THE NATURALIZATION AND APOTHEOSIS OF WALTER NOLL

by Clifford Truesdell, 1993

Prologue

The oldest kind of mechanics is continuum mechanics. Brilliant achievements in it were made by Archimedes, Stevin, Newton, and others. Theses were, for the most part, solutions of rather restricted, special problems.

It seems to me that in the late 1940s new kinds of continuum mechanics began to be envisioned and explored, different in spirit from the older kinds. The maximal, often difficult use of general principles in place of particular, neat instances became attractive. In a word, rigorous mathematical analysis based upon consequences of fairly general principles --- the laws of a fairly general mechanics, became appealing in preference to the guess work that the ten gurus of applied mechanics practiced.

I cannot attach a name of a leader of a group to this newer mechanics. My own experience bought me in 1950 to the Graduate Institute for Applied Mathematics at Indiana University, which had just been founded by T.Y. Thomas. The other professors were Eberhart Hopf and David Gilbarg. All students in the Institute had full scholarships and no duties except to study mathematics and do research in it. When I arrived, I found among the seven or eight graduate students two brilliant men, both guided by Gilbarg and in their last year, writing their theses: Serrin and Ericksen. Their departure left a great vacancy in the small body of students. While some good men remained, the atmosphere had lost its elan and charge.

Despite the very small sample, I began to speculate upon it. Ericksen had been a naval officer in the war in Korea; his undergraduate training had thereby been delayed. He was a grown man. Serrin came upon the scene without having first to waste his time in military service and so could carry through his studies without interruption. He was a youth. The one quality both Serrin and Ericksen shared, besides talent, was their willingness to work hard.

The second echelon of students, while qualified and diligent, did not see that a young mathematician needs to work nearly all the time, much longer than the labor unionist's forty hours per week, and cannot give up all his evenings to his wife's friends.

What could be done. Where could we find graduate students well grounded and intensely devoted to mathematical thinking? In my first trip to Europe after the war, 1949, I had visited some of the destroyed cities of Germany, where persons were living in the cellars, the only remains of their former houses, when it took a day's ingenuity and hard work to capture an egg. They were reconstructing as best they could, with primitive tools and scarcely any supplies. There was no laziness.

I thought of Georg Hamel, whom I knew only from his papers and books. He was the only German mathematician expert in mechanics who had survived the quantum revolution, the nazis, and the denazification, and he had welcomed our journal and joined its board actively. I asked him if he could send us an advanced graduate student. In fact, he had retired, and he passed the request on to his successor, Istvan Szabo. Walter Noll does not have entirely happy recollections of Szabo, but since I did not become Szabo's assistant, I could later, more than once, admire his taste in German wines.

Noll's arrival in Bloomington

Walter Noll arrived in Bloomington on September 19. 1953. The walk-in should have thundered; perhaps it did, but silently.

That year, Gilbarg received and sorted the new students. He asked each of them which member of the faculty he might like to work with, and Walter chose me, so that upon his promised return to his post in Berlin as an assistant in mechanical engineering he should have gained some knowledge of the foundations of mechanics, which he then possessed but scantly. Not knowing this reason of his, I suggested that Walter might calculate solutions in a theory called hypo-elasticity, which I had recently invented, or establish asymptotic expansions for solutions in my booklet call An Essay Toward a Unified Theory of Special Functions. Good student that he was, Walter followed neither suggestion. The next day or two, he asked me if it were possible to obtain the doctorate in the course of one year. I looked into the University's catalogue, something I had never done before, and found out yes, but the rules, inflexible, would require Walter to pass a qualifying oral examination in the next few days, write a doctoral thesis which five selected professors would approve in general, then undergo a final, oral examination before an expanded committee -- all in English, of course, and within temporal bounds specified by the catalogue. Walter completed his work for the doctorate in less than ten months.

The qualifying examination was set for October 14th.

I show you the list of subject Walter offered.

List of Subjects
(offered by Noll for the preliminary doctorate examination, 1953)

Analysis

1) Theory of functions of a complex variable, conformal mapping, elliptic functions.

2) Ordinary differential equations in the real domain.

3) Ordinary differential equations in the complex domain and special functions (hypergeometric, Bessel's, Malthieu's functions).

4) Partial differential equations of first and second order, potential theory.

5) Integral equations

6) Calculus of variations.

7) Laplace and Fourier transforms.

8) Vector and tensor-analysis

9) Theory of measure and integration.

10) Banach- and Hilbert-space, functional analysis.

11) Kernel functions.

12) Lie groups.

13) Theory of distributions (L. Schwartz).

Algebra

1) Linear algebra (vector-spaces, matrices, tensors, etc.).

2.) Higher algebra (groups, fields, rings, Galois theory, representations, algebras).

3) Topological groups.

4) Theory of numbers (valuations, algebraic number fields).

Geometry and Topology

1) Differential geometry and Riemannien geometry.

2) Projective geometry.

3) Point-set topology.

4) Algebraic topology.

5) Non-Euclidean geometry.

Mechanics and Physics

1) Technical mechanics

2) Theoretical mechanics

3) Theory of elasticity

4) Electrodynamics

5) Thermodynamics

6) Optics

7) Quantum-mechanics

Unclassified

1) Set-theory

2) Symbolic logics

3) Numerical mathematics

4) Theory of games

I think only the language offered any difficulty, with minor exception I shall mention presently.

The members of a committee for the qualifying examination were Gilbarg, Gustin, a physicist, and myself. Walter handled the questions easily except for one by Gustin and one by me. I asked him to derive the laws of continuum mechanics from Newton's laws. He tried to fight with the problem, but he quickly saw the arguments he was employing were of the kind parroted by physicists. That was the only time I knew a little more than Walter did. The question was important because the time had come to clear away the Newtonian myth, centuries old, especially propagated by Mach, and to face general mechanics as an independent science, described and developed by Euler's laws of balance. As we all know, even today, the physicists have never learned them, and with the help of computers probably they never will.

Noll's thesis is Bloomington

For his thesis Noll selected a topic generalizing certain ideas of Maxwell and Zaremba: the class of constitutive relations

stress rate = f (velocity gradient, stress, density).

His title was **On the Continuity of the Solid and Fluid States**. It includes some isotropic elastic bodies, viscous fluids, and other special instances. I did not work on it except to bring to Walter's attention the paper of S. Zaremba, "Sur une conception nouvelle dans un fluide en mouvement" (1937); Zaremba had published the basic ideas in 1903. Various other persons made three-dimensional extensions of Maxwell's idea, but they were not invariant under changes of frame. Walter saw the need for this invariance, which he called "The principle of isotropy of space". Later he was to call it "the principle of objectivity" and finally *the principle of material frame-indifference*. In fact such a principle had been enunciated by Oldroyd in 1950, but we did not perceive it. (In 19A of ***The***

Non-linear Field Theories of Mechanics, which Noll and I wrote jointly and published in 1965, is a brief history of the two forms of the principle: the Hooke-Poisson-Cauchy form and the Zaremba-Jaumann form, the former being the one employed by Noll in his thesis.)
In the last part Noll provides some exact solutions of the differential equation; in particular, simple shearing flow and Poiseuille flow are studied in detail. Walter replaced Zaremba's approximate solutions by exact ones. He wrote as follows:

> In this paper no terms are ever neglected, and all solutions...are valid for finite motions and not only for small displacements or small rotations. This does not mean that they necessarily represent accurately the behavior of real materials, but the least that can be expected from a mathematical theory is that it is consistent, exact and rigorous. Although I believe that this paper is a small step toward an axiomatic foundation of continuum mechanics, there is much to be done in this direction. The final form of such an axiomatic foundation should include also thermodynamic principles, which are left out of consideration here. Modern linear algebra...is used as the mathematical tool in his work. In particular, points, vectors and linear transformations are used instead of coordinates, components and matrices.... Not only do I believe that in this way the mathematical deductions become shorter and more elegant, but I think also that such a treatment is more intuitive.

I was so old-fashioned then that I insisted the main equations be repeated in co-ordinate notations: a year or two earlier, I had been sometimes forced by editors to use Cartesian co-ordinates, and in fact decades passed before some journals of applied mathematics encouraged the worst possible general notation, namely Cartesian tensors.

Among the innovations Walter introduced is the following theorem: ***A resilient material is hypo-elastic if and only if, for a given initial stress, the stress at a final state depends only on the paths by which the material points reach the final state and not upon the rate at which they traverse these paths.***

The thesis, which is 78 pages long, is still good reading.

Walter's thesis was approved on July 26 by Gilbarg, Gustin, Hlavaty (a differential geometer) and me, as well as two physicists. Walter's defense of his thesis on August 9 was argued by the preceding group augmented by Whaples, an algebraist. I recall Walter's long argument with Whaples, about how to define a determinant in a space without co-ordinates. Walter received the doctorate on September 7. His thesis was published the next year in Volume 4 of the ***Journal of Rational Mechanics and Analysis***.

I had lectured in the fall on hydrodynamics. During the following semester and summer I conducted my lectures as a seminar in statistical mechanics. In January Dietrich Morgenstern, a friend of Walter's and also an assistant of Szabo's came to join us. He already had the Ph.D. and was preparing for Habilitation. In our seminar he presented the second existence theorem in the kinetic theory (the only preceding one having been proved by Carleman (1933)): ***General Existence and Uniqueness Proof for Spatially Homogeneous Solutions of the Maxwell-Boltzmann in the case of Maxwsellian Molecules***, Proceedings of the National Academy of Sciences U.S.A. 40,719-721 (1954) (his assumption is valid only for pseudo-molecules but they need not be Maxwellian). Later he published ***Analytical Studies Related to the Maxwell-Boltzmann Equation***, Journal of Rational Mechanics and Analysis 4,534-555 (1955). There he did not achieve a valuable conclusion. As some other analysts do, he concluded that since his analytical apparatus failed to deliver his desired theorem of existence, he wrote

> "I venture to express doubt that assertions of equal extent can hold for the general case, as long as the theory is based upon the classical Maxwell-Boltzman equation,"

and therefore he proposed

> "a new fundamental equation which follows from the usual physical argument with at least as much justice as does the classical one -- that is, none."

Hell is scattered with dead analysts who fiddled with changing equations to make them fit their analyses. In fact I had learned that Morgenstern regarded natural science as just a hoax, to be replaced by statistics. While he could not prove a good existence theorem, I published one in 1956 for Maxwellian molecules, and Arkeryd obtained somewhat broader conclusions in 1972.

During his last days in Bloomington, Walter wrote a major paper on statistical mechanics***: Die Herleitung der Grundgleichungen der Thermomechanik der Kontinua aus der statistischen Mechanik***, Journal of Rational Mechanics and Analysis 4, 627-646 (1955). That paper unscrambles one which Irving & Kirkwood had published in 1950. Kirkwood meanwhile had died, and Irving, who had been a fellow student of mine from Cal Tech, had gone off into some sort of secret work. The paper by Irving & Kirkwood was full of delta functions, infinite series, and other gobbledegook dear to hearts of physicists and quite useless for the purpose at hand. The argument is set in phase-space. Walter was able to replace the original argument by a simpler one, to obtain the stress tensor and the energy flux by explicit integration, to show that the external forces need not have a potential and may depend upon the speeds of the particles, which need not be alike. The counterparts of the fields of continuum mechanics were obtained, and so were the differential equations they satisfy. In that sense, many aspects of continuum mechanics were derived rigorously from a very general kind of statistical mechanics. It must not be forgotten, nevertheless, that the general fields of continuum mechanics themselves are not unique, and so that quantities derived from them by statistical mechanics cannot be unique either.

I have always been amazed at the neglect physicists bestow upon such precise mathematical statements as can be derived. Apparently they hate mathematics and rigor.

After Noll's paper on statistical mechanics, which I had explained to an audience of physicists in 1957 in Marburg, authors in physics went right on repeating the clumsy treatment of Irving Kirkwood. When a generation later, Walter's paper in statistical mechanics received attention and further development, it was by a mathematician: Mario Pitteri.

Concluding my remarks about Walter's stay in Bloomington, I mention that Charlotte and I entertained him frequently. Charlotte remembers inviting him to dinner three times a week for the whole year. I recall making him listen to phonograph records, especially of German and Italian music, and occasionally I recited some English and German poetry. Walter was not convinced that one or another kind of music or literature could be recognizably different in nationality. Apparently in Germany he had not experienced anything that might be called humane studies or musical performance.

Walter had been given a year's leave of absence from the Technische Hochschule, and he returned to his post there.

Noll's early life.

Since we now know so much about Walter in his maturity, I may, like an ancient bard, tell how he came to be and the vicissitudes through which he struggled. For what follows now I have a co-author: Walter himself.

Walter was born on January 7, 1925 in Berlin-Biesdorf, a district in the North-East of Berlin. His parents, Franz Noll and Martha Noll née Janssen, had lived in Berlin since about 1915. His Father's parents had immigrated from Holland to Germany in about 1890; his mother had grown up in a rural part of Northwest Germany. Both parents had eight years of elementary education. His mother was employed as a maid until she married; his father had two years of trade school and then became a tool-and-die maker. From the time he was a young man he was active in the German Social democratic Party. In 1917, during the first world war, he gave a speech at a rally of striking munitions workers. For that he had to spend the last nine months of the war in prison. When, later, the Nazis arose, he was strongly against them. When they came to power, he predicted that there would be a second world war, that the U.S. would again fight against Germany, and that Germany would again be defeated. Even when Walter was a child, he never doubted that his father's prediction would come true. The thought that he might not survive the impending disaster was never far from his mind.

In 1931 Walter entered the elementary school in Biesdorf, and soon thereafter the family built a house in Miesdorf bei Zeuthen. That was their good fortune because in the new community the family's political views were less obvious than before. The Nazis seized power in 1933. Fortunately the Nolls escaped persecution. Walter remembers his parents' discussing how their former family physician had been seized by storm troopers and beaten to death. That illustrated how Germany was being dominated by a criminal regime, long before the Western countries had grown aware of it. Other incidents of the same kind taught him never to talk about political matters except to his immediate family.

In 1935 Walter entered a secondary school near Zeuthen. At that time children deemed to be talented enough to progress to a university were sent to special academic schools. While his performance in the elementary school was mediocre, Walter's teacher thought him to have a good chance of success in the higher school. The only subject that attracted him there at first was French. He never had any formal instruction in English. He claims his facility with arithmetic and numbers almost never exceeded the mediocre. When, in his third year, geometry was added to the curriculum, life became less boring. The teacher, whose nickname given by the students was "Bommel", seems to have read the famous book called "Flatland". Bommel told the students about that and also about a four-dimensional world. When he asked for questions, Walter said "Since all matter consists of atoms and molecules, and since these are three-dimensional, there cannot be any two-dimensional objects." Bommel did not know how to answer that, so he ordered Walter to sit down. The next time the class met, Bommel asked him to report on the content of the preceding class. He ostentatiously gave Walter an A, which was remarkable because he had never given anybody anything. Walter feels the A was a reward for falling silent. He states that at this time he was extremely timid and shy, and his report cards contained comments such as "Walter is fairly bright, but he should attempt to become more outgoing."

Walter will never forget his teacher of physics in the fourth year of high school. He was named Rudolf Hohensee. Hohensee was the first person to show Walter a true understanding of physics and mathematics. He recognized Walter's talent for these subjects, and he helped him with advice and special instruction, inviting him often on Sundays to his apartment to tell him about subjects that were not covered in the classroom. By that time Walter could solves tricky problems involving constructions of triangles with ruler and compass. Hohensee's wife was also a teacher of mathematics and physics, and the Hohensees become close friends of Walter's. I was pleased to meet them during one of my visits to Berlin. Walter dedicated his Bloomington doctoral thesis to Hohensee. Hohensee is still living.

When the war began in '39, Walter was 14 years old. The best teachers were drafted into the armed forces. He studied, one way and another, by himself. He claims that he read ***Kant's Critique of Pure Reason*** at that time. I do not doubt that he began it, but I think it would be impolite if I should ask him now, in his hoary age, whether he read to the end of it. On March 23, 1943, he obtained his abitur (English translation: get out), with the designation "gut". In those days only 10% of the candidates for the abitur received it.

In this period many of Walter's classmates were killed.
In June of 1943 Walter was drafted for basic military service. A minor injury led to an infection which kept him in a hospital for a long period, and then his papers were misfiled and he was left at home for six months, during which he attended classes at the University of Berlin: Modern Algebra, Advanced Calculus, and Infinite Series, all three from real mathematics books. Finally he was enlisted into the air force signal corps, but he contracted diphtheria in the late autumn of 1944 and spent six weeks in

isolation. On May 1 he was taken prisoner of war by a British soldier in Lubeck, and he was released on July 7.

There was chaos in Germany at the time. Walter did farm work and later factory work. He devoted a large part of his free time to systemic self-study of English, for even then he felt that one could not get anywhere without knowing that language. By February of 1946 his knowledge of it was solid enough that he never again needed systematic study.

On March 12, 1946, he returned to Berlin as a student at the Technical University. In order to enroll, a student had to work 100 hours removing rubble, which lay about in great heaps. Life in Berlin was very hard for the next two years. Walter froze during the winters, and he and his family starved all the time. He found it difficult to study because his mind was always on food. At one period he participated in a discussion group meeting every two weeks in the residence of Herman B. Wells, the president of Indiana University and adviser for education in Germany. After the meetings the students were invited to stay for sandwiches, coffee, and cake. It was difficult -- he says -- for us starving students to behave in a civilized manner and not fight over the food like a pack of wolves.

Noll in the German and French Universities

Walter chose the Technical University because the Humboldt University was not to open until the following semester. Nonetheless, it was possible for a student to make mathematics his main subject in the Technical University, but there it was necessary to follow four semesters of engineering mechanics. Liese Hohensee, who at the time was an assistant, introduced Walter to Istvan Szabo. Szabo recognized that Walter was a very good student and arranged for him to be a teaching assistant. Walter followed courses also at the Humboldt University and the Free University. Walter considers that he learned as much, if not more, by systematic study of books than by attending courses.

In the late Summer of 1948 Walter participated in a course organized by the French military government. It was the time of the blockade, but the French put the students onto a plane to get them out of Berlin. The course marked the end of starvation; from that time on, Walter never again experienced severe hunger. Walter enjoyed a British exchange program for German students. There was work for two months, mainly gathering potatoes, but also there were weekend trips to Glasgow, Edinburgh, and other places.

While in the Spring and Summer of 1949 Walter continued his studies in Berlin, in October he entered upon a fellowship from the French Government which paid for tuition and room and board in Paris. He learned a lot of mathematics there, especially through participation in seminars and by independent reading. He took part in a seminar at the Ecole Normale Superieure on “Fonctions analytiques de plusieures variables”, conducted by Henri Cartan with participants Jean-Pierre Serre and Armand Borel. He states,

> the discussions were far above my head, and the experience did something for my humility.

Paris brought to Walter his first contact with the work of N. Bourbaki. He saved enough money to buy all the volumes of the ***Elements de Mathematiques*** yet published; later he studied some of the volumes systematically. In the Spring and Summer of 1950 Walter acquired “Certificats d’Etudes Superieures” through three examinations at the Sorbonne. The titles of these are not so simple as they sound. The crucial part of such an examination was the written portion, which took seven hours, all in one day. Only fifty-nine of the 296 passed the examination in “Calcul differential et integral”; of the forty-nine, only four received the grade “tres bien” and another four, “bien”, which was given to Walter. Through the examination Walter acquired the degree of “licencie es sciences”.

Returning to Berlin in August of 1950, Walter continued his studies and prepared for the doctorate. On May 5 he acquired the degree of “Diplom-Ingenieur” with the mark “Ausgezeichnet”. Despite the title, it was a degree in mathematics. The topic was “reproducing kernels, with applications to the theory of analytic functions”. Walter has always claimed that his study of pure mathematics before applications has helped him more than anything else.

A selection from Noll’s introduction to his German thesis runs as follows:

> "In the past hundred years mathematics went the way of more and more specialization and subdivision. Recently, however, a tendency for methodical unification has emerged. It has been realized that only a few fundamental structures are the basis of mathematics. The discovery of these structures leads to

> a deeper insight and above all to simplification. This development is necessary, for how else could the accumulating profusion of mathematical knowledge be preserved and transmitted?"

Walter has spent much of his life developing the science of mechanics. I leave to the audience to decide where and how stands mechanics among the fundamental structures. That, I should think, would require Walter to define the fundamental structures. Did he ever do that?

Szabo had meanwhile become senior professor of mechanics, and he gave Walter a two-year contract, renewable for two more, a regular civil service position. His principal duty was to write scientific books by Szabo. Walter put considerable effort into revision of “Hütte”, a handbook for engineers, and a major part of another of Szabo’s books. Walter states,

> "Had I been in the United States, I surely would have been named co-author of these books. On the other hand, Szabo gave me more autonomy than his other assistants, who had much more involvement than I with the unpleasant task of conducting exercise sessions for engineering mechanics courses."

Walter confesses that in this period he spent too much time playing GO.

Walter’s later work

I began my biography of Walter when he had already matured, as the old chroniclers did in their epics, and after that I recounted his youthful adventures. Now pulling the skeins together, I close by stressing some of his work after he left Bloomington.

While he was still in Bloomington, Walter had remarked how easy it was to get a university job in America, and he had received two offers before returning to Germany. Walter’s contract in Berlin could not be renewed beyond the one obligatory year, and the prospects for a university position there seemed not promising. Also Walter disliked the European ever present consciousness of rank and seniority, and he preferred the frankness of the American university scene. In Bloomington he had played with the idea of eventually returning to the United States as a permanent resident. When, in 1955, Walter decided to transplant himself to the land of the free and the home of the brave, my recommendations to the nabobs and mugwumps of organized mathematics came to naught. Nonetheless, Walter had made up his mind, and he accepted an ill-paid post in a minor university located in a slum of Los Angeles.

The next year Walter accepted an offer from the Carnegie Institute of Technology. I am told that Leighton, the head of the department then, began his letter to Walter with the words "You have been nominated by Professors Truesdell and Gilbarg for appointment in our department." Later heads were not always so friendly.

In 1957 Walter wrote a long report with the title *On the Foundation of the Mechanics of Continuous Media*. In it appear for the first time the terms *principle of objectivity, principle of determinism for the stresses*, and *simple material*, but much more than that, it is an attempt to create a mathematical-conceptual framework for general mechanics. Three-dimensional continuum mechanics is the expressed setting, but point masses and other kinds of continuum mechanics are implicit for development. When it was new, this report was read by many students of continuum mechanics. It opened a new prospect on Mechanics and its branches through its compact and condensed style and its efficient, brief treatment and complete solutions of examples.

This report was never published. Over the following years various papers based directly or indirectly upon it, culminating in various unexpected new ideas. The only part of the report not taken up and developed later was the theory of Maxwellian fluids, which offers some difficulties.

More than thirty years have passed since the appearance of Walter's report. Few copies were issued, and fewer remain. They are not to be found in libraries. I show you a reprint, and at the registration desk you may take one for yourself.

The most influential, early published paper of Walter's was **A Mathematical Theory of the Mechanical Behavior of Continuous Media**, *Archive for Rational Mechanics and Analysis, Volume 2, 197-226 (1958)*. That paper, and the following ones jointly with Bernard Coleman, I am sure you all know well. There are two others from the same period which deserve more notice than they have received. First, The Foundations of Classical Mechanics in the Light of Recent Advances in Continuum Mechanics, presented at a symposium in 1957 but not published until 1959. The paper begins by answering the question that I had put to Walter in his qualifying examination:

> "It is a widespread belief even today that classical mechanics is a dead subject, that its foundations were made clear long ago, and that all that remains to be done is to solve special problems. This is not so. It is true that the mechanics of systems of a finite number of mass points has been on a sufficiently rigorous basis since Newton's day.
>
> Many textbooks on theoretical mechanics dismiss continuous bodies with the remark that they can be regarded as the limiting case of a particle system with an increasing number of particles. They cannot. The erroneous belief that they can had the unfortunate effect that no serious attempt was made for a long period to put classical continuum mechanics on a rigorous axiomatic basis. Only the recent advances in the theory of materials other than perfect fluids and linearly elastic solids have revived the interest in the foundations of classical mechanics. A clarification of these foundations is of importance also for the following reason. It is known that continuous matter is really made up of elementary particles. The basis laws governing the elementary particles are those of quantum mechanics. The science that provides the link between these basic laws and the laws describing the behavior of gross matter is statistical mechanics. At the present time this link is quite weak, partly because the mathematical difficulties are formidable, and partly because the basic laws themselves are not yet completely clear. A rigorous theory of continuum mechanics would give at least some precise information on what kind of gross behavior the basic laws ought to predict.
>
> I want to give here a brief outline of an axiomatic scheme for continuum mechanics, and I shall attempt to introduce the same level of rigor and clarity as is now customary in pure mathematics."

Walter then defines a system of body forces and a system of contact forces. Proved theorems follow:

Theorem I. There is a vector-valued function S, defined for all oriented surfaces c in a body, such that $C_p(c) = S(c)$ whenever c is a piece of the boundary.

Definition. A system of forces for a body is a family of vector-valued measures F_B such that, for each part of the body B, F_B is defined on the subsets of B and have decompositions

$$F_B = B_B + C_B$$

and (B_B) is a system of body forces and (C_B) a system of contact forces.

Theorem II. For any two separate parts P and Q of a body,

$$F_{P,Q}(P) = - F_{Q,P}(Q)$$

i.e. the resultant mutual force exerted on P by Q is equal and opposite to the resultant mutual force exerted on Q by P.

Theorem III. The contact force acting across c is opposite to the contact force acting across -c.

Thus the "reaction principle" becomes a proved theorem.
Further development delivers the stress principle.
The invariance of constitutive relations under change of frame, here attributed to Oldroyd, is reasserted and clarified.

Finally I mention another important paper, read in 1959 but not published until 1963***: La Mecanique Classique, Basee sur un Axiome d'Objectivite.*** This paper introduces a general system of forces. The fundamental axiom: for every body the working is frame-indifferent. The resultant force on every body is null, and so is the resultant torque. The two conclusions are necessary and sufficient for the truth of the fundamental axiom. (The laws of inertia are absorbed in this statement.) For the first time, forces in general are brought out as elements of a system. For each body, the function F_B is assumed to be a vector-valued measure. Particular classes of bodies require particular treatments.

The assumption holds trivially for the analytical dynamics of mass-points; in contrast, for the usual kinds of three-dimensional continuum mechanics requires a good deal of analysis. Only recently has a natural axiom been found:

(on a transparency, C.T. shows here a copy of **his Axiom on Forces in Continuum Mechanics** stated on p.156 of his book A FIRST COURSE IN RATIONAL CONTINUUM MECHANICS, Second Edition, 1991)

The treatment of this axiom began with Noll; Gurtin and Williams did much to develop it; and, finally, an essential, completing definition was given recently by Noll & Virga.

In my opinion the most important of all the discoveries about the foundations of mechanics is the theory of systems of forces. All the nonsense of traditional mechanics is swept away; the hard matter remains through mathematical arguments. The Principia has been famous from the start for its failure to tell the reader what Newton means by a force. The stupidest and commonest explanation even today is "a force is a push or a pull in a given direction." Famous philosophers have written pages and books in attempts to provide acceptable explanation of force. Mathematicians have turned their backs on clarity by introducing Lagrangians and Hamiltonians and variational principles so as to evade the true notion of force. Walter's definition replaces those evasions by a mathematical structure something like a Boolean algebra.

Finally, something must be said about Walter as a teacher. His doctoral students have been few; he is adored by some persons, of whom I confess to be one. While Weierstrass lectured to an auditorium of 100, Riemann's lectures were heard rarely by a many as thirteen listeners, and he seems not be have had much contact with students. Weierstrass attracted many hearers; Riemann had few immediate disciples, but his ulterior influence did not reach its acme until the first third of this century. Walter's following is scant but devoted. I think of him sometimes akin to St. Anthony of Padua, who, when the multitudes refused their ears, went down to the seashore and preached to the fish. I will not distinguish the fish, nor will I estimate their progress; such was his eloquence, it is told, they leapt out of the water to hear him. During his life, Anthony dwelt in a poor, bare hut. Canonization comes only posthumously, of course, after a sufficient number of miracles have been proved attributable to the corpse or the image. The cult of St. Anthony has, over the centuries, constructed a vast church with many chapels and statues. Were that kind of memorial to be elevated upon the tomb of a mathematicians' bones, his soul might not be pleased. Certainly he would not consider it a reward to be reborn as late president of the American Mathematical Society. More likely, he might hope to be remembered as are Riemann and Cantor.

I must end with Sanctifiable Walter's foreseen demise - inevitable, for Charon's bark awaits every mortal - but only the Eumenides can anticipate a terminus. Some of us - I speak now of youths and maidens, nay, even babes - may live on to learn from Walter his new concepts of symbols and relations for mathematics in general. Of course it cannot be written, but it is to be inoculated electronically into sufficiently rarefied brains. To be concrete, we might descend to grave-stones. A common mortal may truthfully and in his testament leave to his relict the choice of style and ornament she desires for his monument, but even popes and emperors have tried and failed, and so also has a great mathematician, for Jacob Bernoulli's marker, in the cloister of Basel's cathedral, though he had yearned for his beloved logarithmic spiral to be graven upon his tablet there, pictures only the common archimedian.

One broader flaw in my foresight of Walter's sanctification remains. In the records of the Roman church I find no saint already christened Walter or Gualtiero. We shall have to expect that Walter will not be elevated to tread pavements of gold but rather to Valhalla, where he will be served by buxom Valkure, proferring foaming beakers of mature mead. Even so, he will not be satisfied. If another warrior in Valhall should praise one of Walter's achievements, he will cry

but I don't do it that way any more!