Monoids, Boolean Algebras, Materially Ordered Sets

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Abstract

In this paper, the interplay between certain mathematical structures is elucidated. First, it is shown that there is a one-to-one correspondence between bounded half-lattices and commutative idempotent monoids (c.i.-monoids). Adding certain additional structural ingredients and axioms, such c.i.-momoids become Boolean algebras. There is a non-trivial one-to-one correspondence between these and what we call *materially ordered sets*, which are half -lattices that satisfy certain additional axioms. Such materially ordered sets can serve as mathematical models for certain physical systems. The correspondence between materially ordered sets and Boolean algebras can be used to show, for example, that the law of action and reaction (Newton's third law) is not an independent axiom but a consequence of fundamental balance laws.

0. Mathematical Structures

A mathematical structure is described by prescribing ingredients and postulating axioms, which are conditions that the ingredients are assumed to satisfy. In most cases, one starts with a single set and endows it with structure by specifying ingredients that are entities involving constructions from this given set. An isomorphism between two structures of the same type is an invertible mapping between the underlying sets that induces a correspondence between the ingredients. An automorphism is an isomorphism from the structured set to itself.

Given a set S endowed with a specified structure and an arbitrary invertible mapping from S to a set T, one can use this mapping to transport the structure from S to T by transporting the ingredients of S to T. The axioms for T are then automatically satisfied. In some of the cases, the set T may coincide with S and then S acquires a second structure of the same type. The mapping is an automorphism only if this second structure coincides with the given one.

These considerations will be illustrated by the content of the remainder of this paper.

Given a set S we define Sub S to be the set of all subsets of S.

Let $f : A \to B$ be a mapping with domain A and codomain B. The **image mapping** of f is the mapping $f_> : \text{Sub } A \to \text{Sub } B$ defined by

$$f_>(U) := \{f(x) \mid x \in U\} \quad \text{for all} \quad U \in \text{Sub } A. \tag{1}$$

Let $S \in \text{Sub } A$ and $T \in \text{Sub } B$ be such that $f_>(S) \subseteq T$. Then the **adjustment** $f|_S^T : S \to T$ of f is defined by

$$f|_S^T(x) := f(x) \quad \text{for all} \quad x \in S. \tag{2}$$

A **pre-monoid** is a set M endowed with structure by the prescription of a mapping $\text{cmb} : M \times M \to M$ called **combination**, which satisfies the *associative axiom*

$$\text{cmb}(\text{cmb}(a, b), c) = \text{cmb}(a, \text{cmb}(b, c)) \quad \text{for all} \quad a, b, c \in M. \tag{3}$$

A **monoid** M is a pre-monoid, with combination cmb, endowed with additional structure by the prescription of a **neutral** $\text{nt} \in M$ which satisfies the *neutrality axiom*

$$\text{cmb}(a, \text{nt}) = \text{cmb}(\text{nt}, a) = a \quad \text{for all} \quad a \in M. \tag{4}$$

Let M be a pre-monoid. It is easy to prove that M contains at most one element nt which satisfies the neutrality axiom (4). If such an element exists, we say that M is **monoidable**, because we can use nt to endow M with the natural structure of a monoid. If M contains no such element, we can add an additional element nt to M and extend the combination mapping to $M \cup \{\mathrm{nt}\}$ by defining

$$\mathrm{cmb}(a, \mathrm{nt}) = \mathrm{cmb}(\mathrm{nt}, a) = a \qquad \text{for all} \quad a \in M \cup \{\mathrm{nt}\}. \tag{5}$$

It is easy to see that (3) is still satisfied. For this reason we deal only with monoids.

A monoid is said to be **commutative** if the additional axiom

$$\mathrm{cmb}(a, b) = \mathrm{cmb}(b, a) \qquad \text{for all} \quad a, b \in M \tag{6}$$

is satisfied and is said to be **idempotent** if the axiom

$$\mathrm{cmb}(a, a) = a \qquad \text{for all} \quad a \in M \tag{7}$$

is satisfied.

Let H be a subset of a monoid M. We say H is a **submonoid** if it is stable under combination and contains nt, i.e. if

$$\mathrm{cmb}_{>}(H \times H) \subseteq H \quad \text{ and } \quad \mathrm{nt} \in H. \tag{8}$$

If this is the case, then the designation of nt as the neutral and $\mathrm{cmb}|^{H}_{H\times H}$ as the combination endows H with the natural structure of a monoid.

An **ordered set** is a set S endowed with structure by the prescription of a relation $\prec$ (read "precedes") which is reflexive, antisymmetric and transitive, i.e. which satisfies the following axioms:

$$a \prec a, \tag{9}$$

$$(a \prec b \text{ and } b \prec a) \Longrightarrow a = b, \tag{10}$$

$$(a \prec b \text{ and } b \prec c) \Longrightarrow a \prec c, \tag{11}$$

for all $a, b, c \in S$. A relation that satisfies all three of the above axioms is said to be an **order** on the set S.

Let T be a subset of an ordered set S. Then T becomes an ordered set by restricting the order of S to T.

We will use the following facts about ordered sets. We say $m \in S$ is a **maximum** of a given ordered set S with respect to the relation $\prec$ if $a \prec m$ for all $a \in S$. It turns out that S can have at most one maximum, so we write $\max_{\prec} S := m$. Given $U \in \mathrm{Sub}\ S$, and $c \in S$, we will write $U \prec c$ if $a \prec c$ for all $a \in U$. There is at most one $c \in S$ such that $U \prec c$ and $U \prec d$ implies $c \prec d$ for all $d \in S$. If such a c exists we say that U has a **supremum** with respect to $\prec$ and write $\sup_{\prec} U := c$.

The reverse relation $\succ$ (read "follows"), defined by $a \succ b :\Longleftrightarrow b \prec a$ for all $a, b \in S$, is also an order. We say that $m \in S$ is a **minimum** for S with respect to $\prec$ if m is a maximum with respect to $\succ$ so that $\min_{\prec} S := \max_{\succ} S$. Similarly, we say that U has an **infimum** with respect to $\prec$ if U has a supremum with respect to $\succ$, and so we have $\inf_{\prec} U := \sup_{\succ} U$. Whenever the maximum, minimum, supremum or infimum are mentioned they will always mean with respect to the relation $\prec$, so the subscript $\prec$ will be dropped from now on.

To summarize, $\sup U$ and $\inf U$, if they exist, are characterized by

$$U \prec \sup U \quad \text{ and } \quad U \prec c \Longrightarrow \sup U \prec c \quad \text{ for all } \quad c \in S \tag{12}$$

and

$$\inf U \prec U \quad \text{ and } \quad c \prec U \Longrightarrow c \prec \inf U \quad \text{ for all } \quad c \in S. \tag{13}$$

If every doubleton has an infimum then so has every non-empty finite set. In particular, we have

$$\inf \{a, b, c\} = \inf \{a, \inf \{b, c\}\} = \inf \{\inf \{a, b\}, c\} \quad \text{for all} \quad a, b, c \in S. \tag{14}$$

If S does not have a maximum we can add to S an element m and extend the relation $\prec$ to $S \cup \{m\}$ by defining

$$a \prec m \quad \text{for all} \quad a \in S \cup \{m\} \tag{15}$$

so that m is the maximum of $S \cup \{m\}$.

Similarly, if S doesn't have a minimum one can add an element to S which will become the minimum by extending $\prec$. Therefore, for the purposes of this paper, we only deal with ordered sets that have both a minimum and a maximum.

Notes:

(0.1) The mathematical infrastructure (notation and terminology) used here is taken, in part, from Chapter 0 of [FDS].

(0.2) When monoids are usually discussed, a multiplicative notation is often used for the combination mapping or, if the monoid is commutative, an additive notation is often used. We believe that an impartial notation should be used because often the combination mapping is neither multiplication nor addition. The symbol e is a common notation for the neutral, but we feel the symbol nt is better suited since it resembles the word "neutral" and so is more descriptive. The symbol $\leq$ is often used when talking about ordered sets but, as will be seen in Example (1.1), in general orders do not agree with the common order associated with $\leq$ on $\mathbb{R}$.

(0.3) A group G is a monoid, with combination cmb and neutral nt, endowed with additional structure by the prescription of a *reversion* mapping $\mathrm{rev} : G \to G$ satisfying the reversion axiom

$$\mathrm{cmb}(\mathrm{rev}(a), a) = \mathrm{nt} = \mathrm{cmb}(a, \mathrm{rev}(a)) \qquad \text{for all} \quad a \in G. \tag{16}$$

Let G be a monoid. It is easy to prove that there can be at most one mapping $\mathrm{rev} : G \to G$ that satisfies (16). If such a mapping exists, we say that G is *groupable* because we can use rev to endow G with the structure of a group. The procedure for converting a pre-monoid into a monoid has no analogue for converting monoids into groups.

(0.4) In our view, the study of monoids and ordered sets is very much neglected in most undergraduate curricula. For example, certain types of monoids can be used to define abstract versions of divisibility, prime elements and prime decompositions. Such treatment unifies what is usually described separately for the multiplicative monoid of the natural numbers and for the multiplicative monoid of polynomials over a field (see [N1]).

(0.5) A detailed presentation of the theory of ordered sets is given in Chs. 6 and 7 of [JS].

Examples:

(0.1) The additive structure of the set $\mathbb{N}$ of natural numbers, whose neutral is 0, is a monoid.

(0.2) The muliplicative structure of the set $\mathbb{N}$ of natural numbers, whose neutral is 1, is a monoid.

(0.3) The set $\mathrm{Map}(S, S)$ of all mappings of a gives set S to itself is a monoid. The combination is composition and the neutral is the identity mapping 1_S of S.

1. Monoids and Lattices

We will call a commutative idempotent monoid a **c.i.-monoid** for short.

Definition: *An ordered set S is called a **bounded half-lattice** if it has a maximum and every doubleton has an infimum.*

Theorem 1: *Let M be a c.i.-monoid. Define the relation $\prec$ on M by*

$$a \prec b \ :\Longleftrightarrow\ \mathrm{cmb}(a, b) = a \quad \text{for all} \quad a, b \in M. \tag{17}$$

Then $\prec$ is an order on M. Moreover, with this order

$$\max S = \mathrm{nt} \quad \text{and} \quad \mathrm{cmb}(a, b) = \inf\{a, b\} \qquad \text{for all} \quad a, b \in M \tag{18}$$

so that M has the structure of a bounded half-lattice. Conversly, if S is a bounded half-lattice ordered by $\prec$, then it becomes a c.i.-monoid by defining

$$\mathrm{nt} := \max S \quad \text{and} \quad \mathrm{cmb}(a, b) := \inf\{a, b\} \qquad \text{for all} \quad a, b \in S \tag{19}$$

Proof: Let a c.i.-monoid M be given and define the relation $\prec$ on it by (17). Since M is idempotent, we have $a \prec a$ for all $a \in M$. Let $a, b \in M$ be given and suppose $a \prec b$ and $b \prec a$. Then by (17) and (6) $a = \text{cmb}(a, b) = \text{cmb}(b, a) = b$ and so $a = b$ and thus (10) holds.

Now let $a, b, c \in M$ be given and suppose $a \prec b$ and $b \prec c$ so that, by (17), $\text{cmb}(a, b) = a$ and $\text{cmb}(b, c) = b$. Then, from the associative axiom (3) for the monoid, we have

$$a = \text{cmb}(a, b) = \text{cmb}(a, \text{cmb}(b, c)) = \text{cmb}(\text{cmb}(a, b), c) = \text{cmb}(a, c) \tag{20}$$

and hence $a \prec c$. Thus, since $a, b, c \in M$ were arbitrary, (11) holds and $\prec$ is an order.

Let $a, b, c \in M$ be given. Since $\text{cmb}(a,\ \text{cmb}(a, b)) = \text{cmb}(\text{cmb}(a, a), b) = \text{cmb}(a, b)$ by (3) and (7), it follows that $\text{cmb}(a, b) \prec a$. Similarly we have $\text{cmb}(a, b) \prec b$, so that $\text{cmb}(a, b)$ is a candidate for $\inf\{a, b\}$. Suppose $c \prec \{a, b\}$ so that $\text{cmb}(c, a) = c = \text{cmb}(c, b)$. Then

$$\text{cmb}(c, \text{cmb}(a, b)) = \text{cmb}(\text{cmb}(c, a), b) = \text{cmb}(c, b) = c. \tag{21}$$

Hence, by (17), $c \prec \text{cmb}(a, b)$ and thus by (13) $\inf\{a, b\} = \text{cmb}(a, b)$.

By the neutrality axiom for cmb involving nt, it is clear from (17) that $a \prec \text{nt}$ for all $a \in M$ so that $\text{nt} = \max M$.

For the converse assertion start by defining $\text{cmb}(a, b) := \inf\{a, b\}$ for all $a, b \in S$. Since the infimum always exists this is a well defined function from $S \times S$ to S. From the definition of infimum it is clear that this operation is commutative, and (14) says it is associative. This mapping is also idempotent since $\inf\{a, a\} = a$ for all $a \in M$. Let nt denote the maximum of S. Then by the definition of infimum, $\inf\{\text{nt}, a\} = a$ for all $a \in S$ and so nt is the neutral element for cmb. □

Definition: *A set M is a **double c.i.-monoid** if it has two commutative idempotent monoid structures, denoted by* $(M, \text{meet}, \text{tp})$ (read tp as "top") *and* $(M, \text{join}, \text{bt})$ (read bt as "bottom"), *such that*

$$\text{meet}(a, b) = a \Longleftrightarrow \text{join}(a, b) = b \tag{22}$$

Definition: *An ordered set S is said to be a **bounded lattice** if it is a bounded half-lattice with respect to the order $\prec$ and also with respect to the reverse order $\succ$.*

Equivalently, a bounded lattice is an ordered set which has a minimum and a maximum and in which every doubleton has an infimum and a supremum.

Corollary 1: *Let M be a double c.i.-monoid. Define the relation $\prec$ on M by*

$$a \prec b \ :\Longleftrightarrow\ \text{meet}(a, b) = a \qquad \text{for all} \quad a, b \in M. \tag{23}$$

Then $\prec$ is an order on M. Moreover, with this order,

$$\max S = \text{tp}, \quad \min S = \text{bt}, \quad \text{meet}(a, b) = \inf\{a, b\} \quad \text{and} \quad \text{join}(a, b) = \sup\{a, b\} \qquad \text{for all} \quad a, b \in S, \tag{24}$$

so that M has has the structure of a bounded lattice. Conversly, if S is a bounded lattice ordered by $\prec$, then it becomes a double c.i.-monoid by defining

$$\text{tp} := \max S, \quad \text{bt} := \min S, \quad \text{meet}(a, b) := \inf\{a, b\} \quad \text{and} \quad \text{join}(a, b) := \sup\{a, b\} \qquad \text{for all} \quad a, b \in S. \tag{25}$$

Proof: From Thm. 1 a c.i.-monoid is a bounded half-lattice so a double c.i.-monoid has one bounded half-lattice structure associated with the operation meet and another with the operation join. Since the meet and join operations are related through (22), the order associated with meet is just the reverse of the order associated with join so using the order associated with meet, M becomes a bounded lattice such that (24) holds.

To prove the converse assertion we use the converse assertion of Thm. 1 and the fact that a bounded lattice has the structure of two bounded half-lattices, one associated with the order and the other associated with the reverse order. Thus S has two c.i.-monoid structures. Notice if $\inf\{a, b\} = a$ we have $\sup\{a, b\} =$

b and vice versa, hence S is a double c.i.-monoid. □

Examples:

(1.1) Consider the set $\mathbb{N}$ of natural numbers. $\mathbb{N}$ has two natural orders. One, of course, is $\leq$ (less then or equal). The other is *div* (read divides), so that n *div* m means n is a divisor of m. Thm. 1 applies because $\mathbb{N}$ with *div* is a bounded lattice and so is also a double c.i.-monoid with $\mathrm{meet}(a,b) = \inf\{a,b\} = \gcd(a,b)$ (greatest common divisor) and $\mathrm{join}(a,b) = \sup\{a,b\} = \mathrm{lcm}(a,b)$ (least common multiple). We also have $\max_{div}\mathbb{N} = 0$ and $\min_{div}\mathbb{N} = 1$.

(1.2) The set of all subspaces of a given linear space V, denoted by Subsp V, is ordered by inclusion. Let $U, W \in \mathrm{Subsp}\ V$ be given. It turns out that $\mathrm{meet}(U,W) := \inf\{U,W\} = U \cap W$ and $\mathrm{join}(U,W) = \sup\{U,W\} = \mathrm{Lsp}(U \cup W)$, the linear span of $U \cup W$. Since $U \cap V = U$ and $\mathrm{Lsp}(U \cup \{0\}) = U$ for all $U \in \mathrm{Subsp}\ V$, we have $V = \max\ V = \mathrm{tp}$ and $\{0\} = \min\ V = \mathrm{bt}$. This is also a double c.i.-monoid.

(1.3) In the previous example, one can replace "linear space" by "monoid", "group", and various other concepts. Then "linear span" has to be replaced by "monoid span","group span" etc. The maximum and top are the whole set while the minimum and bottom will consist of the singleton of the appropriate neutral.

2. Boolean Algebras

Definition: *A commutative monoid is given the structure of a* ***Boolean algebra*** *by specifying an additional ingredient, a mapping* $\mathrm{cpt} : M \to M$ *called the* ***counterpart*** *mapping. In the context of a Boolean algebra we use* meet *for the combination mapping and* tp *for the neutral element. The ingredients of the Boolean algebra are required to satisfy the following additional axioms:*

(BA1) $$\mathrm{cpt} \circ \mathrm{cpt} = 1_M,$$

(BA2) $$\mathrm{meet}(a, \mathrm{cpt}(a)) = \mathrm{cpt}(\mathrm{tp}) \quad \text{for all} \quad a \in M,$$

and

(BA3) $$\mathrm{meet}(a, \mathrm{cpt}(\mathrm{meet}(\mathrm{cpt}(b), \mathrm{cpt}(c)))) = \mathrm{cpt}(\mathrm{meet}(\mathrm{cpt}(\mathrm{meet}(a,b)), \mathrm{cpt}(\mathrm{meet}(a,c)))) \quad \text{for all } a,b,c \in M.$$

We define

$$\mathrm{bt} := \mathrm{cpt}(\mathrm{tp}), \quad \mathrm{join}(a,b) := \mathrm{cpt}(\mathrm{meet}(\mathrm{cpt}(a),\mathrm{cpt}(b))) \quad \text{for all} \quad a, b \in M \tag{26}$$

and use the notation

$$a \wedge b := \mathrm{meet}(a,b), \tag{27}$$

$$a \vee b := \mathrm{join}(a,b), \tag{28}$$

$$a^{\mathrm{cpt}} := \mathrm{cpt}(a). \tag{29}$$

Then a complete set of the axioms can be written in the form:

$$(a^{\mathrm{cpt}})^{\mathrm{cpt}} = a$$

$$a \wedge b = b \wedge a, \tag{30}$$

$$(a \wedge b) \wedge c = a \wedge (b \wedge c), \tag{31}$$

$$a \wedge \mathrm{tp} = a, \tag{32}$$

$$a \wedge a^{\mathrm{cpt}} = \mathrm{bt}, \tag{33}$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \tag{34}$$

valid for all $a, b, c \in M$. In this way, we see that (BA3) is just the distributive law between the join and meet operations.

Theorem (The Duality Principle): *Let M be a Boolean algebra. Then M is naturally endowed with a second Boolean algebra structure with combination mapping* join*, neutral element* bt *and counterpart mapping* cpt*. Moreover, this new structure is obtained from the old one by the isomorphism* cpt*.*

Proof: From (BA1) it is evident that cpt is a mapping from M onto itself which is its own inverse. Thus, it can be used to transfer the existing Boolean algebra structure on M to another one. The definition (26) expresses the assertion that meet and tp are transported by cpt to join and bt, and vice versa. □

The main consequence of this result is that any formula quantified over all of M remains valid if every join is replaced by meet and top is replaced by bottom, and vice versa. We will refer to this new version of the equation as the *dual* of the original equation. For example, the dual of (33) is $a \vee a^{\mathrm{cpt}} = \mathrm{tp}$.

Proposition 1: *The following statements are true for all $a, b \in M$:*

$$a \wedge a = a \tag{35}$$

$$a \wedge \mathrm{bt} = \mathrm{bt} \tag{36}$$

$$a \wedge b = a \Longleftrightarrow a \wedge b^{\mathrm{cpt}} = \mathrm{bt} \tag{37}$$

$$a \wedge (a \vee b) = a \tag{38}$$

Proof: Let $a \in M$ be given. Then by the dual of (32), then the dual of (33), then (34), then (33) and finally the dual of (32) we have

$$a = a \ \wedge \ \mathrm{tp} = a \ \wedge (a \vee a^{\mathrm{cpt}}) = (a \ \wedge \ a) \vee (a \ \wedge \ a^{\mathrm{cpt}}) = (a \ \wedge \ a) \vee \mathrm{bt} = a \wedge a$$

which proves (35).

Let $a \in M$ be given. Then by (33), (31), (36) and (33) we obtain

$$a \ \wedge \mathrm{bt} \quad = \quad a \wedge (a \wedge a^{\mathrm{cpt}}) = (a \wedge a) \wedge a^{\mathrm{cpt}} = a \wedge a^{\mathrm{cpt}} = \mathrm{bt}$$

which proves (36).

Assume that $a \wedge b = a$. Then by (31), then (33) and finally (36) we have

$$a \wedge b^{\mathrm{cpt}} = (a \wedge b) \wedge b^{\mathrm{cpt}} = a \wedge (b \wedge b^{\mathrm{cpt}}) = a \wedge \mathrm{bt} = \mathrm{bt}.$$

Now assume that $a \wedge b^{\mathrm{cpt}} = \mathrm{bt}$. Then by (32), then the dual of (33), then (34) and finally the dual of (32) we have

$$a = a \wedge \mathrm{tp} = a \wedge (b \vee b^{\mathrm{cpt}}) = (a \wedge b) \vee (a \wedge b^{\mathrm{cpt}}) = (a \wedge b) \vee \mathrm{bt} = a \wedge b$$

hence (37) holds.

Let $a, b \in M$ be given. Then by the dual of (32), then (37), then the dual of (34), then the dual of (32) and finally (30) we obtain

$$a = a \ \vee \ \mathrm{bt} = a \vee (b \ \wedge \mathrm{bt}) = (a \vee b) \wedge (a \ \vee \mathrm{bt}) = (a \vee b) \wedge a = a \wedge (a \vee b)$$

which proves (38). □

Corollary 2: *A Boolean algebra M has the structure of a double c.i.-monoid with respect to* (M, meet, tp) *and* (M, join, bt).

Proof: From (36), (30) and the Duality Principle, it is clear that M has two monoid structures that are both commutative and idempotent. Let $a, b \in M$ be given and suppose $a \wedge b = a$. Then, by substitution and the dual of (38), $b \vee a = b \vee (a \wedge b) = b$. The implication $a \vee b = b \Longrightarrow b \wedge a = a$ follows from the Duality Principle. Hence (22) is valid, so M is a double c.i.-monoid. □

There is also the notion of a sub-Boolean algebra. Let M be a Boolean algebra and N a subset of M. N is a **subalgebra** if it is a submonoid of the underlying monoid associated with M and if $\mathrm{cpt}_{>}N \subseteq N$.

Notes:

(2.1) In the literature there are many different variations on the definition of a Boolean algebra. The most common one requires two binary operations on the set M and that each element have a counterpart. Also, it assumes almost twice the number of axioms as does our definition. This is because the concept of a counterpart mapping, which is used to prove the Duality Principle, is not introduced. Thus, for every axiom that is assumed one also has to assume its dual. By initially assuming a counterpart mapping we were able to prove the Duality Principle, while most textbooks just note it as being true. In some of the literature (38) of the previous theorem is assumed as an axiom instead of the neutrality law for the underlying monoid. This property is often called the absorption law (see page 6 of [RS]). We feel that the concept of a monoid, which is very basic and natural in mathematics, is the proper foundation for a Boolean algebra and that this absorption law is not natural. As far as we know, this is a new definition for a Boolean algebra.

(2.2) Once again we use impartial terminology for the concept of a Boolean algebra. What we call "the counterpart" is often called "the complement". However, as Example (3.2) below will show, the counterpart may differ from the set-theoretic complement. Often times 0 is used to denote the bottom and 1 to denote the top (see [J]).

Examples:

(2.1) Consider the set Sub S of all subsets of a given set S. Then Sub S is a Boolean algebra with the meet of two sets defined to be their intersection and the counterpart mapping to be the complementation.

(2.2) Let S be a set and let $\mathrm{Pred}(S)$ be the set of all predicates on S, i.e. the set of all statements, however defined, about elements of S. If meet := "and", tp := "true" and cpt := "not" then $\mathrm{Pred}(S)$ is a Boolean algebra if $=$ is replaced by the logical equivalence $\Longleftrightarrow$. The corresponding order is the implication $\Longrightarrow$.

(2.3) Given any Boolean algebra M the subset $\{\mathrm{bt}, \mathrm{tp}\}$ is a subalgebra. This is called the *trivial* subalgebra. Let $a \in M$ with $a \notin \{\mathrm{bt}, \mathrm{tp}\}$ be given and consider $\{\mathrm{bt}, a, a^{\mathrm{cpt}}, \mathrm{tp}\}$. This is also a subalgebra of M.

(2.4) Let S be an infinite set and let M be the set of all subsets of S that are either finite or cofinite (having a finite complement). Then M is a subalgebra of Sub S.

3. Materially Ordered Sets

Definition: *An ordered set M with order $\prec$ is said to be* ***materially ordered*** *if the following axioms are satisfied:*

(MO1) $$M \text{ has a maximum ma and a minimum mn.}$$

(MO2) $$\text{Every doubleton has an infimum.}$$

(MO3) $$\text{For every } a \in M \text{ there is exactly one member of } M, \text{denoted by } a^{\mathrm{rem}}, \text{ such that } \inf\{a, a^{\mathrm{rem}}\} = \mathrm{mn} \text{ and } \sup\{a, a^{\mathrm{rem}}\} = \mathrm{ma}.$$

(MO4) $$(\inf\{a, b^{\mathrm{rem}}\} = \mathrm{mn}) \implies a \prec b \qquad \text{for all} \quad a, b \in M.$$

In the context of material orders, ma is called the **material all** and mn is called the **material nothing**. Given $a \in M$, the element a^{rem} is called the **remainder** of a.

Remark: The concept of a materially ordered set was first introduced by one of us in the context of an axiomatic foundation of physical systems (see [N2]). Here M is considered to consist of the whole system and all of its parts. Given $a, b \in M$, $a \prec b$ is read "a is a part of b". The maximum ma is the "material all", i.e. the whole system, and the minimum mn is the "material nothing". The $\inf\{a, b\}$ is the "common part" of a and b, and a^{rem} is the part of the whole system ma that remains after a has been removed. With this in mind, the two conditions (MO3) and (MO4) are very natural. □

Theorem 2: *Let M be a Boolean algebra and $\prec$ the order induced by meet according to Thm. 1. Then $\prec$ is a material order with* $\mathrm{ma} := \mathrm{tp}$ *and* $\mathrm{mn} := \mathrm{bt}$.

Proof: By Cor. 2 M has the structure of a double c.i.-monoid and thus also has the structure of a bounded lattice by Thm. 1. Therefore (MO1) and (MO2) hold. Let $a, b \in M$ be given and assume $\inf\{a, b^{\mathrm{cpt}}\} = \mathrm{bt}$. By (19) with cmb := meet and (27) this is equivalent to $a \wedge b^{\mathrm{cpt}} = \mathrm{bt}$. By (37) this is equivalent to $a \wedge b = a$ and so, by (17), $a \prec b$.

Let $a \in M$ be given. Using (33) and its dual it is evident that a^{cpt} satisfies $\inf\{a, a^{\mathrm{cpt}}\} = \mathrm{bt}$ and $\sup\{a, a^{\mathrm{cpt}}\} = \mathrm{tp}$. It must be shown that a^{cpt} is the only element with this property. Let $b \in M$ satisfy $a \wedge b = \mathrm{bt}$ and $a \vee b = \mathrm{tp}$. Using the dual of (32), the dual of (34), then the dual of (33), and finally (32), we obtain

$$\begin{aligned} a^{\mathrm{cpt}} &= a^{\mathrm{cpt}} \vee \mathrm{bt} \;=\; a^{\mathrm{cpt}} \vee (a \wedge b) \\ &= (a^{\mathrm{cpt}} \vee a) \wedge (a^{\mathrm{cpt}} \vee b) \;=\mathrm{tp} \wedge (a^{\mathrm{cpt}} \vee b) \\ &= a^{\mathrm{cpt}} \vee b. \end{aligned}$$

Taking the dual of the final equation gives $a = a \wedge b^{\mathrm{cpt}}$, so that $a \prec b^{\mathrm{cpt}}$ by (17). In a similar manner, using $a \vee b = \mathrm{tp}$, one obtains $b^{\mathrm{cpt}} \prec a$ and thus $b = a^{\mathrm{cpt}}$ by (10). Thus, (MO3) holds with $a^{\mathrm{rem}} := a^{\mathrm{cpt}}$. □

Theorem 3: *Let M be a materially ordered set. Then M has the structure of a Boolean algebra with* $\mathrm{meet}(a, b) := \inf\{a, b\}$ *and* $\mathrm{cpt}(a) := a^{\mathrm{rem}}$ *for all $a, b \in M$, and* $\mathrm{tp} := \mathrm{ma}$.

This result is not trivial and will be proved in a sequence of lemmas. It follows from Thm. 1 that if $\mathrm{meet}(a, b) := \inf\{a, b\}$ then M has the structure of a commutative monoid with ma as the neutral element and whose combination mapping is meet. We use the notation $a \wedge b := \mathrm{meet}(a, b) = \inf\,\{a, b\}$ and $a \vee b := \sup\,\{a, b\}$. Since (MO3) holds, one can define the function $\mathrm{rem} := (a \mapsto a^{\mathrm{rem}}) : M \to M$.

Lemma 1: *The mapping* $\mathrm{rem} := (a \mapsto a^{\mathrm{rem}}) : M \to M$ *satisfies* (BA1) *and* (BA2). *Namely,* $\mathrm{rem} \circ \mathrm{rem} = 1_M$ *and* $a \wedge a^{\mathrm{rem}} = \mathrm{mn}$.

Proof: Let $a \in M$ be given and consider $a^{\mathrm{rem}} \in M$. Then by (MO3) there is a $(a^{\mathrm{rem}})^{\mathrm{rem}} \in M$ such that $a^{\mathrm{rem}} \wedge (a^{\mathrm{rem}})^{\mathrm{rem}} = \mathrm{mn}$ and $a^{\mathrm{rem}} \vee (a^{\mathrm{rem}})^{\mathrm{rem}} = \mathrm{ma}$. However, a also satisfies this property so by the uniqueness guaranteed by (MO3), $a = (a^{\mathrm{rem}})^{\mathrm{rem}}$. Thus $\mathrm{rem} \circ \mathrm{rem} = 1_M$. It follows immediately from (MO3) that (BA2) holds. □

Lemma 2: *Let $a, b \in M$ be given. Then the following are true:*

$$a \prec b \iff a \wedge b^{\mathrm{rem}} = \mathrm{mn} \tag{39}$$

$$a \prec b \iff b^{\mathrm{rem}} \prec a^{\mathrm{rem}} \tag{40}$$

$$a \vee b \;\text{ exists} \quad \text{and} \quad a \vee b = (a^{\mathrm{rem}} \wedge b^{\mathrm{rem}})^{\mathrm{rem}} \tag{41}$$

Proof: By (17) $a \prec b$ is equivalent to $a \wedge b = a$. Thus (39) follows directly from (37).

Using (39) we have $a \prec b$ if and only if $a \wedge b^{\mathrm{rem}} = \mathrm{mn} = b^{\mathrm{rem}} \wedge a$, which is equivalent to $b^{\mathrm{rem}} \prec a^{\mathrm{rem}}$ by (39) again. This proves (40).

Equation (40) states that the function rem is order reversing and hence changes infima to suprema and vice vera. Thus, since we have assumed that every doubleton has an infimum, every doubleton also has a supremum and so (41) holds. □

All that is left to be shown is that the distributive law holds.

Lemma 3: *Given $a_1, a_2, b \in M$, we have $b \wedge (a_1 \vee a_2) = (b \wedge a_1) \vee (b \wedge a_2)$.*

Proof: Notice that by the characterization of infimum and supremum, (12) and (13), $a_i \wedge b \prec b$ and $a_i \wedge b \prec a_i \prec (a_1 \vee a_2)$ for $i = 1, 2$. Thus, $(a_1 \wedge b) \vee (a_2 \wedge b) \prec b$ and $(a_1 \wedge b) \vee (a_2 \wedge b) \prec a_1 \vee a_2$ which together yield

$$(a_1 \wedge b) \vee (a_2 \wedge b) \prec b \wedge (a_1 \vee a_2). \tag{42}$$

In order to show that (42) remains valid if $\prec$ is replaced by $\succ$, the property (MO4) of materially ordered sets is crucial. Start by defining $c := (a_1 \wedge b) \vee (a_2 \wedge b)$ so that

$$\{a_1 \wedge b, a_2 \wedge b\} \prec c. \tag{43}$$

With this fact, along with (39), we obtain $a_1 \wedge b \wedge c^{\text{rem}} = a_2 \wedge b \wedge c^{\text{rem}} = \text{mn}$. Using Lem. 2 along with (12), we obtain

$$\begin{aligned} a_1 \wedge b \wedge c^{\text{rem}} = a_2 \wedge b \wedge c^{\text{rem}} = \text{mn} \quad &\Rightarrow \quad \{a_1, a_2\} \prec (b \wedge c^{\text{rem}})^{\text{rem}} \\ &\Rightarrow \quad a_1 \vee a_2 \prec (b \wedge c^{\text{rem}})^{\text{rem}} \\ &\Rightarrow \quad (a_1 \vee a_2) \wedge b \wedge c^{\text{rem}} = \text{mn} \\ &\Rightarrow \quad (a_1 \vee a_2) \wedge b \prec c. \end{aligned}$$

Hence, $b \wedge (a_1 \vee a_2) \prec (a_1 \wedge b) \vee (a_2 \wedge b)$, and so together with (42) and (10) we obtain the desired result. □

Proof of Thm. 3 Putting together the comments after the statement of Thm. 3, Lem. 1, Lem. 2 and Lem. 3 we have the promised result. □

Corollary 3: *Let M be a commutative monoid. Then there is at most one mapping* $\text{cpt} : M \to M$ *that satisfies* (BA1)-(BA3).

Proof: First of all, from Prop. 1, if M is not idempotent then M cannot be made into a Boolean algebra and hence there doesn't exist a mapping that satisfies (BA1)-(BA3). If M is idempotent then one can define an order on M by (17). Either this order is material or it isn't. If it is then by Thm. 3 M has the structure of a Boolean algebra and hence there exists a mapping that satisfies (BA1)-(BA3). This mapping is unique by (M03). If the order is not material then there doesn't exist a mapping that satisfies (BA1)-(BA3) because if there did exist such a mapping then by Thm. 1 M would have the structure of a materially ordered set, which is a contradiction. □

If there exists a mapping $\text{cpt} : M \to M$ that satisfies properties (BA1)-(BA3), then M is said to be **Booleanable** because this mapping endows M with the structure of a Boolean algebra.

Pitfall: Given a materially ordered set M it is possible to have a proper subset N of M that is also materially ordered. From Thm. 3, both M and N also have the structure of a Boolean algebra; however, N need not be a subalgebra of M. For an example of such a proper subset of M consider M_p, with $p \neq$ma, as defined in Thm. 4.

Notes:

(3.1) A proof of a result very similar to Cor. 3 can be found on p. 474 in [J]. The proof found there uses just the Boolean algebra structure and so is different from the proof presented here.

Examples:

(3.1) Let a set S be given. Then Sub S is materially ordered by inclusion.

(3.2) Let $\mathcal{T}$ be a topological space. Then the set of regularly open sets $\text{Ro}(\mathcal{T})$, i.e., the set of open sets that are equal to the interior of their closure, is materially ordered by inclusion. Also the set of regularly closed sets $\mathrm{Rc}(\mathcal{T})$, the set of closed sets that are equal to the closure of their interior, is materially ordered by inclusion. The remainder mapping for $\mathrm{Ro}(\mathcal{T})$ is the interior of the complement and, for $\mathrm{Rc}(\mathcal{T})$, the remainder mapping is the closure of the complement. It is not hard to see that, in general, neither the open nor the closed sets are materially ordered by inclusion.

(3.3) Let V be a (genuine) inner product space. Then Subsp V is not a Boolean algebra. One might think that by defining meet and join as in example (1.2) and then defining $\mathrm{cpt}(U) := U^{\perp}$, where $U^{\perp}$ denotes the orthogonal supplement of U, one might obtain the structure of a Boolean algebra. However, the axiom (BA3) is not satisfied, although the others are. Looking at it from the point of view of a materially ordered set, (MA3) is not satisfied.

4. Application to Physics

As was noted in the previous section, materially ordered sets have applications to physics. In this section we will outline an application that was first introduced in [N2].

Let M be a materially ordered set and V a linear space. We say that the parts p and q are **separate** if $p \wedge q = \mathrm{mn}$. We use the notation

$$(M^2)_{\mathrm{sep}} := \{(p, q) \in M^2 \mid p \wedge q = \mathrm{mn}\}. \tag{44}$$

A function $H : M \to V$ is said to be **additive** if

$$H(p \vee q) = H(p) + H(q) \ \text{ for all } (p, q) \in (M^2)_{\mathrm{sep}}. \tag{45}$$

Theorem 4: *Let M be a materially ordered set and $p \in M$. Then $M_p := \{q \in M \mid q \prec p\}$ is a materially ordered set and the remainder mapping in M_p is given by*

$$\mathrm{rem}_p := (a \mapsto a^{\mathrm{rem}} \wedge p). \tag{46}$$

Proof: Since M_p is a subset of M it is also ordered by $\prec$ and $\mathrm{mn} \prec a \prec p$ for all $a \in M_p$ so it has a minimum and a maximum. Since the infimum of every doubleton in M exists by (MO2), it is clear that every doubleton in M_p has an infimum. It is a straight forward calculation using (48) to show that $a \wedge a^{\mathrm{rem}_p} = \mathrm{mn}$ and $a \vee a^{\mathrm{rem}_p} = p$. Property (MO4) also follows from the fact that M_p is a subset of a materially ordered set. $\square$

A function $I : (M^2)_{\mathrm{sep}} \to V$ is said to be an **interaction** in M if, for all $p \in M$, both $I(\,\cdot\,, p^{\mathrm{rem}}) : M_p \to V$ and $I(p^{\mathrm{rem}}, \,\cdot\,) : M_p \to V$ are additive.

The **resultant** $\mathrm{R}_I : M \to V$ of a given interaction I in M is defined by

$$\mathrm{R}_I(p) := I(p, p^{\mathrm{rem}}). \tag{47}$$

We say that a given interaction is **skew** if

$$I(q, p) = -I(p, q) \ \text{ for all } (p, q) \in (M^2)_{\mathrm{sep}}. \tag{48}$$

Let $(a, b) \in (M^2)_{\mathrm{sep}}$ be given. Then a simple calculation, using the additivity of $I(\,\cdot\,, a^{\mathrm{rem}})$ and $I(\,\cdot\,, b^{\mathrm{rem}})$, shows that

$$I(a, b) + I(b, a) = I(a, a^{\mathrm{rem}}) + I(b, b^{\mathrm{rem}}) - I(a \vee b, (a \vee b)^{\mathrm{rem}}). \tag{49}$$

Theorem 5: *An interaction is skew if and only if its resultant is additive.*

Proof: The result follows immediately from (51) and (49). $\square$

Remark: The previous theorem is fairly easy to prove based on the Boolean algebra structure of M according to Thm. 3. It states that the law of action and reaction, which is referred to as Newton's Third Law when the interactions are interpreted as forces, is true if and only if the resultant is additive. The fact that the resultant caused by the interaction is additive is a consequence of basic balance laws. Thus, if one assumes appropriate balance laws, one can prove Newton's Third Law instead of assuming it. Such a balance law can be formulated as follows: "The resultant of a given interaction is balanced by the action of the exterior world on the system." □

Examples:

(4.1) Let a finite set S be given. Then $M := \text{Sub } S$ can be thought of as a system of discrete particles.

(4.2) It is more difficult to construct a continuous system. Let E be a Euclidean space and D a subset of E. D is said to be a **fit region** in E if it satisfies the following properties:

(F1) D is a bounded subset of E
(F2) D is regularly open (see example (3.2))
(F3) D has negligible boundary, i.e. $\text{Bdy}(D)$ can be covered by a finite number of balls whose total volume can be made arbitrarily small
(F4) D has finite perimeter as defined, for example, in [NV].

Denote the set of all fit regions in E by $\text{Fr}(E)$. Let $B \in \text{Fr}(E)$ be given and define $M := \{P \in \text{Sub } B \mid P \in \text{Fr}(E)\}$. Then one can prove that M is materially ordered by inclusion. The counterpart mapping is $\text{cpt}(P) = \text{int}(B \backslash P)$ and hence is different from the complement $B \backslash P$ of P in B. The proof that this is indeed materially ordered is highly non-trivial (see [NV]).

The Boolean algebra structure of M is a subalgebra of the Boolean algebra of the regularly open subsets of B.

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References

[FDS] Noll, W. :*Finite-Dimensional Spaces: Algebra, Geometry, and Analysis, Vol.I*, 393 pages, posted on the website *math.cmu.edu/∼ wn0g/noll* . (Originally published by Martinus Nijhoff Publishers in 1987.)

[J] Jacobson, N., *Basic Algebra I*, second edition, W. H. Freeman and Company 1985.

[N1] Noll, W., *Monoids*, lecture notes, 1992, posted on the website http://www.math.cmu.edu/∼wn0g/noll.

[N2] Noll, W., *Lectures on the Foundations of Continuum Mechanics and Thermodynamics*, Archive for Rational Mechanics and Anaysis **52** 62-92 1973. (An earlier version was presented in in 1965 in a published lecture.)

[NV] Noll, W. & Virga, E., *Fit Regions and Functions of bounded variation*, Archive for Rational Mechanics and Analysis **102**, 1-21 1988.

[JS] Schäffer, J., *Basic Language of Mathematics*, unpublished notes.

[RS] Sikorski, R., *Boolean Algebras*, second edition, Springer-Verlag 1964.