(1_{\mathcal{A}}) = 1_{\Phi(\mathcal{A})}. \tag{12.1}$$

It easily follows from (I3), (F1) and (F2) that every isofunctor Φ satisfies

$$\Phi(\phi^{\leftarrow}) = \big(\Phi(\phi)\big)^{\leftarrow} \quad \text{for all} \quad \phi \in \mathrm{Dom}\,\Phi. \tag{12.2}$$

One can construct new isofunctors from given isofunctors in the same way as new mappings are constructed from given mappings. (See, for example, Sect. 03

and 04, [FDS].) Thus, if Φ and Ψ are isofunctors such that $\operatorname{Cod}\Phi = \operatorname{Dom}\Psi$, one can define the **composite isofunctor** $\Psi \circ \Phi : \operatorname{Dom}\Phi \to \operatorname{Cod}\Psi$ by

$$(\Psi \circ \Phi)(\phi) := \Psi(\Phi(\phi)) \quad \text{for all} \quad \phi \in \operatorname{Dom}\Phi \tag{12.3}$$

Also, given isofunctors Φ and Ψ, one can define the **product-isofunctor**

$$\Phi \times \Psi : \operatorname{Dom}\Phi \times \operatorname{Dom}\Psi \longrightarrow \operatorname{Cod}\Phi \times \operatorname{Cod}\Psi$$

of Φ and Ψ by

$$(\Phi \times \Psi)(\phi, \psi) := (\Phi(\phi), \Psi(\psi)) \tag{12.4}$$

for all $\phi \in \operatorname{Dom}\Phi$ and all $\psi \in \operatorname{Dom}\Psi$.

Product-isofunctors of arbitrary families of isofunctors are defined in a similar way. In particular, if a isofunctor Φ and an index set I are given, we define the I-**power-isofunctor** $\Phi^{\times I} : (\operatorname{Dom}\Phi)^I \to (\operatorname{Cod}\Phi)^I$ of Φ by

$$\Phi^{\times I}(\phi_i \mid i \in I) = (\Phi(\phi_i) \mid i \in I) \tag{12.5}$$

for all families $(\phi_i \mid i \in I)$ in $\operatorname{Dom}\Phi$. We write $\Phi^{\times n} := \Phi^{\times n^]}$ when $n \in$.

We now assume that an isocategory ISO with object-class OBJ is given.

The **identity-isofunctor** $\operatorname{Id} : \mathrm{ISO} \to \mathrm{ISO}$ of ISO is defined by

$$\operatorname{Id}(\phi) = \phi \quad \text{for all} \quad \phi \in \mathrm{ISO}. \tag{12.6}$$

We then have

$$\operatorname{Id}(\mathcal{A}) = \mathcal{A} \quad \text{for all} \quad \mathcal{A} \in OBJ. \tag{12.7}$$

If I is an index set, then the identity-isofunctor of ISO^I is $\operatorname{Id}^{\times I}$. In particular, the identity-isofunctor of $\mathrm{ISO} \times \mathrm{ISO}$ is $\operatorname{Id} \times \operatorname{Id}$.

Given an object $\mathcal{C} \in OBJ$. The **trivial-isofunctor** $\operatorname{Tr}_{\mathcal{C}} : \mathrm{ISO} \to \mathrm{ISO}$ **for** $\mathcal{C}$ is defined by

$$\operatorname{Tr}_{\mathcal{C}}(\phi) = 1_{\mathcal{C}} \quad \text{for all} \quad \phi \in \mathrm{ISO}. \tag{12.8}$$

We then have

$$\operatorname{Tr}_{\mathcal{C}}(\mathcal{A}) = \mathcal{C} \quad \text{for all} \quad \mathcal{A} \in OBJ. \tag{12.9}$$

One often needs to consider a variety of "accounting isofunctors" whose domain and codomain isocategories are obtained from ISO by product formation. For example, the **switch-isofunctor** $\operatorname{Sw} : \mathrm{ISO}^2 \to \mathrm{ISO}^2$ is defined by

$$\operatorname{Sw}(\phi, \psi) := (\psi, \phi) \quad \text{for all} \quad \phi, \psi \in \mathrm{ISO}. \tag{12.10}$$

Given any index set I, the **equalization-isofunctor** $\operatorname{Eq}_I : \mathrm{ISO} \to \mathrm{ISO}^I$ is defined by

$$\operatorname{Eq}_I(\phi) := (\phi \mid i \in I) \quad \text{for all} \quad \phi \in \mathrm{ISO}. \tag{12.11}$$

We write $\mathrm{Eq}_n := \mathrm{Eq}_{n]}$ when $n \in$.

Let a index set I and a family $(\Phi_i \mid i \in I)$ of isofunctors, with $\mathrm{Dom}\,\Phi_i =$ ISO for all $i \in I$, be given. We then identify the family $(\Phi_i \mid i \in I)$ with the **termwise-formation isofunctor**

$$(\Phi_i \mid i \in I) : \mathrm{ISO} \to \bigtimes_{i \in I} \mathrm{Cod}\,\Phi_i$$

defined by

$$(\Phi_i \mid i \in I) := \bigtimes_{i \in I} \Phi_i \circ \mathrm{Eq}_I,$$

so that

$$(\Phi_i \mid i \in I)(\phi) = \bigtimes_{i \in I} \Phi_i(\phi), \quad \text{for all} \quad \phi \in \mathrm{ISO}. \tag{12.12}$$

In particular, if $I = 2^{]}$, we then identify the pair (Φ_1, Φ_2) with the **pair-formation isofunctor** $(\Phi_1, \Phi_2) : \mathrm{ISO} \to \mathrm{Cod}\,\Phi_1 \times \mathrm{Cod}\,\Phi_2$.

Let isofunctors Φ and Ψ, both from ISO to ISO$'$, be given. A **natural assignment** α **form** Φ **to** Ψ is a rule that associates with each object $\mathcal{F}$ of ISO a mapping

$$\alpha_{\mathcal{F}} : \Phi(\mathcal{F}) \to \Psi(\mathcal{F}),$$

such that

$$\Psi(\chi) \circ \alpha_{\mathrm{Dom}\,\chi} = \alpha_{\mathrm{Cod}\,\chi} \circ \Phi(\chi) \qquad \text{for all} \quad \chi \in \mathrm{ISO}; \tag{12.13}$$

i.e. the diagram

$$\begin{array}{ccc}
\Phi(\mathrm{Dom}\,\chi) & \xrightarrow{\alpha_{\mathrm{Dom}\,\chi}} & \Psi(\mathrm{Dom}\,\chi) \\
{\scriptstyle \Phi(\chi)}\big\downarrow & & \big\downarrow{\scriptstyle \Psi(\chi)} \\
\Phi(\mathrm{Cod}\,\chi) & \xrightarrow[\alpha_{\mathrm{Cod}\,\chi}]{} & \Psi(\mathrm{Cod}\,\chi)
\end{array}$$

is commutative. We write $\alpha : \Phi \longrightarrow \Psi$ to indicate that Φ is the **domain isofunctor**, denoted by Dmf_α, and Ψ is the **codomain isofunctor**, denoted by Cdf_α.

One can construct new natural assignments from given ones in the same way as new mappings from given ones. Let natural assignments $\alpha : \Phi \to \Psi$ and $\beta : \Psi \to \Theta$ be given. We can define the **composite assignment** $\beta \circ \alpha : \Phi \to \Theta$, by assigning to each object $\mathcal{F}$ of $\mathrm{Dom}\,\Phi = \mathrm{Dom}\,\Psi$ the mapping $(\beta \circ \alpha)_{\mathcal{F}} := \beta_{\mathcal{F}} \circ \alpha_{\mathcal{F}}$. If α, β are natural assignment, one can define the **product-assignment** $\alpha \times \beta$ by assigning to each pair $(\mathcal{F}, \mathcal{G})$ of objects the mapping $(\alpha \times \beta)_{(\mathcal{F},\mathcal{G})} := \alpha_{\mathcal{F}} \times \beta_{\mathcal{G}}$.

Given a natural assignment $\alpha : \Phi \to \Psi$ and a isofunctor Θ such that $\mathrm{Cod}\,\Theta = \mathrm{Dom}\,\Phi = \mathrm{Dom}\,\Psi$, one can define the **composite assignment**

$\alpha \circ \Theta : \Phi \circ \Theta \to \Psi \circ \Theta$ by assigning to each object $\mathcal{F}$ of $\operatorname{Dom} \Phi = \operatorname{Dom} \Psi$ the mapping $(\alpha \circ \Theta)_{\mathcal{F}} := \alpha_{\Theta(\mathcal{F})}$.

13. Tensor Functors

We say that an isocategory ISO is **concrete** if ISO consists of mappings, the object-class *OBJ* consists of sets, and if domain and codomain, composition, identity and inverse have the meanning they are usually given for sets and mappings. (See, e.g. Sect. 01 – 04 of [FDS]).

Examples of concrete isocategory

The following are some concrete isocategories to be used in this book:

(A) The category FIS whose object-class *FS* consists of all finite dimensional flat spaces over and whose ISOmorphism-class FIS consists of all flat isomorphism from one such space onto another or itself.

(B) Fix a field and we consider the concrete isocategory whose object-class *LS* consists of all finite dimensional linear spaces over and whose ISOmorphism-class LIS consists of all linear isomorphism from one such space onto another or itself.

(C) Given $s \in$, the category DIF^s whose object-class *DF* consists of all C^s manifolds and whose ISOmorphism-class DIF^s consists of all diffeomorphism from one such manifold onto another or itself.

From now on, *in this section*, we will deal only with LIS and the categories obtained from it by product formation, such as $\mathrm{LIS}^m \times \mathrm{LIS}^n$ when $m, n \in$. We use the term **tensor functor of degree** $n \in$ for functor from LIS^n to LIS. (Under this definition, composition of tensor functors is somewhat strange: the second one of those functors must be of degree 1!!!!!!!!!!!!!)

Examples of tensor functor

Here is a list of important tensor functors used in linear algebra and differential geometry:

(1) The **product-space functor** $\mathrm{Pr} : \mathrm{LIS}^2 \to \mathrm{LIS}$. It is defined by

$$\mathrm{Pr}(\mathbf{A}, \mathbf{B}) := \mathbf{A} \times \mathbf{B} \quad \text{for all} \quad (\mathbf{A}, \mathbf{B}) \in \mathrm{LIS}^2. \tag{13.1}$$

We have $\mathrm{Pr}(\mathcal{V}, \mathcal{W}) := \mathcal{V} \times \mathcal{W}$ (the *product-space* of $\mathcal{V}$ and $\mathcal{W}$) for all $\mathcal{V}, \mathcal{W} \in LS$.

(2) Given $k \in$, the k-**lin-map-functor** $\mathrm{Lin}_k : \mathrm{LIS}^k \times \mathrm{LIS} \to \mathrm{LIS}$. It assigns to each list $(\mathcal{V}_i \,|\, i \in k^{]}\,)$ in LS and each $\mathcal{W} \in LS$ the linear space

$$\mathrm{Lin}_k((\mathcal{V}_i \,|\, i \in k^{]}\,), \mathcal{W}) := \mathrm{Lin}_k\Big(\mathop{\times}_{i \in k^{]}} \mathcal{V}_i \,,\, \mathcal{W}\Big) \tag{13.2}$$

of all k-multilinear mappings from $\times_{i \in k^{]}} \mathcal{V}_i$ to $\mathcal{W}$, and it assigns to every list $(\mathbf{A}_i \,|\, i \in k^{]}\,)$ in LIS and each $\mathbf{B} \in \mathrm{LIS}$ the linear mapping

$$\mathrm{Lin}_k((\mathbf{A}_i \,|\, i \in k^{]}\,)\,,\, \mathbf{B}) \tag{13.3}$$

from $\mathrm{Lin}_k\big(\times_{i \in k^{]}} \mathrm{Dom}\, \mathbf{A}_i, \mathrm{Dom}\, \mathbf{B}\big)$ to $\mathrm{Lin}_k\big(\times_{i \in k^{]}} \mathrm{Cod}\, \mathbf{A}_i, \mathrm{Cod}\, \mathbf{B}\big)$ defined by

$$\mathrm{Lin}_k((\mathbf{A}_i \,|\, i \in k^{]}\,), \mathbf{B})\mathbf{T} := \mathbf{B}\mathbf{T} \circ \mathop{\times}_{i \in k^{]}} \mathbf{A}_i^{-1} \tag{13.4}$$

for all $\mathbf{T} \in \mathrm{Lin}\big(\times_{i \in k^{]}} \mathrm{Dom}\, \mathbf{A}_i, \mathrm{Dom}\, \mathbf{B}\big)$.

When $k = 1$, $\mathrm{Lin}_1 : \mathrm{LIS} \times \mathrm{LIS} \to \mathrm{LIS}$ is called the **lin-map-functor** and abreviated by $\mathrm{Lin} := \mathrm{Lin}_1$.

(3) Given $k \in$, the k-**multilin-functor** $\mathrm{Ln}_k : \mathrm{LIS}^2 \to \mathrm{LIS}$. It is defined by

$$\mathrm{Ln}_k := \mathrm{Lin}_k \circ (\mathrm{Eq}_k \times \mathrm{Id}). \tag{13.5}$$

We have

$$\mathrm{Ln}_k(\mathbf{A}, \mathbf{B})\mathbf{T} := \mathbf{B}\mathbf{T} \circ (\mathbf{A}^{-1})^{\times k} \tag{13.6}$$

for all $\mathbf{A}, \mathbf{B} \in \mathrm{LIS}$ and all $\mathbf{T} \in \mathrm{Lin}_k((\mathrm{Dom}\, \mathbf{A})^k, \mathrm{Dom}\, \mathbf{B})$. and

$$\mathrm{Ln}_k(\mathcal{V}, \mathcal{W}) := \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W}) \tag{13.7}$$

for all $\mathcal{V}, \mathcal{W} \in LS$

There are two very important "*subfunctors*" (see [E-M]), Sm_k and Sk_k, given in following. The **symmetric-k-multilin-functor** $\mathrm{Sm}_k : \mathrm{LIS}^2 \to \mathrm{LIS}$ assigns to every pair of linear spaces $(\mathcal{V}, \mathcal{W}) \in LS^2$ the linear sapce

$$\mathrm{Sm}_k(\mathcal{V}, \mathcal{W}) := \mathrm{Sym}_k(\mathcal{V}^k, \mathcal{W}) \tag{13.8}$$

of all *symmetric* k-multilinear mappings from $\mathcal{V}^k$ to $\mathcal{W}$. It is clear that

$$\mathrm{Sm}_k(\mathbf{A}, \mathbf{B})\mathbf{T} := \mathbf{B}\mathbf{T} \circ (\mathbf{A}^{-1})^{\times k} \tag{13.9}$$

for all $\mathbf{A}, \mathbf{B} \in \mathrm{LIS}$ and all $\mathbf{T} \in \mathrm{Sym}_k((\mathrm{Dom}\, \mathbf{A})^k, \mathrm{Dom}\, \mathbf{B})$. The **skew-$k$-multilin-functor** $\mathrm{Sk}_k : \mathrm{LIS}^2 \to \mathrm{LIS}$ is defined in the same manner as Sm_k, except that $\mathrm{Sym}_k(\mathcal{V}^k, \mathcal{W})$ in (13.8) is replaced by the linear space $\mathrm{Skew}_k(\mathcal{V}^k, \mathcal{W})$ of all *skew* k-multilinear mappings from $\mathcal{V}^k$ to $\mathcal{W}$.

(4) Given $n \in$, the k-**linform-functor** Lnf_k, the k-**symform-functor** Smf_k, the k-**skewform-functor** Skf_k, all from LIS to LIS. They are defined by

$$\mathrm{Lnf}_k := \mathrm{Ln}_k \circ (\mathrm{Id}, \mathrm{Tr})\ ,\ \mathrm{Smf}_k := \mathrm{Sm}_k \circ (\mathrm{Id}, \mathrm{Tr})\ ,\ \mathrm{Skf}_k := \mathrm{Sk}_k \circ (\mathrm{Id}, \mathrm{Tr}). \tag{13.10}$$

Given $\mathcal{V} \in LS$, we have

$$\mathrm{Lnf}_k(\mathcal{V}) := \mathrm{Lin}_k(\mathcal{V}^k,), \tag{13.11}$$

the space of all k-multilinear forms on $\mathcal{V}^k$. We have

$$\mathrm{Lnf}_k(\mathbf{A})\boldsymbol{\omega} := \boldsymbol{\omega} \circ (\mathbf{A}^{-1})^{\times k} \quad \text{for all} \quad \boldsymbol{\omega} \in \mathrm{Lin}_k((\mathrm{Dom}\,\mathbf{A})^k,) \tag{13.12}$$

and all $\mathbf{A} \in$ LIS. The formulas (13.11) and (13.12) remain valid if Lin is replaced by Sym or Skew and Lnf by Smf or Skf correspondingly.

When $k = 1$, we have $\mathrm{Lnf}_1 = \mathrm{Smf}_1 = \mathrm{Skf}_1$ which is called the **duality-functor** and denoted by $\mathrm{Dl} : \mathrm{LIS} \to \mathrm{LIS}$.

(5) The **lineon-functor** $\mathrm{Ln} : \mathrm{LIS} \to \mathrm{LIS}$. It is defined by

$$\mathrm{Ln} := \mathrm{Lin} \circ \mathrm{Eq}_2. \tag{13.13}$$

We have

$$\mathrm{Ln}(\mathcal{V}) := \mathrm{Lin}(\mathcal{V}, \mathcal{V}) \quad \text{for all} \quad \mathcal{V} \in LS \tag{13.14}$$

and

$$\mathrm{Ln}(\mathbf{A})\mathbf{T} := \mathbf{A}\mathbf{T}\mathbf{A}^{-1} \quad \text{for all} \quad \mathbf{A} \in \mathrm{LIS} \ \text{ and } \ \mathbf{T} \in \mathrm{Ln}(\mathrm{Dom}\,\mathbf{A}). \tag{13.15}$$

It is clear that $\mathrm{Lin}_1 = \mathrm{Ln}_1$, however, $\mathrm{Ln}_1 \neq \mathrm{Ln}$! Notation?

Remark : In much of the literature (see [K-N], Sect. 2 of Ch.I or [M-T-W], §3.2) the use of the term "tensor" is limited to tensor functors of the form $\mathbf{T}^r_s := \mathrm{Lin} \circ (\mathrm{Lnf}_s, \mathrm{Lnf}_r) : \mathrm{LIS} \to \mathrm{LIS}$ with $r, s \in$, or to tensor functors that are naturally equivalent to one of this form. Given $\mathcal{V} \in LS$ a member of the linear space $\mathbf{T}^r_s(\mathcal{V})$ is called a "tensor of contravariant order r and covariant order s."
∎

Let a family of tensor functors $(\Phi_i \mid i \in k^{]})$ and a tensor functor Ψ with $\mathrm{Dom} \times_{i \in k^{]}} \Phi_k = \mathrm{LIS}^k = \mathrm{Dom}\,\Psi$ be given. We say that a natural assignment $\beta : \times_{i\in k^{]}} \Phi_k \to \Psi$ is a k-**linear assignment** if, for every $\mathcal{F} \in LS^k$, the mapping

$$\beta_{\mathcal{F}} : \mathop{\times}_{i \in k^{]}} \Phi_i(\mathcal{F}_i) \to \Psi(\mathcal{F}) \tag{13.16}$$

is k-linear.

The following are examples for bilinear natural assignments.

(6) Given $k \in$, the **alternating assgnment** Alt : $\mathrm{Ln}_k \to \mathrm{Sk}_k$ it assigns each pair $(\mathcal{V}, \mathcal{W}) \in LS^2$ the mapping

$$\mathrm{Alt}_{(\mathcal{V},\mathcal{W})}\mathbf{A} := \sum_{\sigma \in \mathrm{Perm}\, k^{]}} (\mathrm{sgn}\,\sigma)\mathbf{A} \circ \mathrm{T}_\sigma \tag{13.17}$$

where Perm $k^{]}$ is the permutation group of $k^{]}$ and T_σ is defined as in (11.3), for all $\mathbf{A} \in \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W})$.

(7) The **tensor product** tpr : Id × Id → Lin ∘ (Dl × Id) ∘ Sw assigns each pair $(\mathcal{V}, \mathcal{W}) \in LS^2$ the mapping

$$\mathrm{tpr}_{(\mathcal{V},\mathcal{W})} : \mathcal{V} \times \mathcal{W} \to \mathrm{Lin}(\mathcal{W}^*, \mathcal{V}) \tag{13.18}$$

defined by

$$\mathrm{tpr}_{(\mathcal{V},\mathcal{W})}(\mathbf{v}, \mathbf{w}) := \mathbf{v} \otimes \mathbf{w} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V} \text{ and } \mathbf{w} \in \mathcal{W}, \tag{13.19}$$

where $\mathbf{v} \otimes \mathbf{w}$ is the tensor product defined according to Def. 1 of Sect. 25, [FDS], with the identification $\mathcal{W} \cong \mathcal{W}^{**}$.

We use $\mathbf{v} \otimes \mathbf{w} \in \mathrm{Lin}(\mathcal{W}^*, \mathcal{V})$ but others use $\mathbf{v} \otimes \mathbf{w} \in \mathrm{Lin}(\mathcal{V}^*, \mathcal{W})$ (see e.g. [B-W]). Our definition of $\otimes$ bring up the switch functor Sw here!!!!!!!!!!!!!!!!!!!

The **wedge product** wpr : Id × Id → Lin ∘ (Dl × Id) ∘ Sw is defined by

$$\mathrm{wpr}_{(\mathcal{V},\mathcal{W})}(\mathbf{v}, \mathbf{w}) := \mathbf{v} \wedge \mathbf{w} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V} \text{ and } \mathbf{w} \in \mathcal{W}, \tag{13.20}$$

where $\mathbf{v} \wedge \mathbf{w}$ is the wedge product defined according to (12.9) of Sect. 12, [FDS], Vol.2, with the identification $\mathcal{W} \cong \mathcal{W}^{**}$.

We have wpr = $\frac{1}{2}$ Alt ∘ tpr. Need more development!!!!!!!!!!!!!!!!!!!

We now assume that the field relative to which LS and LIS are defined in above is the field of real number. Given $\mathcal{V}, \mathcal{W} \in LS$, the set

$$\mathrm{Lis}(\mathcal{V}, \mathcal{W}) := \left\{ \mathbf{A} \in \mathrm{LIS} \,\middle|\, \mathrm{Dom}\,\mathbf{A} = \mathcal{V}, \mathrm{Cod}\,\mathbf{A} = \mathcal{W} \right\} \tag{13.21}$$

is then an open subset of the linear space $\mathrm{Lin}(\mathcal{V}, \mathcal{W})$. (See, for example, the Differentiation Theorem for Inversion Mappings in Sect.68 of [FDS].).

Let a tensor functor Φ be given. For every pair of objects $(\mathcal{V}, \mathcal{W})$ of Dom Φ, we define the mapping

$$\Phi_{(\mathcal{V},\mathcal{W})} : \mathrm{Lis}(\mathcal{V}, \mathcal{W}) \to \mathrm{Lis}(\Phi(\mathcal{V}), \Phi(\mathcal{W})) \tag{13.22}$$

by

$$\Phi_{(\mathcal{V},\mathcal{W})}(\mathbf{A}) := \Phi(\mathbf{A}) \quad \text{for all} \quad \mathbf{A} \in \mathrm{Lis}(\mathcal{V}, \mathcal{W}). \tag{13.23}$$

Indeed, we can view (13.22) as a bilinear assignment from $\mathrm{Lin} = \mathrm{Ln}_1$ to $\mathrm{Lin} \circ (\Phi \times \Phi)$. The one to be used in (13.27)

$$\Phi_{(\mathcal{V},\mathcal{V})} : \mathrm{Lis}(\mathcal{V}) \to \mathrm{Lis}(\Phi(\mathcal{V}))$$

is a linear assignment from Ln to $\mathrm{Ln} \circ \Phi$ and hence whose gradient is also a linear assignment from Ln to $\mathrm{Ln} \circ \Phi$!!!!!!!!!!!!!!!!

We say that the tensor functor Φ is **analytic** if $\Phi_{(\mathcal{V},\mathcal{W})}$ is an analytic mapping for every pair of objects $(\mathcal{V}, \mathcal{W})$ of $\mathrm{Dom}\,\Phi$. We say that a natural assignment $\alpha : \Phi \to \Psi$ is an **analytic** assignment if the mapping $\alpha_{\mathcal{F}} : \Phi(\mathcal{F}) \to \Psi(\mathcal{F})$ is an analytic mapping for every object $\mathcal{F}$ of $\mathrm{Dom}\,\Phi$. All the tensor functors listed in above are in fact analytic. (The fact that they are of class C^∞ can easily be inferred from the results of Ch.6 of [FDS]. Proofs that they are analytic can be inferred, for example, from the results that will be presented in Ch.2 of Vol.2 of [FDS].)

Theorem : *Let an analytic tensor functor* Φ *be given and associate with each* $\mathcal{V} \in \mathrm{Dom}\,\Phi$ *the mapping*

$$\overset{\bullet}{\Phi}_{\mathcal{V}} : \mathrm{Ln}(\mathcal{V}) \to \mathrm{Ln}(\Phi(\mathcal{V})) \tag{13.24}$$

defined by

$$\overset{\bullet}{\Phi}_{\mathcal{V}} := \nabla_{\mathbf{1}_{\mathcal{V}}} \Phi_{(\mathcal{V},\mathcal{V})}. \tag{13.25}$$

(The gradient-notation used here is explained in [FDS], Sect.63.) *Then* $\overset{\bullet}{\Phi}$ *is a linear assignment from* Ln *to* $\mathrm{Ln} \circ \Phi$*. We call* $\overset{\bullet}{\Phi}$ *the* **derivative** *of* Φ*.*

Proof: Let a pair of objects $(\mathcal{V}, \mathcal{W})$ of $\mathrm{Dom}\,\Phi$ and $\mathbf{A} \in \mathrm{Lis}(\mathcal{V}, \mathcal{W})$ be given. It follows from (13.23), from axiom (F1), and from (12.2) that

$$\Phi_{(\mathcal{W},\mathcal{W})}(\mathbf{A}\mathbf{L}\mathbf{A}^{-1}) = \Phi(\mathbf{A})\Phi_{(\mathcal{V},\mathcal{V})}(\mathbf{L})\Phi(\mathbf{A})^{-1} \tag{13.26}$$

for all $\mathbf{L} \in \mathrm{Lis}(\mathcal{V}, \mathcal{V})$. By (13.15) we may write (13.26) as

$$\big(\Phi_{(\mathcal{W},\mathcal{W})} \circ \mathrm{Ln}(\mathbf{A})\big)(\mathbf{L}) = \big(\mathrm{Ln}(\Phi(\mathbf{A})) \circ \Phi_{(\mathcal{V},\mathcal{V})}\big)(\mathbf{L}) \tag{13.27}$$

for all $\mathbf{L} \in \mathrm{Lis}(\mathcal{V}, \mathcal{V})$. Taking the gradient of (13.27) with respect to $\mathbf{L}$ at $\mathbf{L} := \mathbf{1}_{\mathcal{V}}$ yields

$$\overset{\bullet}{\Phi}_{\mathcal{W}} \circ \mathrm{Ln}(\mathbf{A}) = (\mathrm{Ln} \circ \Phi)(\mathbf{A}) \circ \overset{\bullet}{\Phi}_{\mathcal{V}}. \tag{13.28}$$

In view of (12.13) it follows that $\overset{\bullet}{\Phi}$ is a natural assignment from Ln to $\mathrm{Ln} \circ \Phi$. The linearity of $\overset{\bullet}{\Phi}$ follows from the definition of gradient. ∎

We now list the derivatives of a few analytic tensor functors. The formulas given are valid for every $\mathcal{V} \in LS$.

(6) $\text{Ln}^{\bullet}_{\mathcal{V}} : \text{Ln}(\mathcal{V}) \to \text{Ln}(\text{Ln}(\mathcal{V}))$ is given by

$$(\text{Ln}^{\bullet}_{\mathcal{V}}\mathbf{L})\mathbf{M} = \mathbf{LM} - \mathbf{ML} \quad \text{for all} \quad \mathbf{L}, \mathbf{M} \in \text{Ln}(\mathcal{V}) \tag{13.29}$$

(This formula is an easy consequence of (13.15) and, [FDS] (68.9).).

(7) Let $k \in$ be given. In order to describe

$$(\text{Lnf}_k)^{\bullet}_{\mathcal{V}} : \text{Ln}(\mathcal{V}) \to \text{Ln}(\text{Lin}_k(\mathcal{V}^k,)), \tag{13.30}$$

we define, for every $\mathbf{L} \in \text{Ln}(\mathcal{V})$ and every $j \in k^{]}$, $D_j(\mathbf{L}) \in (\text{Ln}(\mathcal{V}))^k$ by

$$(D_j(\mathbf{L}))_i := \left\{ \begin{array}{lll} \mathbf{L} & if & i = j \\ \\ \mathbf{1}_{\mathcal{V}} & if & i \neq j \end{array} \right\} \qquad \text{for all} \quad i \in k^{]}. \tag{13.31}$$

We then have

$$((\text{Lnf}_k)^{\bullet}_{\mathcal{V}}\mathbf{L})\boldsymbol{\omega} = -\sum_{j \in k^{]}} \boldsymbol{\omega} \circ D_j(\mathbf{L}) \quad \text{for all} \quad \boldsymbol{\omega} \in \text{Lin}_k(\mathcal{V}^k,) \tag{13.32}$$

and all $\mathbf{L} \in \text{Ln}(\mathcal{V})$. The formula (13.32) remains valid if Lnf is replaced by Smf or Skf and Lin by Sym or Skew, correspondingly.

The General Chain Rule for gradients (see [FDS], Sect.63) and the definition (13.25) immediately lead to the following

Chain Rule for Analytic Tensor Functors

Let Φ and Ψ be analytic tensor functors. Then the composite functor $\Psi \circ \Phi$ is also an analytic tensor functor and we have

$$(\Psi \circ \Phi)^{\bullet} = (\Psi^{\bullet} \circ \Phi) \circ \Phi^{\bullet}, \tag{13.33}$$

where the composite assignments on the right are explained in the end of Sect.12.

For example, (13.33) shows that, for each $\mathcal{V} \in LS$,

$$(\text{Ln} \circ \text{Ln})^{\bullet}_{\mathcal{V}} : \text{Ln}(\mathcal{V}) \to \text{Ln}(\text{Ln}(\text{Ln}(\mathcal{V})))$$

is given by

$$(\text{Ln} \circ \text{Ln})^{\bullet}_{\mathcal{V}} = \text{Ln}^{\bullet}_{\text{Ln}(\mathcal{V})}\text{Ln}^{\bullet}_{\mathcal{V}}. \tag{13.34}$$

In view of (13.29.) above, (13.34) gives

$$\begin{aligned} \big(((\text{Ln} \circ \text{Ln})^{\bullet}_{\mathcal{V}}\mathbf{L})\mathbf{K}\big)\mathbf{M} &= \big((\text{Ln}^{\bullet}_{\mathcal{V}}\mathbf{L})\mathbf{K} - \mathbf{K}(\text{Ln}^{\bullet}_{\mathcal{V}}\mathbf{L})\big)\mathbf{M} \\ &= \mathbf{L}(\mathbf{KM}) - (\mathbf{KM})\mathbf{L} - \mathbf{K}(\mathbf{LM} - \mathbf{ML}) \end{aligned} \tag{13.35}$$

for all $\mathcal{V} \in LS$, all $\mathbf{K} \in \mathrm{Ln}(\mathrm{Ln}(\mathcal{V}))$, and all $\mathbf{L}, \mathbf{M} \in \mathrm{Ln}(\mathcal{V})$.

If Φ and Ψ are analytic tensor functors so is $\mathrm{Pr} \circ (\Phi, \Psi)$ and we have

$$(\mathrm{Pr} \circ (\Phi, \Psi))^{\bullet}_{\mathcal{V}} = (\Phi^{\bullet}_{\mathcal{V}} \mathbf{L}) \times \mathbf{1}_{\Psi(\mathcal{V})} + \mathbf{1}_{\Psi(\mathcal{V})} \times (\Phi^{\bullet}_{\mathcal{V}} \mathbf{L}) \tag{13.36}$$

for all $\mathcal{V} \in LS$ and all $\mathbf{L} \in \mathrm{Ln}(\mathcal{V})$.

Let α be an analytic assignment of degree $n \in$. If we associate with each $\mathcal{V} \in LS$ the mapping $(\nabla\alpha)_{\mathcal{V}} := \nabla(\alpha_{\mathcal{V}})$, the gradient of the mapping $\alpha_{\mathcal{V}}$, then $\nabla\alpha$ is again an analytic assignment of degree n and we have $\mathrm{Dmf}_{\nabla\alpha} = \mathrm{Dmf}_{\alpha}$ and $\mathrm{Cdf}_{\nabla\alpha} = \mathrm{Lin} \circ (\mathrm{Dmf}_{\alpha}, \mathrm{Cdf}_{\alpha})$. We call $\nabla\alpha$ the **gradient** of α.

Let tensor functors Φ_1, Φ_2, Ψ, all of degree $n \in$ but not necessarily analytic, be given. Each bilinear assignment $\beta : \mathrm{Pr} \circ (\Phi_1, \Phi_2) \to \Psi$ is then analytic and its gradient $\nabla\beta : \mathrm{Pr} \circ (\Phi_1, \Phi_2) \to \mathrm{Lin} \circ (\mathrm{Pr} \circ (\Phi_1, \Phi_2), \Psi)$ is given by

$$\big((\nabla\beta)_{\mathcal{V}}(\mathbf{v}_1, \mathbf{v}_2)\big)(\mathbf{u}_1, \mathbf{u}_2) = \beta_{\mathcal{V}}(\mathbf{v}_1, \mathbf{u}_2) + \beta_{\mathcal{V}}(\mathbf{u}_1, \mathbf{v}_2) \tag{13.37}$$

for all $\mathcal{V} \in LS$, all $\mathbf{v}_1, \mathbf{u}_1 \in \Phi_1(\mathcal{V})$, and all $\mathbf{v}_2, \mathbf{u}_2 \in \Phi_2(\mathcal{V})$.

If α is an analytic assignment of degree $n \in$ and if Φ is any isofunctor from LIS^k to LIS^n with $k \in$, then $\alpha \circ \Phi$ is an analytic assignment of degree k and we have $\nabla(\alpha \circ \Phi) = (\nabla\alpha) \circ \Phi$.

14. Short Exact Sequences

Let a pair $(\mathbf{I},\mathbf{P})$ of mappings be given such that $\operatorname{Cod}\mathbf{I} = \operatorname{Dom}\mathbf{P}$. We often write

$$\mathcal{U} \xrightarrow{\mathbf{I}} \mathcal{V} \xrightarrow{\mathbf{P}} \mathcal{W} \quad \text{or} \quad \mathcal{W} \xleftarrow{\mathbf{P}} \mathcal{V} \xleftarrow{\mathbf{I}} \mathcal{U} \tag{14.1}$$

to indicate that $\mathcal{U} = \operatorname{Dom}\mathbf{I}$, $\mathcal{V} = \operatorname{Cod}\mathbf{I} = \operatorname{Dom}\mathbf{P}$ and $\operatorname{Cod}\mathbf{P} = \mathcal{W}$. If $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are linear spaces and if $\mathbf{I}$ is injective linear mapping, $\mathbf{P}$ is surjective linear mapping with

$$\operatorname{Rng}\mathbf{I} = \operatorname{Null}\mathbf{P},$$

we say that $(\mathbf{I},\mathbf{P})$, or (14.1), is a **short exact sequence** *. In the literature, a short exact sequence is often expressed as

$$\mathbf{0} \longrightarrow \mathcal{U} \xrightarrow{\mathbf{I}} \mathcal{V} \xrightarrow{\mathbf{P}} \mathcal{W} \longrightarrow \mathbf{0}.$$

Let a short exact sequence $\mathcal{U} \xrightarrow{\mathbf{I}} \mathcal{V} \xrightarrow{\mathbf{P}} \mathcal{W}$ be given.

Notation: *The set of all linear right-inverses of* $\mathbf{P}$ *is denoted by*

$$\operatorname{Riv}(\mathbf{P}) := \{\, \mathbf{K} \in \operatorname{Lin}(\mathcal{W},\mathcal{V}) \mid \mathbf{PK} = \mathbf{1}_{\mathcal{W}} \,\}, \tag{14.2}$$

and the set of all linear left-inverses of $\mathbf{I}$ *is denoted by*

$$\operatorname{Liv}(\mathbf{I}) := \{\, \mathbf{D} \in \operatorname{Lin}(\mathcal{V},\mathcal{U}) \mid \mathbf{DI} = \mathbf{1}_{\mathcal{U}} \,\}. \tag{14.3}$$

Proposition 1: *There is a bijection* $\mathbf{\Lambda} : \operatorname{Riv}(\mathbf{P}) \to \operatorname{Liv}(\mathbf{I})$ *such that, for every* $\mathbf{K} \in \operatorname{Riv}(\mathbf{P})$

$$\mathcal{U} \underset{\mathbf{\Lambda}(\mathbf{K})}{\longleftarrow} \mathcal{V} \underset{\mathbf{K}}{\longleftarrow} \mathcal{W} \tag{14.4}$$

is again a short exact sequence. We have

$$\mathbf{KP} + \mathbf{I}\mathbf{\Lambda}(\mathbf{K}) = \mathbf{1}_{\mathcal{V}} \quad \text{for all} \quad \mathbf{K} \in \operatorname{Riv}(\mathbf{P}). \tag{14.5}$$

Proof: It is easily seen that $(\mathbf{K} \mapsto \operatorname{Rng}\mathbf{K})$ is a bijection from $\operatorname{Riv}(\mathbf{P})$ to the set of all supplements of $\operatorname{Null}\mathbf{P} = \operatorname{Rng}\mathbf{I}$ in $\mathcal{V}$. Also, $(\mathbf{D} \mapsto \operatorname{Null}\mathbf{D})$ is a bijection from $\operatorname{Liv}(\mathbf{I})$ to the set of all supplements of $\operatorname{Rng}\mathbf{I} = \operatorname{Null}\mathbf{P}$ in $\mathcal{V}$. The mapping $\mathbf{\Lambda}$ is the composite of the first of these bijections with the inverse of the second one.

* The term short exact sequence comes from the more general concept of an "exact sequence" which is not needed here.

Let $\mathbf{K} \in \operatorname{Riv}(\mathbf{P})$ be given. Both $\mathbf{KP}$ and $\mathbf{I\Lambda}(\mathbf{K})$ are idempotents with $\operatorname{Rng}\mathbf{KP} = \operatorname{Rng}\mathbf{K}$ and $\operatorname{Rng}\mathbf{I\Lambda}(\mathbf{K}) = \operatorname{Rng}\mathbf{I}$. Since $\operatorname{Rng}\mathbf{K}$ and $\operatorname{Rng}\mathbf{I}$ are supplementary in $\mathcal{V}$, it follows that

$$\mathbf{KP} + \mathbf{I\Lambda}(\mathbf{K}) = \mathbf{1}_{\mathcal{V}}. \tag{14.6}$$

Since $\mathbf{K} \in \operatorname{Riv}(\mathbf{P})$ was arbitrary, the assertion follows. ∎

Proposition 2: $\operatorname{Riv}(\mathbf{P})$ *is a flat in* $\operatorname{Lin}(\mathcal{W}, \mathcal{V})$ *whose direction space is*

$$\{\, \mathbf{IL} \mid \mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U}) \,\},$$

$\operatorname{Liv}(\mathbf{I})$ *is a flat in* $\operatorname{Lin}(\mathcal{V},\mathcal{U})$ *whose direction space is*

$$\{\, -\mathbf{LP} \mid \mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U}) \,\}.$$

Proof: Given $\mathbf{K}, \mathbf{K}' \in \operatorname{Riv}(\mathbf{P})$, we have $\mathbf{1}_{\mathcal{W}} = \mathbf{P}\mathbf{K} = \mathbf{P}\mathbf{K}'$ and hence $\mathbf{P}(\mathbf{K} - \mathbf{K}') = \mathbf{0}$. It follows that $\operatorname{Rng}(\mathbf{K} - \mathbf{K}') \subset \operatorname{Null}\mathbf{P} = \operatorname{Rng}\mathbf{I}$ and hence $\mathbf{K} - \mathbf{K}' = \mathbf{IL}$ for some $\mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U})$. On the other hand, given $\mathbf{K} \in \operatorname{Riv}(\mathbf{P})$ and $\mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U})$, we have $\mathbf{P}(\mathbf{IL}) = \mathbf{0}$ and hence $\mathbf{1}_{\mathcal{W}} = \mathbf{PK} = \mathbf{P}(\mathbf{K} + \mathbf{IL})$, which implies $\mathbf{K} + \mathbf{IL} \in \operatorname{Riv}(\mathbf{P})$. These facts show that $\operatorname{Riv}(\mathbf{P})$ is a flat in $\operatorname{Lin}(\mathcal{W}, \mathcal{V})$ with direction space $\{\, \mathbf{IL} \mid \mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U}) \,\}$.

Similar arguments show that $\operatorname{Liv}(\mathbf{I})$ is a flat in $\operatorname{Lin}(\mathcal{V},\mathcal{U})$ with direction space $\{\, -\mathbf{LP} \mid \mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U}) \,\}$. ∎

Proposition 3: *Let* $\mathbf{K}$ *and* $\mathbf{K}'$ *in* $\operatorname{Riv}(\mathbf{P})$ *be given and determine* $\mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U})$ *such that* $\mathbf{K} - \mathbf{K}' = \mathbf{IL}$. *Then*

$$\mathbf{\Lambda}(\mathbf{K}) - \mathbf{\Lambda}(\mathbf{K}') = -\mathbf{LP}. \tag{14.7}$$

Proof: It follows from (14.5) that $\mathbf{KP} + \mathbf{I\Lambda}(\mathbf{K}) = \mathbf{1}_{\mathcal{V}} = \mathbf{K}'\mathbf{P} + \mathbf{I\Lambda}(\mathbf{K}')$ and hence

$$\mathbf{I}(\mathbf{\Lambda}(\mathbf{K}) - \mathbf{\Lambda}(\mathbf{K}')) = -(\mathbf{K} - \mathbf{K}')\mathbf{P}.$$

Since $\mathbf{K} - \mathbf{K}' = \mathbf{IL}$ and $\mathbf{I}$ is injective, we obtain $\mathbf{\Lambda}(\mathbf{K}) - \mathbf{\Lambda}(\mathbf{K}') = -\mathbf{LP}$. ∎

It follows from the injectivity of $\mathbf{I}$ and from the surjectivity of $\mathbf{P}$ that both the direction space $\{\mathbf{I}\}\operatorname{Lin}(\mathcal{W},\mathcal{U})$ of $\operatorname{Riv}(\mathbf{P})$ and the direction space $\operatorname{Lin}(\mathcal{W},\mathcal{U})\{\mathbf{P}\}$ of $\operatorname{Liv}(\mathbf{I})$ are naturally isomorphic to $\operatorname{Lin}(\mathcal{W},\mathcal{U})$. Hence we may and will consider $\operatorname{Lin}(\mathcal{W},\mathcal{U})$ to be the external translation space (see Conventions and Notations) of both $\operatorname{Riv}(\mathbf{P})$ and $\operatorname{Liv}(\mathbf{I})$. We have

$$\dim \operatorname{Riv}(\mathbf{P}) = (\dim \mathcal{W})(\dim \mathcal{U}) = \dim \operatorname{Liv}(\mathbf{I}). \tag{14.8}$$

Proposition 4: *The mapping* $\boldsymbol{\Lambda} : \mathrm{Riv}(\mathbf{P}) \to \mathrm{Liv}(\mathbf{I})$*, as described in Prop. 1, is a flat isomorphism whose gradient* $\nabla\boldsymbol{\Lambda} \in \mathrm{Lin}(\mathrm{Lin}(\mathcal{W},\mathcal{U}))$ *is* $-\mathbf{1}_{\mathrm{Lin}(\mathcal{W},\mathcal{U})}$*, so that*

$$\nabla\boldsymbol{\Lambda}(\mathbf{L}) = -\mathbf{L} \quad \text{for all} \quad \mathbf{L} \in \mathrm{Lin}(\mathcal{W},\mathcal{U}). \tag{14.9}$$

Proof: It follows from Prop. 2 and the identification $\mathrm{Lin}(\mathcal{W},\mathcal{U})\{\mathbf{P}\} \cong \mathrm{Lin}(\mathcal{W},\mathcal{U})$ that $\boldsymbol{\Lambda} : \mathrm{Riv}(\mathbf{P}) \to \mathrm{Liv}(\mathbf{I})$ is a flat isomorphism with $\nabla\boldsymbol{\Lambda} = -\mathbf{1}_{\mathrm{Lin}(\mathcal{W},\mathcal{U})}$. ∎

__**Notation:**__ *Let* $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$ *be given. We define the mapping*

$$\boldsymbol{\Gamma}^{\mathbf{K}} : \mathrm{Riv}(\mathbf{P}) \to \mathrm{Lin}\,(\mathcal{W},\mathcal{U})$$

by

$$\boldsymbol{\Gamma}^{\mathbf{K}}(\mathbf{K}') := -\boldsymbol{\Lambda}(\mathbf{K})\mathbf{K}' \qquad \text{for all} \quad \mathbf{K}' \in \mathrm{Riv}(\mathbf{P}). \tag{14.10}$$

Proposition 5: *For every* $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$*, the mapping* $\boldsymbol{\Gamma}^{\mathbf{K}} : \mathrm{Riv}(\mathbf{P}) \to \mathrm{Lin}\,(\mathcal{W},\mathcal{U})$ *is a flat isomorphism whose gradient* $\nabla\boldsymbol{\Gamma}^{\mathbf{K}} \in \mathrm{Lin}(\mathrm{Lin}(\mathcal{W},\mathcal{U}))$ *is* $-\mathbf{1}_{\mathrm{Lin}(\mathcal{W},\mathcal{U})}$*; i.e.*

$$\nabla\boldsymbol{\Gamma}^{\mathbf{K}}(\mathbf{L}) = -\mathbf{L} \quad \text{for all} \quad \mathbf{L} \in \mathrm{Lin}(\mathcal{W},\mathcal{U}).$$

Proof: Let $\mathbf{K}_1, \mathbf{K}_2 \in \mathrm{Riv}(\mathbf{P})$ be given; then we determine $\mathbf{L} \in \mathrm{Lin}(\mathcal{W},\mathcal{U})$ such that $\mathbf{K}_1 - \mathbf{K}_2 = \mathbf{IL}$. It follows from (14.10) and $\boldsymbol{\Lambda}(\mathbf{K})\mathbf{I} = \mathbf{1}_{\mathcal{U}}$ that

$$\boldsymbol{\Gamma}^{\mathbf{K}}(\mathbf{K}_1) - \boldsymbol{\Gamma}^{\mathbf{K}}(\mathbf{K}_2) = -\boldsymbol{\Lambda}(\mathbf{K})(\mathbf{K}_1 - \mathbf{K}_2) = -\boldsymbol{\Lambda}(\mathbf{K})(\mathbf{IL}) = -\mathbf{L}.$$

Since $\mathbf{K}_1, \mathbf{K}_2 \in \mathrm{Riv}(\mathbf{P})$ were arbitrary, the assertion follows. ∎

Proposition 6: *We have*

$$\begin{aligned} \mathbf{K} - \mathbf{K}' &= \mathbf{I}\,\boldsymbol{\Gamma}^{\mathbf{K}}(\mathbf{K}') \\ \boldsymbol{\Lambda}(\mathbf{K}) - \boldsymbol{\Lambda}(\mathbf{K}') &= -\boldsymbol{\Gamma}^{\mathbf{K}}(\mathbf{K}')\mathbf{P} \end{aligned} \tag{14.11}$$

and hence $\boldsymbol{\Gamma}^{\mathbf{K}'}(\mathbf{K}) = -\boldsymbol{\Gamma}^{\mathbf{K}}(\mathbf{K}')$ *for all* $\mathbf{K}, \mathbf{K}' \in \mathrm{Riv}(\mathbf{P})$*. Moreover,*

$$\boldsymbol{\Gamma}^{\mathbf{K}_1}(\mathbf{K}_3) - \boldsymbol{\Gamma}^{\mathbf{K}_2}(\mathbf{K}_3) = \boldsymbol{\Gamma}^{\mathbf{K}_1}(\mathbf{K}_2) \tag{14.12}$$

for all $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3 \in \mathrm{Riv}(\mathbf{P})$*.*

Proof: In view of (14.5) and (14.10), we have

$$\mathbf{K} - \mathbf{K}' = (\mathbf{KP} - \mathbf{1}_{\mathcal{V}})\mathbf{K}' = -(\mathbf{I}\,\boldsymbol{\Lambda}(\mathbf{K}))\mathbf{K}' = \mathbf{I}\,\boldsymbol{\Gamma}^{\mathbf{K}}(\mathbf{K}')$$

for all $\mathbf{K}', \mathbf{K} \in \mathrm{Riv}(\mathbf{P})$. The second equation $(14.11)_2$ follows from $(14.11)_1$ and Prop. 2 with $\mathbf{L}$ replaced by $\mathbf{\Gamma^K}(\mathbf{K}')$.

We observe from (14.11) that

$$\begin{aligned} \mathbf{I}\,\mathbf{\Gamma}^{\mathbf{K}_1}(\mathbf{K}_2) = \mathbf{K}_1 - \mathbf{K}_2 &= (\mathbf{K}_1 - \mathbf{K}_3) - (\mathbf{K}_2 - \mathbf{K}_3) \\ &= \mathbf{I}(\mathbf{\Gamma}^{\mathbf{K}_1}(\mathbf{K}_3) - \mathbf{\Gamma}^{\mathbf{K}_2}(\mathbf{K}_3)) \end{aligned}$$

for all $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3 \in \mathrm{Riv}(\mathbf{P})$. Since $\mathbf{I}$ is injective, (14.12) follows. ∎

Remark: We consider $\mathrm{Lin}(\mathcal{W},\mathcal{U})$ to be the external translation space of $\mathrm{Riv}(\mathbf{P})$. Given $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$, in view of $(14.11)_1$, we have

$$\mathbf{\Gamma^K}(\mathbf{K}') = \mathbf{K} - \mathbf{K}' \quad \text{for all} \quad \mathbf{K}' \in \mathrm{Riv}(\mathbf{P}).$$

Roughly speaking, the flat isomorphism $\mathbf{\Gamma^K} : \mathrm{Riv}(\mathbf{P}) \to \mathrm{Lin}(\mathcal{W},\mathcal{U})$ identify $\mathrm{Riv}(\mathbf{P})$ with $\mathrm{Lin}(\mathcal{W},\mathcal{U})$ by choosing $\mathbf{K}$ as the "*zero*" (or "*orgin*"). ∎

15. Brackets and Twists

We assume now that linear spaces $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{Z}$ and a short exact sequence

$$\mathrm{Lin}(\mathcal{W},\mathcal{Z}) \quad \overset{\mathbf{I}}{\longrightarrow} \quad \mathcal{V} \quad \overset{\mathbf{P}}{\longrightarrow} \quad \mathcal{W} \tag{15.1}$$

are given. Recall from Prop. 1 of Sec. 14 that to every linear right-inverse $\mathbf{K}$ of $\mathbf{P}$ there corresponds exactly one linear left-inverse $\mathbf{\Lambda}(\mathbf{K})$ of $\mathbf{I}$ such that

$$\mathrm{Lin}(\mathcal{W},\mathcal{Z}) \quad \underset{\mathbf{\Lambda}(\mathbf{K})}{\longleftarrow} \quad \mathcal{V} \quad \underset{\mathbf{K}}{\longleftarrow} \quad \mathcal{W} \tag{15.2}$$

is again a short exact sequence. In view of the identification

$$\mathrm{Lin}\big(\mathcal{W}, \mathrm{Lin}\,(\mathcal{W},\mathcal{Z})\big) \cong \mathrm{Lin}_2\,(\mathcal{W}^2,\mathcal{Z}) \tag{15.3}$$

we may identify the external translation space $\mathrm{Lin}\big(\mathcal{W}, \mathrm{Lin}\,(\mathcal{W},\mathcal{Z})\big)$ of $\mathrm{Riv}(\mathbf{P})$ with $\mathrm{Lin}_2\,(\mathcal{W}^2,\mathcal{Z})$.

Assumption : From now on, we assume that in this section, a flat $\mathcal{F}$ in $\mathrm{Riv}(\mathbf{P})$ with direction space $\{\mathbf{I}\}\mathrm{Sym}_2\,(\mathcal{W}^2,\mathcal{Z})$ is given. Here $\mathrm{Sym}_2\,(\mathcal{W}^2,\mathcal{Z})$ is regarded as a subspace of $\mathrm{Lin}_2\,(\mathcal{W}^2,\mathcal{Z}) \cong \mathrm{Lin}\big(\mathcal{W}, \mathrm{Lin}\,(\mathcal{W},\mathcal{Z})\big)$.

Proposition 1: *For every* $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{F}$,

$$(\mathbf{\Lambda}(\mathbf{K}_1)\mathbf{v})(\mathbf{P}\mathbf{v}') - (\mathbf{\Lambda}(\mathbf{K}_1)\mathbf{v}')(\mathbf{P}\mathbf{v}) = (\mathbf{\Lambda}(\mathbf{K}_2)\mathbf{v})(\mathbf{P}\mathbf{v}') - (\mathbf{\Lambda}(\mathbf{K}_2)\mathbf{v}')(\mathbf{P}\mathbf{v}) \tag{15.4}$$

holds for all $\mathbf{v}, \mathbf{v}' \in \mathcal{V}$.

Proof: Let $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{F}$ be given. Then we determine $\mathbf{L} \in \mathrm{Sym}_2(\mathcal{W}^2, \mathcal{Z})$ such that $\mathbf{K}_1 - \mathbf{K}_2 = \mathbf{IL}$. It follows from Prop.3 of Sect.14 that

$$(\mathbf{\Lambda}(\mathbf{K}_1)\mathbf{v})(\mathbf{Pv}') - (\mathbf{\Lambda}(\mathbf{K}_2)\mathbf{v})(\mathbf{Pv}') = -\mathbf{L}(\mathbf{Pv}, \mathbf{Pv}')$$

holds for all $\mathbf{v}, \mathbf{v}' \in \mathcal{V}$. By interchanging $\mathbf{v}$ and $\mathbf{v}'$ and observing that $\mathbf{L}$ is symmetric, we conclude that (15.4) follows. ▮

__Definition__: *In view of Prop. 1, the $\mathcal{F}$-**bracket** $\mathbf{B}_{\mathcal{F}} \in \mathrm{Skw}_2(\mathcal{V}^2, \mathcal{Z})$ can be defined such that*

$$\mathbf{B}_{\mathcal{F}}(\mathbf{v}, \mathbf{v}') := (\mathbf{\Lambda}(\mathbf{K})\mathbf{v})(\mathbf{Pv}') - (\mathbf{\Lambda}(\mathbf{K})\mathbf{v}')(\mathbf{Pv}) \quad \text{for all} \quad \mathbf{v}, \mathbf{v}' \in \mathcal{V} \tag{15.5}$$

is valid for all $\mathbf{K} \in \mathcal{F}$. Using the identification (15.3) *we also have*

$$\mathbf{B}_{\mathcal{F}} \in \mathrm{Lin}\big(\mathcal{V}, \mathrm{Lin}(\mathcal{V}, \mathcal{Z})\big).$$

Proposition 2: *The $\mathcal{F}$-**bracket** $\mathbf{B}_{\mathcal{F}} \in \mathrm{Lin}\big(\mathcal{V}, \mathrm{Lin}(\mathcal{V}, \mathcal{Z})\big)$ satisfies*

$$\begin{aligned} \mathbf{B}_{\mathcal{F}}(\mathbf{I}\,\mathbf{M}) = \mathbf{M}\,\mathbf{P} \qquad & \text{for all} \quad \mathbf{M} \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z}), \\ (\mathbf{B}_{\mathcal{F}}\mathbf{v})\mathbf{K} = \mathbf{\Lambda}(\mathbf{K})\mathbf{v} \qquad & \text{for all} \quad \mathbf{K} \in \mathcal{F} \text{ and all } \mathbf{v} \in \mathcal{V}. \end{aligned} \tag{15.6}$$

If $\dim \mathcal{Z} \neq 0$, *then* $\mathbf{B}_{\mathcal{F}}$ *is injective; i.e.* $\mathrm{Null}\,\mathbf{B}_{\mathcal{F}} = \{\mathbf{0}\}$.

Proof: The equations $(15.6)_1$ and $(15.6)_2$ follow from Definition (15.5) together with $\mathbf{\Lambda}(\mathbf{K})\,\mathbf{I} = \mathbf{1}_{\mathrm{Lin}(\mathcal{W},\mathcal{Z})}$ and $\mathbf{PK} = \mathbf{1}_{\mathcal{W}}$, respectively.

Let $\mathbf{v} \in \mathrm{Null}\,\mathbf{B}_{\mathcal{F}}$ be given, so that $\mathbf{B}_{\mathcal{F}}\,\mathbf{v} = \mathbf{0}$ and hence

$$\mathbf{0} = \big(\mathbf{B}_{\mathcal{F}}\mathbf{v}\big)\,\mathbf{IM} = \mathbf{B}_{\mathcal{F}}\,(\mathbf{v}, \mathbf{IM}) = -\big(\mathbf{B}_{\mathcal{F}}(\mathbf{I}\,\mathbf{M})\big)\mathbf{v}$$

for all $\mathbf{M} \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z})$. Using $(15.6)_1$, it follows that $-\mathbf{MPv} = \mathbf{0}$ for all $\mathbf{M} \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z})$, which can happen, when $\dim \mathcal{Z} \neq 0$, only if $\mathbf{Pv} = \mathbf{0}$ and hence $\mathbf{v} \in \mathrm{Null}\,\mathbf{P} = \mathrm{Rng}\,\mathbf{I}$. Thus we may choose $\mathbf{M}' \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z})$ such that $\mathbf{v} = \mathbf{IM}'$ and hence $\mathbf{B}_{\mathcal{F}}(\mathbf{IM}') = \mathbf{0}$. Using $(15.6)_1$ again, it follows that $\mathbf{M}'\,\mathbf{P} = \mathbf{0}$. Since $\mathbf{P}$ is surjective , we conclude that $\mathbf{M}' = \mathbf{0}$ and hence $\mathbf{v} = \mathbf{0}$. Since $\mathbf{v} \in \mathrm{Null}\,\mathbf{B}_{\mathcal{F}}$ was arbitrary, it follows that $\mathrm{Null}\,\mathbf{B}_{\mathcal{F}} = \{\mathbf{0}\}$. ▮

__Definition__: *The $\mathcal{F}$-**twist***

$$\mathbf{T}_{\mathcal{F}} : \mathrm{Riv}(\mathbf{P}) \to \mathrm{Skw}_2(\mathcal{W}^2, \mathcal{Z}) \tag{15.7}$$

is defined by

$$\mathbf{T}_{\mathcal{F}}(\mathbf{K}) := -\mathbf{B}_{\mathcal{F}} \circ (\mathbf{K} \times \mathbf{K}) \qquad \text{for all} \quad \mathbf{K} \in \mathrm{Riv}(\mathbf{P}), \tag{15.8}$$

where $\mathbf{B}_{\mathcal{F}}$ *is the* $\mathcal{F}$*-bracket defined by* (15.5).

Proposition 3: *For every* $\mathbf{H} \in \mathcal{F}$*, we have*

$$\mathbf{T}_{\mathcal{F}} = \mathbf{\Gamma}^{\mathbf{H}} - \mathbf{\Gamma}^{\mathbf{H}\sim} \tag{15.9}$$

where $^{\sim}$ *denotes the value-wise switch, so that* $\mathbf{\Gamma}^{\mathbf{H}\sim}(\mathbf{K})(\mathbf{s},\mathbf{t}) = \mathbf{\Gamma}^{\mathbf{H}}(\mathbf{K})(\mathbf{t},\mathbf{s})$ *for all* $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$ *and all* $\mathbf{s},\mathbf{t} \in \mathcal{W}$.

Proof: Let $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$ and $\mathbf{s},\mathbf{t} \in \mathcal{W}$ be given. By (15.8) and (15.5), we see that for every $\mathbf{H} \in \mathcal{F}$ we have

$$\begin{aligned}\mathbf{T}_{\mathcal{F}}(\mathbf{K})(\mathbf{s},\mathbf{t}) &= -\mathbf{B}_{\mathcal{F}}(\mathbf{Ks},\mathbf{Kt}) \\ &= -\mathbf{\Lambda}(\mathbf{H})(\mathbf{Ks})\mathbf{P}(\mathbf{Kt}) + \mathbf{\Lambda}(\mathbf{H})(\mathbf{Kt})\mathbf{P}(\mathbf{Ks}).\end{aligned} \tag{15.10}$$

We conclude from $\mathbf{P}\,\mathbf{K} = \mathbf{1}_{\mathcal{W}}$, (15.10) and (14.10) that

$$\mathbf{T}_{\mathcal{F}}(\mathbf{K})(\mathbf{s},\mathbf{t}) = \mathbf{\Gamma}^{\mathbf{H}}(\mathbf{K})(\mathbf{s},\mathbf{t}) - \mathbf{\Gamma}^{\mathbf{H}}(\mathbf{K})^{\sim}(\mathbf{s},\mathbf{t}).$$

Since $\mathbf{s},\mathbf{t} \in \mathcal{W}$ and $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$ were arbitrary, (15.9) follows. ∎

Remark: It is clear from (15.9) and (11.6) that

$$\mathbf{T}_{\mathcal{F}} = 2\ \mathrm{Alt} \circ \mathbf{\Gamma}^{\mathbf{H}} \qquad \text{for all} \quad \mathbf{H} \in \mathcal{F}.$$

The numerical factor 2 is conventional which reduces numerical factors in calculations. ∎

Proposition 4: *The* $\mathcal{F}$*-torsion* $\mathbf{T}_{\mathcal{F}}$ *is a surjective flat mapping whose gradient*

$$\nabla \mathbf{T}_{\mathcal{F}} \in \mathrm{Lin}\left(\mathrm{Lin}_2\left(\mathcal{W}^2, \mathcal{Z}\right), \mathrm{Skw}_2\left(\mathcal{W}^2, \mathcal{Z}\right)\right)$$

is given by

$$(\nabla \mathbf{T}_{\mathcal{F}})\mathbf{L} = \mathbf{L}^{\sim} - \mathbf{L} \tag{15.11}$$

for all $\mathbf{L} \in \mathrm{Lin}_2\left(\mathcal{W}^2, \mathcal{Z}\right)$.

Proof: Let $\mathbf{H} \in \mathcal{F}$ be given. It follows from (15.8) and (15.5)

$$\mathbf{T}_{\mathcal{F}}\left(\mathbf{H} - \tfrac{1}{2}\mathbf{IL}\right) = \mathbf{L} \qquad \text{for all} \quad \mathbf{L} \in \mathrm{Skw}_2\left(\mathcal{W}^2, \mathcal{Z}\right)$$

and hence $\mathbf{T}_{\mathcal{F}}$ is surjective.

Prop. 3 together with Prop. 4 in Sec. 14 shows that the $\mathcal{F}$-torsion $\mathbf{T}_{\mathcal{F}}$ is a flat mapping whose gradient is given by (15.11). ∎

In view of definitions (15.8), (15.5) and (15.11), we have $\mathbf{T}_{\mathcal{F}}{}^{<}(\{\mathbf{0}\}) = \mathcal{F}$.

Definition: *We say that* $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$ *is* $\mathcal{F}$-**twist-free** (*or* $\mathcal{F}$-**symmetric**) *if* $\mathbf{T}_{\mathcal{F}}(\mathbf{K}) = \mathbf{0}$, *i.e. if* $\mathbf{K} \in \mathcal{F}$.

$\mathcal{F}$ is a flat in $\mathrm{Riv}(\mathbf{P})$ with the (external) direction space $\mathrm{Sym}_2(\mathcal{W}^2, \mathcal{Z})$ and hence

$$\dim \mathbf{T}_{\mathcal{F}}{}^{<}(\{\mathbf{0}\}) = \dim \mathrm{Sym}_2(\mathcal{W}^2, \mathcal{Z}) = \frac{n(n+1)}{2} m, \tag{15.12}$$

where $n := \dim \mathcal{W}$ and $m := \dim \mathcal{Z}$. The mapping

$$\mathbf{S}_{\mathcal{F}} := \left(\mathbf{1}_{\mathrm{Riv}(\mathbf{P})} + \tfrac{1}{2}\mathbf{I}\mathbf{T}_{\mathcal{F}}\right)\Big|^{\mathbf{T}_{\mathcal{F}}{}^{<}(\{\mathbf{0}\})} \tag{15.13}$$

is the projection of $\mathrm{Riv}(\mathbf{P})$ onto $\mathbf{T}_{\mathcal{F}}^{<}(\{\mathbf{0}\})$ with $\mathrm{Null}\,\nabla\mathbf{S}_{\mathcal{F}} = \mathrm{Skw}_2(\mathcal{W}^2, \mathcal{Z})$. If $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$, we call

$$\mathbf{S}_{\mathcal{F}}(\mathbf{K}) = \mathbf{K} + \frac{1}{2}\mathbf{I}\big(\mathbf{T}_{\mathcal{F}}(\mathbf{K})\big)$$

the $\mathcal{F}$-**symmetric part** of $\mathbf{K}$.