Chapter 1

Linear Spaces

This chapter is a brief survey of basic linear algebra. It is assumed that the reader is already familiar with this subject, if not with the exact terminology and notation used here. Many elementary proofs are omitted, but the experienced reader will have no difficulty supplying these proofs for himself or herself.

In this chapter the letter $\mathbb{F}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. (Actually, $\mathbb{F}$ could be an arbitrary field. To preserve the validity of certain remarks, $\mathbb{F}$ should be infinite.)

11 Basic Definitions

Definition 1: A linear space (*over* $\mathbb{F}$) *is a set* $\mathcal{V}$ *endowed with structure by the presciption of*

(i) *an operation* add: $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$, *called the* **addition** *in* $\mathcal{V}$,

(ii) *an operation* sm: $\mathbb{F} \times \mathcal{V} \to \mathcal{V}$, *called the* **scalar multiplication** *in* $\mathcal{V}$,

(iii) *an element* $\mathbf{0} \in \mathcal{V}$ *called the* **zero** of $\mathcal{V}$,

(iv) *a mapping* opp: $\mathcal{V} \to \mathcal{V}$ *called* the **opposition** *in* $\mathcal{V}$, *provided that the following axioms are satisfied:*

(A1) add $(\mathbf{u}, \text{add}(\mathbf{v},\mathbf{w})) = \text{add}(\text{add}\,(\mathbf{u},\mathbf{v}), \mathbf{w})$ *for all* $\mathbf{u}, \mathbf{v},\mathbf{w} \in \mathcal{V}$,

(A2) $\text{add}(\mathbf{u},\mathbf{v}) = \text{add}(\mathbf{v},\mathbf{u})$ *for all* $\mathbf{u},\mathbf{v} \in \mathcal{V}$,

(A3) $\text{add}(\mathbf{u}, \mathbf{0}) = \mathbf{u}$ *for all* $\mathbf{u} \in \mathcal{V}$,

(A4) $\text{add}(\mathbf{u}, \text{opp}(\mathbf{u})) = \mathbf{0}$ *for all* $\mathbf{u} \in \mathcal{V}$, (S1) $\mathrm{sm}(\xi, \mathrm{sm}(\eta, \mathbf{u})) = \mathrm{sm}(\xi\eta, \mathbf{u})$ *for all* $\xi, \eta \in \mathbb{F}$, $\mathbf{u} \in \mathcal{V}$,

(S2) $\mathrm{sm}(\xi + \eta, \mathbf{u}) = \mathrm{add}(\mathrm{sm}(\xi, \mathbf{u}), \mathrm{sm}(\eta, \mathbf{u}))$ *for all* $\xi, \eta \in \mathbb{F}$, $\mathbf{u} \in \mathcal{V}$,

(S3) $\mathrm{sm}(\xi, \mathrm{add}(\mathbf{u},\mathbf{v})) = \mathrm{add}(\mathrm{sm}(\xi, \mathbf{u}), \mathrm{sm}(\xi, \mathbf{v}))$ *for all* $\xi \in \mathbb{F}$, $\mathbf{u},\mathbf{v} \in \mathcal{V}$,

(S4) $\mathrm{sm}\ (1\mathbf{u}) = \mathbf{u}$ *for all* $\mathbf{u} \in \mathcal{V}$.

The prescription of add, $\mathbf{0}$, and opp, subject to the axioms (A1)-(A4), endows $\mathcal{V}$ with the structure of a commutative group (See Sect. 06). Thus, one can say that a linear space is a commutative group endowed with additional structure by the prescription of a scalar multiplication sm: $\mathbb{F} \times \mathcal{V} \to \mathcal{V}$ subject to the conditions (S1)-(S4).

The zero $\mathbf{0}$ of $\mathcal{V}$ and the opposition opp of $\mathcal{V}$ are uniquely determined by the operation add (see the remark on groupable pre-monoids in Sect. 06). In other words, if add, sm, $\mathbf{0}$, and opp endow $\mathcal{V}$ with the structure of a linear space and if add, sm, $\mathbf{0}'$, and opp$'$ also endow $\mathcal{V}$ with such a structure, then $\mathbf{0}' = \mathbf{0}$ and $\mathrm{opp}' = \mathrm{opp}$ and hence the structures coincide. This fact enables one to say that the prescription of two operations, add: $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and sm: $\mathbb{F} \times \mathcal{V} \to \mathcal{V}$, endow $\mathcal{V}$ with the structure of a linear space if there exist $\mathbf{0} \in \mathcal{V}$ and opp: $\mathcal{V} \to \mathcal{V}$ such that the conditions (A1)-(S4) are satisfied.

The following facts are easy consequences of the (A1)-(S4):

(I) For every $\mathbf{u},\mathbf{v} \in \mathcal{V}$ there is exactly one $\mathbf{w} \in \mathcal{V}$ such that $\mathrm{add}(\mathbf{u},\mathbf{w}) = \mathbf{v}$; in fact, $\mathbf{w}$ is given by $\mathbf{w} := \mathrm{add}(\mathbf{v}, \mathrm{opp}\ (\mathbf{u}))$.

(II) $\mathrm{sm}(\xi - \eta, \mathbf{u}) = \mathrm{add}(\mathrm{sm}\ (\xi, \mathbf{u}), \mathrm{opp}(\mathrm{sm}(\eta, \mathbf{u})))$ *for all* $\xi, \eta \in \mathbb{F}$, $\mathbf{u} \in \mathcal{V}$.

(III) $\mathrm{sm}(\xi, \mathrm{add}(\mathbf{u}, \mathrm{opp}(\mathbf{v}))) = \mathrm{add}(\mathrm{sm}(\xi, \mathbf{u}), \mathrm{opp}(\mathrm{sm}(\xi, \mathbf{v})))$ for all $\xi \in \mathbb{F}$, $\mathbf{u},\mathbf{v} \in \mathcal{V}$.

(IV) $\mathrm{sm}(-\xi, \mathbf{u}) = \mathrm{opp}(\mathrm{sm}(\xi,\mathbf{u}))$ for all $\xi \in \mathbb{F}$, $\mathbf{u} \in \mathcal{V}$.

(V) If $\xi \in \mathbb{F}$ and $\mathbf{u} \in \mathcal{V}$, then $\mathrm{sm}(\xi, \mathbf{u}) = \mathbf{0}$ if and only if $\xi = 0$ or $\mathbf{u} = \mathbf{0}$.

The field $\mathbb{F}$ acquires the structure of a linear space if we prescribe add, sm, $\mathbf{0}$ and opp for $\mathbb{F}$ by putting $\mathrm{add}(\xi, \eta) := \xi + \eta$, $\mathrm{sm}(\xi, \eta) := \xi\eta$, $\mathbf{0} := 0$, opp $(\xi) := -\xi$, so that addition, zero, and opposition in the linear space $\mathbb{F}$ have their ordinary meaning while scalar multiplication reduces to ordinary multiplication. The axioms (A1)-(S4) reduce to rules of elementary arithmetic in $\mathbb{F}$.

It is customary to use the following simplified notations in an arbitrary linear space $\mathcal{V}$:

$$\begin{array}{rcll} \mathbf{u}+\mathbf{v} & := & \text{add}(\mathbf{u},\mathbf{v}) & \text{when} \quad \mathbf{u},\mathbf{v} \in \mathcal{V}, \\ \xi\,\mathbf{u} & := & \text{sm}\,(\xi,\,\mathbf{u}) & \text{when} \quad \xi \in \mathbb{F},\, \mathbf{u} \in \mathcal{V}, \\ -\mathbf{u} & := & \text{opp}(\mathbf{u}) & \text{when} \quad \mathbf{u} \in \mathcal{V}, \\ \mathbf{u}-\mathbf{v} & := & \mathbf{u}+(-\mathbf{v}) = \text{add}\,(\mathbf{u},\,\text{opp}\,(\mathbf{v})) & \text{when} \quad \mathbf{u},\mathbf{v} \in \mathcal{V}. \end{array}$$

We will use these notations most of the time. If several linear spaces are considered at the same time, the context should make it clear which of the several addition, scalar multiplication, or opposition operations are meant when the symbols $+, -$, or juxtaposition are used. Also, when the symbols $\mathbf{0}$ or 0 are used, they denote the zero of whatever linear space the context requires. With the notations above, the axioms (A1)-(S4) and the consequences (I)-(V) translate into the following familiar rules, valid for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathcal{V}$ and all $\xi, \eta \in \mathbb{F}$:

$$\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}, \tag{11.1}$$

$$\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}, \tag{11.2}$$

$$\mathbf{u}+\mathbf{0} = \mathbf{u}, \tag{11.3}$$

$$\mathbf{u}-\mathbf{u} = \mathbf{0}, \tag{11.4}$$

$$\xi(\eta\mathbf{u}) = (\xi\eta)\mathbf{u}, \tag{11.5}$$

$$(\xi+\eta)\mathbf{u} = \xi\mathbf{u}+\eta\mathbf{u}, \tag{11.6}$$

$$\xi(\mathbf{u}+\mathbf{v}) = \xi\mathbf{u}+\xi\mathbf{v}, \tag{11.7}$$

$$1\mathbf{u} = \mathbf{u}, \tag{11.8}$$

$$\mathbf{u}+\mathbf{w}=\mathbf{v} \;\Leftrightarrow\; \mathbf{w}=\mathbf{v}-\mathbf{u}, \tag{11.9}$$

$$(\xi-\eta)\mathbf{u} = \xi\mathbf{u}-\eta\mathbf{u}, \tag{11.10}$$

$$\xi(\mathbf{u}-\mathbf{v}) = \xi\mathbf{u}-\xi\mathbf{v}, \tag{11.11}$$

$$\xi\mathbf{u}=\mathbf{0} \;\Leftrightarrow\; (\xi=0 \text{ or } u=\mathbf{0}). \tag{11.12}$$

Since the linear space $\mathcal{V}$ is a commutative monoid described with additive notation, we can consider the addition of arbitrary families with finite support of elements of $\mathcal{V}$ as explained in Sect. 07.

Notes 11

(1) The term "vector space" is very often used for what we call a "linear space". The trouble with "vector space" is that it leads one to assume that the elements are "vectors" in some sense, while in fact thay very often are objects that could not be called "vectors" by any stretch of the imagination. I prefer to use "vector" only when it has its original geometric meaning (see Def. 1 or Sect. 32).

(2) Sometimes, one finds the term "origin" for what we call the "zero" of a linear space.

12 Subspaces

Definition 1: *A non-empty subset* $\mathcal{U}$ *of a linear space* $\mathcal{V}$ *is called a* **subspace** *of* $\mathcal{V}$ *if it is stable under the addition* add *and scalar multiplication* sm *in* $\mathcal{V}$, *i.e., if*

$$\mathrm{add}_{>}(\mathcal{U} \times \mathcal{U}) \subset \mathcal{U} \quad \text{and} \quad \mathrm{sm}_{>}(\mathbb{F} \times \mathcal{U}) \subset \mathcal{U}.$$

It is easily proved that a subspace $\mathcal{U}$ of $\mathcal{V}$ must contain the zero $\mathbf{0}$ of $\mathcal{V}$ and must be invariant under opposition, so that

$$\mathbf{0} \in \mathcal{U}, \quad \mathrm{opp}_{>}(\mathcal{U}) \subset \mathcal{U}.$$

Moreoever, $\mathcal{U}$ acquires the natural structure of a linear space if the addition, scalar multiplication, and opposition in $\mathcal{U}$ are taken to be the adjustments $\mathrm{add}|_{\mathcal{U}\times\mathcal{U}}^{\mathcal{U}}$, $\mathrm{sm}|_{\mathbb{F}\times\mathcal{U}}^{\mathcal{U}}$, and $\mathrm{opp}_{|\mathcal{U}}$, respectively, and if the zero of $\mathcal{U}$ is taken to be the zero $\mathbf{0}$ of $\mathcal{V}$.

Let a linear space $\mathcal{V}$ be given. Trivial subspaces of $\mathcal{V}$ are $\mathcal{V}$ itself and the **zero-space** $\{\mathbf{0}\}$. The following facts are easily proved:

Proposition 1: *The collection of all subspaces of* $\mathcal{V}$ *is intersection stable; i.e., the intersection of any collection of subspaces of* 0 $\mathcal{V}$ *is again a subspace of* $\mathcal{V}$.

We denote the span-mapping (see Sect. 03) associated with the collection of all subspaces of $\mathcal{V}$ by Lsp and call its value Lsp $\mathcal{S}$ at a given $\mathcal{S} \in \mathrm{Sub}\,\mathcal{V}$ the **linear span** of $\mathcal{S}$. In view of Prop. 1, Lsp $\mathcal{S}$ is the smallest subspace of $\mathcal{V}$ that includes $\mathcal{S}$ (see Sect. 03). If Lsp $\mathcal{S} = \mathcal{V}$ we say that $\mathcal{S}$ **spans** $\mathcal{V}$ or that $\mathcal{V}$ is **spanned by** $\mathcal{S}$. A subset of $\mathcal{V}$ is a sub*space* if and only if it coincides with its own linear span. The linear span of the empty subset of $\mathcal{V}$ is the zero-space $\{\mathbf{0}\}$ of $\mathcal{V}$, i.e., $\mathrm{Lsp}\emptyset = \{\mathbf{0}\}$. The linear span of a singleton $\{\mathbf{v}\}$, $\mathbf{v} \in \mathcal{V}$, is the set of all scalar multiples of $\mathbf{v}$, which we denote by $\mathbb{F}\mathbf{v}$:

$$\mathrm{Lsp}\{\mathbf{v}\} = \mathbb{F}\mathbf{v} := \{\xi\mathbf{v} \mid \xi \in \mathbb{F}\}. \tag{12.1}$$

We note that the notations for member-wise sums of sets introduced in Sects. 06 and 07 can be used, in particular, if the sets are subsets of a linear space.

Proposition 2: *If* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *are subspaces of* $\mathcal{V}$, *so is their sum and*

$$\mathrm{Lsp}(\mathcal{U}_1 \cup \mathcal{U}_2) = \mathcal{U}_1 + \mathcal{U}_2. \tag{12.2}$$

More generally, if $(\mathcal{U}_i \mid i \in I)$ *is a finite family of subspaces of* $\mathcal{V}$, *so is its sum and*

$$\operatorname{Lsp}\left(\bigcup_{i\in I} \mathcal{U}_i\right) = \sum_{i\in I} \mathcal{U}_i. \tag{12.3}$$

Proposition 3: *If* $(\mathcal{S}_i \mid i \in I)$ *is a finite family of arbitrary subsets of* $\mathcal{V}$, *then*

$$\operatorname{Lsp}\left(\bigcup_{i\in I} \mathcal{S}_i\right) = \sum_{i\in I} \operatorname{Lsp} \mathcal{S}_i. \tag{12.4}$$

Definition 2: *We say that two subspaces* $\mathcal{U}_i$ *and* $\mathcal{U}_2$ *of* $\mathcal{V}$ *are* **disjunct**, *and that* $\mathcal{U}_2$ *is* **disjunct from** $\mathcal{U}_1$, *if* $\mathcal{U}_1 \cap \mathcal{U}_2 = \{\mathbf{0}\}$.

We say that two subspaces $\mathcal{U}_1$ and $\mathcal{U}_2$ *of* $\mathcal{V}$ *are* **supplementary** *in* $\mathcal{V}$ *and that* $\mathcal{U}_2$ *is a* **supplement of** $\mathcal{U}_1$ *in* $\mathcal{V}$ *if* $\mathcal{U}_1 \cap \mathcal{U}_2 = \{\mathbf{0}\}$ *and* $\mathcal{U}_1 + \mathcal{U}_2 = \mathcal{V}$.

Proposition 4: *Let* $\mathcal{U}_1, \mathcal{U}_2$ *be subspaces of* $\mathcal{V}$. *Then the following are equivalent:*

(i) $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *are supplementary in* $\mathcal{V}$.

(ii) *To every* $\mathbf{v} \in \mathcal{V}$ *corresponds exactly one pair* $(\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ *such that* $\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2$.

(iii) $\mathcal{U}_2$ *is maximal among the subspaces that are disjunct from* $\mathcal{U}_1$, i.e., $\mathcal{U}_2$ *is disjunct from* $\mathcal{U}_1$ *and not properly included in any other subspace disjunct from* $\mathcal{U}_1$.

Pitfall: One should not confuse "disjunct" with "disjoint", or "supplement" with "complement". Two subspaces, even if disjunct, are never disjoint. The complement of a subspace is never a subspace, and there is no relation between this complement and any supplement. Moreover, supplements are not unique unless the given subspace is the whole space $\mathcal{V}$ or the zero-space $\{\mathbf{0}\}$. ■

To say that two subspaces $\mathcal{U}_1$ and $\mathcal{U}_2$ are disjunct is equivalent to saying they are supplementary in $\mathcal{U}_1 + \mathcal{U}_2$.

Notes 12

(1) The phrase "subspace generated by" is often used for what we call "linear span of". The modifier "linear" is very often omitted and the linear span of a subset $\mathcal{S}$ of $\mathcal{V}$ is often simply denoted by Sp $\mathcal{S}$. I think it is important to distinguish carefully between *linear* spans and other kinds of spans. (See the discussion of span-mappings in Sect. 03 and the definition of flat spans in Prop. 7 of Sect. 32).

(2) In many textbooks the term "disjoint" is used when we say "disjunct" and the term "complementary" when we say "supplementary". Such usage clashes with the set-theoretical meanings of "disjoint" and "complementary" and greatly increases the danger of becoming a victime of the Pitfall above. It is for this reason that I introduced the terms "disjunct" and "supplementary" after consulting a thesaurus. Later, I realized that "supplementary" had already been used by others. In particular, Bourbaki has used "supplémentaire" since 1947, which I read in 1950 but had forgotten.

(3) Some people write $\mathcal{U}_1 \oplus \mathcal{U}_2$ instead of merely $\mathcal{U}_1 + \mathcal{U}_2$ when the two subspaces $\mathcal{U}_1$ and $\mathcal{U}_2$ are disjunct, and then call $\mathcal{U}_1 \oplus \mathcal{U}_2$ the "direct sum" rather than merely the "sum" of $\mathcal{U}_1$ and $\mathcal{U}_2$. This is really absurd; it indicates confusion between a property (namely disjunctness) of a pair $(\mathcal{U}_1, \mathcal{U}_2)$ of two subspaces and a property of their sum. The sum $\mathcal{U}_1 + \mathcal{U}_2$, as a subspace of $\mathcal{V}$, has no special properties, because $\mathcal{U}_1$ and $\mathcal{U}_2$ cannot be recovered from $\mathcal{U}_1 + \mathcal{U}_2$ (except when they are all zero-spaces).

13 Linear Mappings

Definition 1: *A mapping* $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ *from a linear space* $\mathcal{V}$ *to a linear space* $\mathcal{V}'$ *is said to be* **linear** *if it preserves addition and scalar multiplication, i.e., if*

$$\mathrm{add}'(\mathbf{L}(\mathbf{u}), \mathbf{L}(\mathbf{v})) = \mathbf{L}(\mathrm{add}(\mathbf{u}, \mathbf{v}) \textit{ for all } \mathbf{u}, \mathbf{v} \in \mathcal{V}$$

and

$$\mathrm{sm}'(\xi, \mathbf{L}(\mathbf{u})) = \mathbf{L}(\mathrm{sm}(\xi, \mathbf{u})) \textit{ for all } \xi \in \mathbb{F}, \mathbf{u} \in \mathcal{V}.$$

where add, sm *denote operations in* $\mathcal{V}$ *and* add$'$, sm$'$ *operations in* $\mathcal{V}'$.

For linear mappings, it is customary to omit parentheses and write $\mathbf{Lu}$ for the value of $\mathbf{L}$ at $\mathbf{u}$. In simplified notation, the rules that define linearity read

$$\mathbf{Lu} + \mathbf{Lv} = \mathbf{L}(\mathbf{u} + \mathbf{v}), \tag{13.1}$$

$$\mathbf{L}(\xi\mathbf{u}) = \xi(\mathbf{Lu}), \tag{13.2}$$

valid for all $\mathbf{u},\mathbf{v} \in \mathcal{V}$ and all $\xi \in \mathbb{F}$. It is easily seen that linear mappings $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ also preserve zero and opposition, i.e., that

$$\mathbf{L0} = \mathbf{0}' \tag{13.3}$$

when $\mathbf{0}$ and $\mathbf{0}'$ denote the zeros of $\mathcal{V}$ and $\mathcal{V}'$ respectively, and that

$$\mathbf{L}(-\mathbf{u}) = -(\mathbf{Lu}) \tag{13.4}$$

for all $\mathbf{u} \in \mathcal{V}$.

the constant mapping $\mathbf{0}'_{\mathcal{V}\to\mathcal{V}'}$, whose value is the zero $\mathbf{0}'$ of $\mathcal{V}'$, is the only constant mapping that is linear. This mapping is called a **zero-mapping** and is denoted simply by $\mathbf{0}$. The identity mapping $\mathbf{1}_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ of any linear space $\mathcal{V}$ is trivially linear.

The following facts are easily proved:

Proposition 1: *The composite of two linear mappings is again linear. More precisely: If $\mathcal{V}, \mathcal{V}'$ and $\mathcal{V}''$ are linear spaces and if $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ and $\mathbf{M} : \mathcal{V}' \to \mathcal{V}''$ are linear mappings, so is $\mathbf{M} \circ \mathbf{L} : \mathcal{V} \to \mathcal{V}''$.*

Proposition 2: *The inverse of an invertible linear mappings is again linear. More precisely: If $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ is linear and invertible, then $\mathbf{L}^{\leftarrow} : \mathcal{V}' \to \mathcal{V}$ is linear.*

If $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ is linear, it is customary to write $\mathbf{Lf} := \mathbf{L} \circ \mathbf{f}$ when $\mathbf{f}$ is any mapping with codomain $\mathcal{V}$. In particular, we write $\mathbf{ML}$ instead of $\mathbf{M} \circ \mathbf{L}$ when both $\mathbf{L}$ and $\mathbf{M}$ are linear.

If $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ is linear and invertible, it is customary to write $\mathbf{L}^{-1} := \mathbf{L}^{\leftarrow}$ for its inverse. By Prop. 2, both $\mathbf{L}$ and $\mathbf{L}^{-1}$ then preserve the linear-space structure. Invertible linear mappings are, therefore, linear-space isomorphisms, and we also call them **linear isomorphisms**. We say that two linear spaces $\mathcal{V}$ and $\mathcal{V}'$ are **linearly isomorphic** if there exists a linear isomorphism from $\mathcal{V}$ to $\mathcal{V}'$.

Proposition 3: *If $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ is linear and if $\mathcal{U}$ and $\mathcal{U}'$ are subspaces of $\mathcal{V}$ and $\mathcal{V}'$, respectively, then the image $\mathbf{L}_{>}(\mathcal{U})$ of $\mathcal{U}$ is a subspace of $\mathcal{V}'$ and the pre-image $\mathbf{L}^{<}(\mathcal{U}')$ of $\mathcal{U}'$ is a subspace of $\mathcal{V}$.*

In particular, $\operatorname{Rng} \mathbf{L} = \mathbf{L}_{>}(\mathcal{V})$ is a subspace of $\mathcal{V}'$ and $\mathbf{L}^{<}(\{\mathbf{0}\})$ is a subspace of $\mathcal{V}$.

Definition: *The pre-image of the zero-subspace of the codomain of a linear mapping $\mathbf{L}$ is called the* **nullspace** *of $\mathbf{L}$ and is denoted by*

$$\text{Null}\,\mathbf{L} := \mathbf{L}^{<}(\{\mathbf{0}\}) = \{\mathbf{u} \in \text{Dom}\,\mathbf{L} \mid \mathbf{L}\mathbf{u} = \mathbf{0}\}. \tag{13.5}$$

Proposition 4: *A linear mapping* $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ *is injective if and only if* Null $\mathbf{L} = \{\mathbf{0}\}$.

If $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ is linear, if $\mathcal{U}$ is a subspace of $\mathcal{V}$, and if $\mathcal{U}'$ is a linear space of which $\mathbf{L}_{>}(\mathcal{U})$ is a subspace, then the adjustment $\mathbf{L}|_{\mathcal{U}}^{\mathcal{U}'} : \mathcal{U} \to \mathcal{U}'$ is evidently again linear. In particular, if $\mathcal{U}$ is a subspace of $\mathcal{V}$, the inclusion mapping $\mathbf{1}_{\mathcal{U}\subset\mathcal{V}}$ is linear.

Proposition 5: *If* $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ *is linear and* $\mathcal{U}$ *is a supplement of* Null $\mathbf{L}$ *in* $\mathcal{V}$, *then* $\mathbf{L}|_{\mathcal{U}}^{\text{Rng}\,\mathbf{L}}$ *is invertible. If* $\mathbf{v}' \in \text{Rng}\,\mathbf{L}$, *then* $\mathbf{v} \in \mathcal{V}$ *is a solution of the linear equation*

$$?\ \mathbf{v} \in \mathcal{V}, \quad \mathbf{L}\mathbf{v} = \mathbf{v}' \tag{13.6}$$

if and only if $\mathbf{v} \in \left(\mathbf{L}\,|_{\mathcal{U}}^{\text{Rng}\,\mathbf{L}}\right)^{-1}\mathbf{v}' + \text{Null}\,\mathbf{L}$.

Notes 13

(1) The terms "linear transformation" or "linear operator" are sometimes used for what we call a "linear mapping".

(2) Some people use the term "kernel" for what we call "nullspace" and they use the notation Ker $\mathbf{L}$ for Null $\mathbf{L}$. Although the nullspace of $\mathbf{L}$ is a special kind of kernel (in the sense explained in Sect. 06), I believe it is useful to have a special term and a special notation to stress that one deals with *linear* mappings rather than homomorphisms in general. Notations such as $\mathcal{N}(\mathbf{N})$ and $\mathbf{N}(\mathbf{L})$ are often used for Null $\mathbf{L}$.

14 Spaces of Mappings, Product Spaces

Let $\mathcal{V}$ be a linear space and let S be any set. The set Map$(S, \mathcal{V})$ of all mappings from S to $\mathcal{V}$ can be endowed, in a natural manner, with the structure of a linear space by defining the operations in Map$(S, \mathcal{V})$ by value-wise application of the operations in $\mathcal{V}$. Thus, if $\mathbf{f},\mathbf{g} \in \text{Map}(S, \mathcal{V})$ and $\xi \in \mathbb{F}$, then $\mathbf{f} + \mathbf{g}$, $-\mathbf{f}$ and $\xi\,\mathbf{f}$ are defined by requiring

$$(\mathbf{f} + \mathbf{g})(s) := \mathbf{f}(s) + \mathbf{g}(s), \tag{14.1}$$

$$(-\mathbf{f})(s) := -(\mathbf{f}(s)), \tag{14.2}$$

$$(\xi\mathbf{f})(s) := \xi(\mathbf{f}(s)), \tag{14.3}$$

to be valid for all $s \in S$. The zero-element of Map$(S, \mathcal{V})$ is the constant mapping $\mathbf{0}_{S\to\mathcal{V}}$ with value $\mathbf{0} \in \mathcal{V}$. We denote it simply by $\mathbf{0}$. It is immediate that the axioms (A1)-(S4) of Sect. 01 for a linear space are, in fact, satisfied for the structure of $\text{Map}(S, \mathcal{V})$ just described. Thus, we can talk about the (linear) **space of mappings** $\text{Map}(S, \mathcal{V})$.

Proposition 1: *Let S and S' be sets, let $h : S' \to S$ be a mapping, and let $\mathcal{V}$ be a linear space. Then*

$$(\mathbf{f} \mapsto \mathbf{f} \circ h) : \ \text{Map}(S, \mathcal{V}) \to \text{Map}(S', \mathcal{V})$$

is a linear mapping, i.e.,

$$(\mathbf{f} + g) \circ h \ = \ (\mathbf{f} \circ h) + (g \circ h), \tag{14.4}$$

$$(\xi \mathbf{f}) \circ h \ = \ \xi(\mathbf{f} \circ h) \tag{14.5}$$

hold for all $\mathbf{f},\mathbf{g} \in \text{Map}(S, \mathcal{V})$ *and all* $\xi \in \mathbb{F}$.

Let $\mathcal{V}, \mathcal{V}'$ be linear spaces. We denote the set of all linear mappings from $\mathcal{V}$ into $\mathcal{V}'$ by

$$\text{Lin}(\mathcal{V}, \mathcal{V}') := \left\{ \mathbf{L} \ \in \ \text{Map}(\mathcal{V}, \mathcal{V}') \mid \mathbf{L} \text{ is linear} \right\}.$$

Proposition 2: $\text{Lin}(\mathcal{V}, \mathcal{V}')$ *is a subspace of* $\text{Map}(\mathcal{V}, \mathcal{V}')$.

Proof: $\text{Lin}(\mathcal{V}, \mathcal{V}')$ is not empty because the zero-mapping belongs to it. We must show that $\text{Lin}(\mathcal{V}, \mathcal{V}')$ is stable under addition and scalar multiplication. Let $\mathbf{L},\mathbf{M} \in \text{Lin}(\mathcal{V}, \mathcal{V}')$ be given. Using first the definition (14.1) for $\mathbf{L} + \mathbf{M}$, then the axioms (A1) and (A2) in $\mathcal{V}'$, then the linearity rule (13.1) for $\mathbf{L}$ and $\mathbf{M}$, and finally the definition of $\mathbf{L} + \mathbf{M}$ again, we obtain

$$\begin{aligned}
(\mathbf{L} + \mathbf{M})(\mathbf{u}) + (\mathbf{L} + \mathbf{M})(\mathbf{v}) &= (\mathbf{Lu} + \mathbf{Mu}) + (\mathbf{Lv} + \mathbf{Mv}) \\
&= (\mathbf{Lu} + \mathbf{Lv}) + (\mathbf{Mu} + \mathbf{Mv}) \\
&= \mathbf{L}(\mathbf{u} + \mathbf{v}) + \mathbf{M}(\mathbf{u} + \mathbf{v}) \\
&= (\mathbf{L} + \mathbf{M})(\mathbf{u} + \mathbf{v})
\end{aligned}$$

for all $\mathbf{u},\mathbf{v} \in \mathcal{V}$. This shows that $\mathbf{L} + \mathbf{M}$ satisfies the linearity rule (13.1). In a similar way, one proves that $\mathbf{L} + \mathbf{M}$ satisfies the linearity rule (13.2) and hence that $\mathbf{L} + \mathbf{M} \in \text{Lin}(\mathcal{V}, \mathcal{V}')$. Since $\mathbf{L},\mathbf{M} \in \text{Lin}(calV, \mathcal{V}')$ were arbitrary, it follows that $\text{Lin}(\mathcal{V}, \mathcal{V}')$ is stable under addition.

The proof that $\text{Lin}(\mathcal{V}, \mathcal{V}')$ is stable under scalar multiplication is left to the reader. ∎

We call $\text{Lin}(\mathcal{V}, \mathcal{V}')$ the **space of linear mappings** from $\mathcal{V}$ to $\mathcal{V}'$. We denote the set of all *invertible* linear mappings from $\mathcal{V}$ to $\mathcal{V}'$, i.e., the set of all linear isomorphisms from $\mathcal{V}$ to $\mathcal{V}'$, by $\mathrm{Lis}(\mathcal{V},\mathcal{V}')$. This set $\mathrm{Lis}(\mathcal{V},\mathcal{V}')$ is a subset of $\mathrm{Lin}(\mathcal{V},\mathcal{V}')$ but not a subspace (except when both $\mathcal{V}$ and $\mathcal{V}'$ are zero-spaces).

Proposition 3: *Let S be a set, let $\mathcal{V}$ and $\mathcal{V}'$ be linear spaces, and let* $\mathbf{L} \in \mathrm{Lin}(\mathcal{V},\mathcal{V}')$ be given. *Then*

$$(\mathbf{f} \mapsto \mathbf{Lf}) : \ \mathrm{Map}(S,\mathcal{V}) \to \ \mathrm{Map}(s,\mathcal{V}')$$

is a linear mapping, i.e.,

$$\mathbf{L}(\mathbf{f}+\mathbf{g}) = \mathbf{Lf}+\mathbf{Lg}, \tag{14.6}$$
$$\mathbf{L}(\xi\mathbf{f}) = \xi(\mathbf{Lf}) \tag{14.7}$$

hold for all $\mathbf{f},\mathbf{g} \in \mathrm{Map}(S,\mathcal{V})$ *and all* $\xi \in \mathbb{F}$.

Let $(\mathcal{V}_1,\mathcal{V}_2)$ be a pair of linear spaces. The set product $\mathcal{V}_1 \times \mathcal{V}_2$ (see Sect. 02) has the natural structure of a linear space whose operations are defined by term-wise application of the operations in $\mathcal{V}_1$ and $\mathcal{V}_2$, i.e., by

$$(\mathbf{u}_1,\mathbf{u}_2)+(\mathbf{v}_1,\mathbf{v}_2) := (\mathbf{u}_1+\mathbf{v}_1,\mathbf{u}_2+\mathbf{v}_2), \tag{14.8}$$
$$\xi(\mathbf{u}_1,\mathbf{u}_2) := (\xi\mathbf{u}_1,\xi\mathbf{u}_2), \tag{14.9}$$
$$-(\mathbf{u}_1,\mathbf{u}_2) := (-\mathbf{u}_1,-\mathbf{u}_2) \tag{14.10}$$

for all $\mathbf{u}_1,\mathbf{v}_1 \in \mathcal{V}_1$, $\mathbf{u}_2,\mathbf{v}_2 \in \mathcal{V}_2$, and $\xi \in \mathbb{F}$. The zero of $\mathcal{V}_1 \times \mathcal{V}_2$ is the pair $(\mathbf{0}_1,\mathbf{0}_2)$, where $\mathbf{0}_1$ is the zero of $\mathcal{V}_1$ and $\mathbf{0}_2$ the zero of $\mathcal{V}_2$. Thus, we may refer to $\mathcal{V}_1 \times \mathcal{V}_2$ as the (linear) **product-space** of $\mathcal{V}_1$ and $\mathcal{V}_2$.

The evaluations $\mathrm{ev}_1 : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathcal{V}_1$ and $\mathrm{ev}_2 : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathcal{V}_2$ associated with the product space $\mathcal{V}_1 \times \mathcal{V}_2$ (see Sect. 04) are obviously linear. So are the **insertion mappings**

$$\begin{aligned} \mathrm{ins}_1 &:= (\mathbf{u}_1 \mapsto (\mathbf{u}_1,\mathbf{0})) : \mathcal{V}_1 \to \mathcal{V}_1 \times \mathcal{V}_2, \\ \mathrm{ins}_2 &:= (\mathbf{u}_2 \mapsto (\mathbf{0},\mathbf{u}_2)) : \mathcal{V}_2 \to \mathcal{V}_1 \times \mathcal{V}_2. \end{aligned} \tag{14.11}$$

If $\mathcal{U}_1$ is a subspace of $\mathcal{V}_1$ and $\mathcal{U}_2$ a subspace of $\mathcal{V}_2$, then $\mathcal{U}_1 \times \mathcal{U}_2$ is a subspace of $\mathcal{V}_1 \times \mathcal{V}_2$.

Pitfall: In general, the product-space $\mathcal{V}_1 \times \mathcal{V}_2$ has many subspaces that are *not* of the form $\mathcal{U}_1 \times \mathcal{U}_2$. ■

Let $\mathcal{W}$ be a third linear space. Recall the identification

$$\text{Map}(\mathcal{W}, \mathcal{V}_1) \times \text{Map}(\mathcal{W}, \mathcal{V}_2) \cong \text{Map}(\mathcal{W}, \mathcal{V}_1 \times \mathcal{V}_2)$$

(see Sect. 04). It turns out that this identification is such that the subspaces $\text{Lin}(\mathcal{W}, \mathcal{V}_1)$, $\text{Lin}(\mathcal{W}, \mathcal{V}_2))$, and $\text{Lin}(\mathcal{W}, \mathcal{V}_1 \times \mathcal{V}_2)$ are matched in the sense that the pairs $(\mathbf{L}_1, \mathbf{L}_2) \in \text{Lin}(\mathcal{W}, \mathcal{V}_1) \times \text{Lin}(\mathcal{W}, \mathcal{V}_2)$ of linear mappings correspond to the lienar mappings in $\text{Lin}(\mathcal{W}, \mathcal{V}_1 \times \mathcal{V}_2)$ defined by term-wise evaluation, i.e., by

$$(\mathbf{L}_1, \mathbf{L}_2)\mathbf{w} := (\mathbf{L}_1\mathbf{w}, \mathbf{L}_2\mathbf{w}) \text{ for all } \mathbf{w} \in \mathcal{W}. \tag{14.12}$$

Thus, (14.12) describes the identification

$$\text{Lin}(\mathcal{W}, \mathcal{V}_1) \times \text{Lin}(\mathcal{W}, \mathcal{V}_2) \cong \text{Lin}(\mathcal{W}, \mathcal{V}_1 \times \mathcal{V}_2).$$

There is also a natural linear isomorphism

$$((\mathbf{L}_1, \mathbf{L}_2) \mapsto \mathbf{L}_1 \oplus \mathbf{L}_2) : \text{Lin}(\mathcal{V}_1, \mathcal{W}) \times \text{Lin}(\mathcal{V}_2, \mathcal{W}) \to \text{Lin}(\mathcal{V}_1 \times \mathcal{V}_2, \mathcal{W}).$$

It is defined by

$$(\mathbf{L}_1 \oplus \mathbf{L}_2)(\mathbf{v}_1, \mathbf{v}_2) := \mathbf{L}_1\mathbf{v}_1 + \mathbf{L}_2\mathbf{v}_2 \tag{14.13}$$

for all $\mathbf{v}_1 \in \mathcal{V}_1, \mathbf{v}_2 \in \mathcal{V}_2$, which is equivalent to

$$\mathbf{L}_1 \oplus \mathbf{L}_2 := \mathbf{L}_1\text{ev}_1 + \mathbf{L}_2\text{ev}_2, \tag{14.14}$$

where ev_1 and ev_2 are the evaluation mappings associated with $\mathcal{V}_1 \times \mathcal{V}_2$ (see Sect. 04).

What we said about a pair of linear spaces easily generalizes to an arbitrary family $(\mathcal{V}_i \mid i \in I)$ of linear spaces. The set product $\times(\mathcal{V}_i \mid i \in I)$ has the natural structure of a linear space whose operations are defined by term-wise application of the operations in the $\mathcal{V}_i, i \in I$. Hence, we may refer to $\times(\mathcal{V}_i \mid i \in I)$ as the **product-space** of the family $(\mathcal{V}_i \mid i \in I)$. Given $j \in I$, the evaluation

$$\text{ev}_j : \mathop{\times}_{i \in I} \mathcal{V}_i \to \mathcal{V}_j,$$

(see (04.9)) is linear. So is the insertion mapping

$$\text{ins}_j : \mathcal{V}_j \to \mathop{\times}_{i \in I} \mathcal{V}_i$$

defined by

$$(\mathrm{ins}_j\mathbf{v})_i := \left\{ \begin{array}{lll} \mathbf{0}\in\mathcal{V}_i & \text{if} & i\neq j \\ \mathbf{v}\in\mathcal{V}_j & \text{if} & i=j \end{array} \right\} \text{ for all } \mathbf{v}\in\mathcal{V}_j. \tag{14.15}$$

It is evident that

$$\mathrm{ev}_k\,\mathrm{ins}_j = \left\{ \begin{array}{llll} \mathbf{0} & \in & \mathrm{Lin}(\mathcal{V}_j,\mathcal{V}_k) & \text{if} \quad j\neq k \\ \mathbf{1}_{\mathcal{V}_j} & \in & \mathrm{Lin}(\mathcal{V}_j,\mathcal{V}_j) & \text{if} \quad j=k \end{array} \right\} \tag{14.16}$$

for all $j,k\in I$.

Let $\mathcal{W}$ be an additional linear space. For families $(\mathbf{L}_i \mid i\in I)$ or linear mappings $\mathbf{L}_i\in\ \mathrm{Lin}(\mathcal{W},\mathcal{V}_i)$ we use termwise evaluation

$$(\mathbf{L}_i \mid i\in I)\mathbf{w} := (\mathbf{L}_i\mathbf{w} \mid i\in I) \text{ for all } \mathbf{w}\in\mathcal{W}, \tag{14.17}$$

which describes the identification

$$\mathop{\times}_{i\in I}\mathrm{Lin}(\mathcal{W},\mathcal{V}_i) \cong \mathrm{Lin}(\mathcal{W},\mathop{\times}_{i\in I}\mathcal{V}_i).$$

If the index set I is finite, we also have a natural isomorphism

$$((\mathbf{L}_i \mid i\in I)\mapsto \bigoplus_{i\in I}\mathbf{L}_i) : \mathop{\times}_{i\in I}\mathrm{Lin}(\mathcal{V}_i,\mathcal{W}) \to \mathrm{Lin}(\mathop{\times}_{i\in I}\mathcal{V}_i,\mathcal{W})$$

defined by

$$\bigoplus_{i\in I}\mathbf{L}_i := \sum_{i\in I}\mathbf{L}_i\mathrm{ev}_i. \tag{14.18}$$

If the spaces in a family indexed on I all coincide with a given linear space $\mathcal{V}$, then the product space reduces to the **power-space** $\mathcal{V}^I$, which consist of all families in $\mathcal{V}$ indexed on I (see Sect. 02). The set $\mathcal{V}^{(I)}$ of all families contained in $\mathcal{V}^I$ that have finite support (see Sect. 07) is easily seen to be a subspace of $\mathcal{V}^I$. Of particular interest is the space $\mathbb{F}^{(I)}$ of all families $\lambda := (\lambda_i \mid i\in I)$ in $\mathbb{F}$ with finite support. Also, if I and J are finite sets, it is useful to consider the linear space $\mathbb{F}^{J\times I}$ of $J\times I$-matrices with terms in $\mathbb{F}$ (see Sect. 02). Cross products of linear mappings, as defined in Sect. 04, are again linear mappings.

Notes 14

(1) The notations $\mathcal{L}(\mathcal{V}, \mathcal{V}')$ and $L(\mathcal{V}, \mathcal{V}')$ for our $\text{Lin}(\mathcal{V}, \mathcal{V}')$ are very common.

(2) The notation $\text{Lis}(\mathcal{V}, \mathcal{V}')$ was apparently first introduced by S. Lang (Introduction to Differentiable Manifolds, Interscience 1966). In some previous work, I used $\text{Invlin}(\mathcal{V}, \mathcal{V}')$.

(3) The product-space $\mathcal{V}_1 \times \mathcal{V}_2$ is sometimes called the "direct sum" of the linear spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ and it is then denoted by $\mathcal{V}_1 \oplus \mathcal{V}_2$. I believe such a notation is superfluous because the set-product $\mathcal{V}_1 \times \mathcal{V}_2$ carries the *natural* structure of a linear space and a special notation to emphasize this fact is redundant. A similar remark applies to product-spaces of families of linear spaces.

15 Linear Combinations, Linear Independence, Bases

Definition 1: *Let* $\mathbf{f} := (\mathbf{f}_i \mid i \in I)$ *be a family of elements in a linear space* $\mathcal{V}$. *The mapping*

$$\text{lnc}_{\mathbf{f}}^{\mathcal{V}} : \mathbb{F}^{(I)} \to \mathcal{V}$$

defined by

$$\text{lnc}_{\mathbf{f}}^{\mathcal{V}} \lambda := \sum_{i \in I} \lambda_i \mathbf{f}_i \tag{15.1}$$

for all $\lambda := (\lambda_i \mid i \in I) \in \mathbb{F}^{(I)}$ *is then called the* **linear-combination mapping** *for* $\mathbf{f}$. *The value* $\text{lnc}_{\mathbf{f}}^{\mathcal{V}} \lambda$ *is called the* **linear combination** *of* $\mathbf{f}$ *with coefficient family* λ. (See (07.10) and (07.11) for the notation used here.)

It is evident from the rules that govern sums (see Sect. 07) that linear combination mappings are linear mappings.

In the special case when $\mathcal{V} := \mathbb{F}$ and when $\mathbf{f}$ is the constant family whose terms are all $\mathbf{1} \in \mathbb{F}$, then $\text{lnc}_{\mathbf{f}}^{\mathbb{F}}$ reduces to the **summation mapping** $\text{sum}_i : \mathbb{F}^{(I)} \to \mathbb{F}$ given by

$$\text{sum}_I \lambda := \sum_{i \in I} \lambda_i \quad \text{for all } \lambda \in \mathbb{F}^{(I)}. \tag{15.2}$$

Let $\mathcal{U}$ be a subspace of the given linear space $\mathcal{V}$ and let $\mathbf{f}$ be a family in $\mathcal{U}$. then $\text{Rng}\, \text{lnc}_{\mathbf{f}}^{\mathcal{V}} \subset \mathcal{U}$ and $\text{lnc}_{\mathbf{f}}^{\mathcal{U}}$ is obtained from $\text{lnc}_{\mathbf{f}}^{\mathcal{V}}$ by adjustment of codomain

$$\text{lnc}_{\mathbf{f}}^{\mathcal{U}} = lnc_{\mathbf{f}}^{\mathcal{V}} |^{\mathcal{U}} \quad \text{if} \quad \text{Rng}\, \mathbf{f} \subset \mathcal{U}. \tag{15.3}$$

Definition 2: *The family* $\mathbf{f}$ *in a linear space* $\mathcal{V}$ *is said to be* **linearly independent** *in* $\mathcal{V}$, **spanning** *in* $\mathcal{V}$, *or a* **basis** of $\mathcal{V}$ *depending on whether the linear combination mapping* $\mathrm{lnc}_{\mathbf{f}}^{\mathcal{V}}$ *is injective, surjective, or invertible, respectively. We say that* $\mathbf{f}$ *is* **linearly dependent** *if it is not linearly independent.*

If $\mathbf{f}$ is a family in a given linear space $\mathcal{V}$ and if there is not doubt what $\mathcal{V}$ is, we simply write $\mathrm{lnc}_{\mathbf{f}} := \mathrm{lnc}_{\mathbf{f}}^{\mathcal{V}}$. Also we then omit "in $\mathcal{V}$" and "of $\mathcal{V}$" when we say that $\mathbf{f}$ is linear independent, spanning, or a basis. Actually, if $\mathbf{f}$ is linearly independent in $\mathcal{V}$, it is also linearly independent in any linear space that includes $\mathrm{Rng}\,\mathbf{f}$.

If $\mathbf{b} := (\mathbf{b}_i \mid i \in I)$ is a basis of $\mathcal{V}$ and if $\mathbf{v} \in \mathcal{V}$ is given, then $\mathrm{lnc}_{\mathbf{b}}^{-1}\mathbf{v} \in \mathbb{F}^{(I)}$ is called the family **components** of $\mathbf{v}$ relative to the basis $\mathbf{b}$. Thus

$$\mathbf{v} = \sum_{i \in I} \lambda_i \mathbf{b}_i \tag{15.4}$$

holds if and only if $(\lambda_i \mid i \in I) \in \mathbb{F}^{(I)}$ is the family of components of $\mathbf{v}$ relative to $\mathbf{b}$.

An application of Prop. 4 of Sect. 13 gives:

Proposition 1: *The family* $\mathbf{f}$ *is linearly independent if and only if* Null $(\mathrm{lnc}_{\mathbf{f}}) = \{\mathbf{0}\}$.

Let I' be a subset of the given index set I. We then define the **insertion mapping**

$$\mathrm{ins}_{I' \subset I} : \mathbb{F}^{(I')} \to \mathbb{F}^{(I)}$$

by

$$(\mathrm{ins}_{I' \subset I}(\lambda))_i = \left\{ \begin{array}{lll} \lambda_i & \text{if} & i \in I' \\ 0 & \text{if} & i \in I \backslash I' \end{array} \right\} \tag{15.5}$$

It is clear that $\mathrm{ins}_{I' \subset I}$ is an injective linear mapping. If $\mathbf{f}$ is a family indexed on I and if $\mathbf{f}|_{I'}$ is its restriction to I', we have

$$\mathrm{lnc}_{\mathbf{f}|_{I'}} = \mathrm{lnc}_{\mathbf{f}} \mathrm{ins}_{I' \subset I}. \tag{15.6}$$

From this formula and the injectivity of $\mathrm{ins}_{I' \subset I}$ we can read off the following:

Proposition 2: *If the family* $\mathbf{f}$ *is linearly independent, so are all its restrictions. If any restriction of* $\mathbf{f}$ *is spanning, so is* $\mathbf{f}$.

Let $\mathcal{S}$ be a subset of $\mathcal{V}$. In view of the identification of a set with the family obtained by self-indexing (see Sect. 02) we can consider the linear combination mapping $\text{lnc}_{\mathcal{S}} : \mathbb{F}^{(\mathcal{S})} \to \mathcal{V}$, given by

$$\text{lnc}_{\mathcal{S}}\lambda := \sum_{\mathbf{u}\in\mathcal{S}} \lambda_{\mathbf{u}}\mathbf{u} \tag{15.7}$$

for all $\lambda := (\lambda_{\mathbf{u}} \mid \mathbf{u} \in \mathcal{S}) \in \mathbb{F}^{(\mathcal{S})}$. Thus the definitions above apply not only to families in $\mathcal{V}$ in general, but also to subsets of $\mathcal{V}$ in particular.

The following facts are easily verified:

Proposition 3: *A family in $\mathcal{V}$ is linearly independent if and only if it is injective and its range is linearly independent. No term of a linearly independent family can be zero.*

Proposition 4: *A family in $\mathcal{V}$ is spanning if and only if its range is spanning.*

Proposition 5: *A family in $\mathcal{V}$ is a basis if and only if it is injective and its ranges is a basis-set.*

The following result shows that the definition of a spanning set as a special case of a spanning family is not in conflict with the definition of a set that spans the space as given in Sect. 12.

Proposition 6: *The set of all linear combinations of a family in $\mathcal{V}$ is the linear span of the range of* $\mathbf{f}$, i.e., $\text{Rng}\,\text{lnc}_{\mathbf{f}} = \text{Lsp}(\text{Rng}\,\mathbf{f})$. *In particular, if $\mathcal{S}$ is a subset of $\mathcal{V}$, then* $\text{Rng}\,\text{lnc}_{\mathcal{S}} = \text{Lsp}\mathcal{S}$.

The following is an immediate consequence of Prop. 6:

Proposition 7: *A family* $\mathbf{f}$ *is linearly independent if and only if it is a basis of* $\text{LspRng}\,\mathbf{f}$.

The next two results are not not hard to prove.

Proposition 8: *A subset $\mathfrak{b}$ of $\mathcal{V}$ is linearly dependent if and only if* $\text{Lsp}\mathfrak{b} = \text{Lsp}(\mathfrak{b}\backslash\{\mathbf{v}\})$ *for some* $\mathbf{v} \in \mathfrak{b}$.

Proposition 9: *If $\mathfrak{b}$ is a linearly independent subset of $\mathcal{V}$ and* $\mathbf{v} \in \mathcal{V}\backslash\mathfrak{b}$, *then* $\mathfrak{b} \cup \{\mathbf{v}\}$ *is linearly dependent if and only if* $\mathbf{v} \in \text{Lsp}\mathfrak{b}$.

By applying Props. 8 and 9, one easily proves the following important result:

Characterization of Bases: *Let* $\mathfrak{b}$ *be subset of* $\mathcal{V}$*. Then the following are equivalent:*

(i) $\mathfrak{b}$ *is a basis of* $\mathcal{V}$*.*

(ii) $\mathfrak{b}$ *is both linearly independent and spanning.*

(iii) $\mathfrak{b}$ *is a maximal linearly independent set, i.e.,* $\mathfrak{b}$ *is linearly independent and is not a proper subset of any other linearly independent set.*

(iv) $\mathfrak{b}$ *is a minimal spanning set, i.e.,* $\mathfrak{b}$ *is spanning and has no proper subset that is also spanning.*

We note that the zero-space $\{\mathbf{0}\}$ has exactly one set basis, namely the empty set.

Pitfall: Every linear space other than a zero-space has infinitely many bases, if it has any at all. Unless the space has structure in addition to its structure as a linear space, all of these bases are of equal standing. The bases form a "democracy". For example, every singleton $\{\xi\}$ with $\xi \in \mathbb{F}^{\times}$ is a basis set of the linear space $\mathbb{F}$. The special role of the number 1 and hence the bases $\{1\}$ comes from the additional structure in $\mathbb{F}$ given by the multiplication. ■

Notes 15

(1) In most textbooks, the definition of "linear combination" is rather muddled. Much of the confusion comes from a failure to distinguish a process (the linear-combination *mapping*) from its result (the linear-combination). Also, most authors fail to make a clear distinction between sets, lists, and families, a distinction that is crucial here. I believe that much precision, clarity, and insight are gained by the use of linear-combination mappings. Most textbooks only talk around these mappings without explicitly using them.

(2) The condition of Prop. 1 is very often used as the definition of linear linear independence.

(3) Many people say "coordinates" instead of "components" of $\mathbf{v}$ relative to the basis of b. I prefer to use the term "coordinate" only when it has the meaning described in Chapter 7.

16 Matrices, Elimination of Unknowns

The following result states, roughly, that linear mappings preserve linear combinations.

Proposition 1: *If $\mathcal{V}$ and $\mathcal{W}$ are linear spaces,* $\mathbf{f} := (\mathbf{f}_i \mid i \in I)$ *a family in* $\mathcal{V}$, $\mathbf{L} : \mathcal{V} \to \mathcal{W}$ *a linear mapping, and* $\mathbf{Lf} := (\mathbf{Lf}_i \mid i \in I)$, *then*

$$\mathrm{lnc}_{(\mathbf{Lf})} = \mathbf{L}\ \mathrm{lnc}_{\mathbf{f}}. \tag{16.1}$$

Applying (16.1) to the case when $\mathbf{f}$ is a basis, we obtain:

Proposition 2: *Let* $\mathbf{b} := (\mathbf{b}_i \mid i \in I)$ *be a basis of the linear space* $\mathcal{V}$ *and let* $\mathbf{g} := (\mathbf{g}_i \mid i \in I)$ *be a family in the linear space* $\mathcal{W}$, *indexed on the same set* I as $\mathbf{b}$. *Then there is exactly one* $\mathbf{L} \in \mathrm{Lin}(\mathcal{V}, \mathcal{W})$ *such that* $\mathbf{Lb} = \mathbf{g}$. *This* $\mathbf{L}$ *is injective, surjective, or invertible depending on whether* $\mathbf{g}$ *is linearly independent, spanning, or a basis, respectively.*

The first part of Prop. 2 states, roughly, that a linear mapping can be specified by prescribing its effect on a basis.

Let I be any index set. We define a family $\delta^I := (\delta_i^I \mid i \in I)$ in $\mathbb{F}^{\mathbf{f}}$ by

$$(\delta_i^I)_k = \delta_{i,k} := \left\{ \begin{array}{lll} 1 & \text{if} & k = i \\ 0 & \text{if} & k \neq i \end{array} \right\} \quad \text{for all } i, k \in I. \tag{16.2}$$

It is easily seen that $\mathrm{lnc}_{\delta^I} = 1_{\mathbb{F}^{(I)}}$, which is, of course, invertible. Hence δ^I is a basis of $\mathbb{F}^{(I)}$. We call it the **standard basis** of $\mathbb{F}^{(I)}$. If $\mathbf{f}:= (\mathbf{f}_i \mid i \in I)$ is a family in a given linear space $\mathcal{V}$, then

$$\mathbf{f}_j = \mathrm{lnc}_{\mathbf{f}} \delta_j^I \quad \text{for all } j \in I. \tag{16.3}$$

If we apply Prop. 2 to the case when I is finite, when $\mathbf{b} := \delta^I$ is the standard basis of $\mathcal{V} := \mathbb{F}^I$, and when $\mathcal{W} := \mathbb{F}^J$ for some finite index set J, we obtain:

Proposition 3: *Let I and J be finite index sets. Then the mapping from* $\mathrm{Lin}(\mathbb{F}^I, \mathbb{J}^I)$ *to* $\mathbb{F}^{J\times I}$ *which assigns to* $M \in \mathrm{Lin}(\mathbb{F}^I, \mathbb{F}^J)$ *the matrix* $((M\delta_i^I)_i \mid (\mathbf{j}, \mathbf{i}) \in J \times I)$ *is a linear isomorphism.*

We use the natural isomorphism described in Prop. 3 to identify $M \in \mathrm{Lin}(\mathbb{F}^I, \mathbb{F}^J)$ with the corresponding matrix in $\mathbb{F}^{J\times I}$, so that

$$M_{j,i} = (M\delta_i^I)_j \quad \text{for all } (j, i) \in J \times I. \tag{16.4}$$

Thus, we obtain the identification

$$\mathrm{Lin}(\mathbb{F}^I, \mathbb{F}^J) \cong \mathbb{F}^{J\times I}. \tag{16.5}$$

Proposition 4: *Let I, J be finite index sets. If* $M \in \mathrm{Lin}(\mathbb{F}^I, \mathbb{F}^J) \cong \mathbb{F}^{J\times I}$ *and* $\lambda \in \mathbb{F}^I$, *then*

$$(M\lambda)_j = \sum_{j\in J} N_{k,j} M_{j,i} \quad \text{for all } i \in I. \tag{16.6}$$

Proposition 5: *Let I, J, K be finite index sets. If $M \in \operatorname{Lin}(\mathbb{F}^I, \mathbb{F}^J) \cong \mathbb{F}^{J\times I}$ and*
$N \in \operatorname{Lin}(\mathbb{F}^J, \mathbb{F}^K) \cong \mathbb{F}^{K\times J}$, then $NM \in \operatorname{Lin}(\mathbb{F}^I, \mathbb{F}^K) \cong \mathbb{F}^{K\times I}$ is given by

$$(NM)_{k,i} = \sum_{j\in J} N_{k,j} M_{j,i} \quad \text{for all } k,i) \in K \times I. \tag{16.7}$$

In the case when $I := n^{]},\ J := m^{]},\ K := p^{]}$ and when the bookkeeping scheme (02.4) is used, then (16.6) and (16.7) can be represented in the forms

$$\begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mn} \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} \sum\limits_{i\in I^{]}} M_{1i}\lambda_i \\ \sum\limits_{i\in n^{]}} M_{2i}\lambda_i \\ \vdots \\ \underset{I\in n^{]}}{sum} M_{mi}\lambda_i \end{pmatrix} \tag{16.8}$$

and

$$\begin{bmatrix} N_{11} & \cdots & N_{1m} \\ \vdots & & \vdots \\ N_{p1} & \cdots & N_{pm} \end{bmatrix} \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mm} \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{j\in m^{]}} N_{ij} M_{j1} & \cdots & \sum_{j\in m^{]}} N_{1j} M_{jn} \\ \vdots & & \vdots \\ \sum_{j\in m^{]}} N_{pj} M_{j1} & \cdots & \sum_{j\in m^{]}} M_{pj} M_{jn} \end{bmatrix} \tag{16.9}$$

In particular, (16.9) states that the composition matrices when regarded as linear mappings corresponds to the familiar "row-by-column" multiplication of matrices.

Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces and let $\mathbf{b} := (\mathbf{b}_i \mid i \in I)$ and
$\mathbf{c} := (\mathbf{c}_j \mid j \in J)$ be finite bases of $\mathcal{V}$ and $\mathcal{W}$, respectively. Then the mapping

$$(\mathbf{L} \mapsto (\operatorname{lnc}_{\mathbf{c}})^{-1}\mathbf{L}\operatorname{lnc}_{\mathbf{b}}) : \operatorname{Lin}(\mathcal{V}, \mathcal{W}) \to \operatorname{Lin}(\mathbb{F}^I, \mathbb{F}^J) \tag{16.10}$$

is a linear isomorphism. Given $\mathbf{L} \in \operatorname{Lin}(\mathcal{V}, \mathcal{W})$ we say that $M := (\operatorname{lnc}_{\mathbf{c}})^{-1}\mathbf{L}\operatorname{lnc}_{\mathbf{b}} \in \operatorname{Lin}(\mathbb{F}^I, \mathbb{F}^J) \cong \mathbb{F}^{J\times I}$ is the **matrix of the linear mapping L relative to the bases b and c**. It is characterized by

$$\mathbf{Lb}_i = \sum_{j \in J} M_{j,i}, \mathbf{c}_j \quad \text{for all} \quad i \in I. \tag{16.11}$$

Let I and J be finite index sets. If $M \in \text{Lin}(\mathbb{F}^I, \mathbb{F}^J) \cong \mathbb{F}^{J \times I}$ and $\mu \in \mathbb{F}^J$ are given and if we consider the problem $?\ \lambda \in \mathbb{F}^I, M\lambda = \mu$, i.e.,

$$?(\lambda_i \mid i \in I) \in \mathbb{F}^I, \quad \sum_{i \in I} M_{j,i}\lambda_i = \mu_j \quad \text{for all} \quad j \in J, \tag{16.12}$$

we say that we have the problem of solving a system of $\#J$ linear equations with $\#I$ unknowns. The following theorem describes a procedure, familiar from elementary algebra, called *eliminations of unknowns*, which enables one to reduce a system of linear equations to one having one equation less and one unknown less.

Theorem on Elimination of Unknowns: *Let I and J be finite sets and let $M \in \mathbb{F}^{J \times I}$ be such that $M_{j_0,i_0} \neq 0$ for given $j_0 \in J, i_0 \in I$. Put $I' := I \backslash \{i_0\}$, $J' := J \backslash \{j_0\}$ and define $M' \in \mathbb{F}^{J' \times I'}$ by*

$$M'_{j,i} := M_{j,i} - \frac{M_{j,i_0} M_{j_0,i}}{M_{j_0,i_0}} \text{ for all } (j,i) \in J' \times I'. \tag{16.13}$$

Let $\mu \in \mathbb{F}^J$ and $\mu' \in \mathbb{F}^{J'}$ be related by

$$\mu'_j = \mu_j - \frac{M_{j,i_0}}{M_{j_0,i_0}} j_0 \text{ for all } j \in J'. \tag{16.14}$$

Then $\lambda \in \mathbb{F}^I$ is a solution of the equation

$$?\ \lambda \in \mathbb{F}^I, \quad M\lambda = \mu$$

if and only if the restriction $\lambda\,|_{I'} \in \mathbb{F}^{I'}$ of λ is a solution of the equation

$$?\ \lambda' \in \mathbb{F}^{I'}, \quad M'\lambda' = \mu'$$

and λ_{i_0} is given by

$$\lambda_{i_0} = \frac{1}{M_{j_0,i_0}} \left(\mu_{j_0} - \sum_{i \in I'} M_{j_0,i}\lambda_i \right). \tag{16.15}$$

Corollary: *If $\text{Lin}(\mathbb{F}^I, \mathbb{F}^J)$ contains an injective mapping, then $\#J \geq \#I$.*

Proof: We proceed by induction over $\#I$. If $\#I = 0$ then the assertion is trivial. Assume, then, that $\#I > 0$ and that the assertion is valid if I is replaced by a subset I' of I having one element less. Assume that $\text{Lin}(\mathbb{F}^I, \mathbb{F}^J)$ contains an injective mapping M, do that $\text{Null } M = \{0\}$. Since $\mathbb{F}^I \neq \{0\}$ we must have $M \neq 0$. Hence we can apply the Theorem with the choice $\mu := 0$ to conclude that $\text{Null } M' = \{0\}$, i.e., that $M' \in \text{Lin}(\mathbb{F}^{I'}, \mathbb{F}^{J'})$ must be injective. By the induction hypothesis, we have $\#J' \geq \#I'$, which means $(\#J) - 1 \geq (\#I) - 1$ and hence implies $\#J \geq \#I$. ■

Notes 16

(1) The notation $\delta_{i,k}$ as defined by (16.2) is often attributed to Kronecker ("the Kronecker deltas").

(2) The standard basis δ^I is sometimes called the "natural basis" of $\mathbb{F}^{(I)}$.

(3) Various algorithms, based on the Theorem on Elimination of Unknowns, for solving systems of linear equations are often called "Gaussian elimination procedures".

(4) A matrix can be interpreted as a non-invertible linear mapping by the identification (16.5) is often called a "singular matrix". In the same way, many people would use the term "non-singular" when we speak of an "invertible" matrix.

17 Dimension

Definition: *We say that a linear space $\mathcal{F}$ is* **finite-dimensional** *if it is spanned by some finite subset. The least among the cardinal numbers of finite spanning subsets of $\mathcal{V}$ is called the* **dimension** *of $\mathcal{V}$ and is denoted by* $\dim \mathcal{V}$.

The following fundamental result gives a much stronger characterization of the dimension than is given by the definition.

Characterization of Dimension: *Let $\mathcal{V}$ be a finite-dimensional linear space and $\mathbf{f} := (\mathbf{f}_i \mid i \in I)$ a family of elements in $\mathcal{V}$.*

(a) *If $\mathbf{f}$ is linearly independent then I is finite and $\#I \leq \dim \mathcal{V}$, with equality if and only if $\mathbf{f}$ is a basis.*

(b) *If $\mathbf{f}$ is spanning, then $\#I \geq \dim \mathcal{V}$, with equality if and only if $\mathbf{f}$ is a basis.*

The proof of this theorem can easily be obtained from the Characterization of Bases (Sect. 15) and the following:

Lemma: *If $(\mathbf{b}_j \mid j \in J)$ is a finite basis of $\mathcal{V}$ and if $(\mathbf{f}_i \mid i \in I)$ is any linearly independent family in $\mathcal{V}$, then I must be finite and $\#I \leq \#J$.*

Proof: Let I' be any finite subset of I. By Prop. 2 of Sect. 15, the restriction $\mathbf{f}|_{I'}$ is still linearly independent, which means that $\text{lnc}_{\mathbf{f}|_{I'}} : \mathbb{F}^{I'} \to \mathcal{V}$ is injective.

Since $\mathbf{b}$ is a basis, $\text{lnc}_{\mathbf{b}}$ is invertible and hence $\text{lnc}_{\mathbf{b}}^{-1}\text{lnc}_{\mathbf{f}|_{I'}} : \mathbb{F}^{I'} \to \mathbb{F}^{J}$ is also injective. By the Corollary of the Theorem on Elimination of Unknowns, it follows that $\#I' \leqq \#J$. Since I' was an arbitrary finite subset of I, it follows that I itself must be finite and $\#I \leqq \#J$. ■

The Theorem has the following immediate consequences:

Corollary 1: *If $\mathcal{V}$ is a finite-dimensional linear space, then $\mathcal{V}$ has bases and every basis of $\mathcal{V}$ has* $\dim \mathcal{V}$ *terms.*

Corollary 2: *Two finite-dimensional spaces $\mathcal{V}$ and $\mathcal{V}'$ are linearly isomorphic if and only if they have the same dimension, i.e.,* $\text{Lis}(\mathcal{V}, \mathcal{V}') \neq \emptyset$ *if and only if* $\dim \mathcal{V} = \dim \mathcal{V}'$.

Now let $\mathcal{V}$ be a finite-dimensional linear space. The following facts are not hard to prove with the help of the Characterization of Dimension.

Proposition 1: *If $\mathfrak{s}$ is a linearly independent subset of $\mathcal{V}$, then $\mathfrak{s}$ must be finite and*

$$\dim(\text{Lsp}\mathfrak{s}) = \#\mathfrak{s}. \tag{17.1}$$

Proposition 2: *Every subspace $\mathcal{U}$ of $\mathcal{V}$ is finite-dimensional and satisfies* $\dim \mathcal{U} \leq \dim \mathcal{V}$, *with equality only if* $\mathcal{U} = \mathcal{V}$.

Proposition 3: *Every subspace of $\mathcal{V}$ has a supplement in $\mathcal{V}$. In fact, if $\mathcal{U}_1$ is a subspace of $\mathcal{V}$, then $\mathcal{U}_2$ is a supplement of $\mathcal{U}_1$ in $\mathcal{V}$ if and only if $\mathcal{U}_2$ is a space of greatest dimension among those that are disjunct from $\mathcal{U}_1$.*

Proposition 4: *Two subspaces $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\mathcal{V}$ are disjunct if and only if*

$$\dim \mathcal{U}_1 + \dim \mathcal{U}_2 = \dim(\mathcal{U}_1 + \mathcal{U}_2). \tag{17.2}$$

Proposition 5: *Two subspaces $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\mathcal{V}$ are supplementary in $\mathcal{V}$ if and only if two of the following three conditions are satisfied:*

(i) $\mathcal{U}_1 \cap \mathcal{U}_2 = \{\mathbf{0}\}$,

(ii) $\mathcal{U}_1 + \mathcal{U}_2 = \mathcal{V}$,

(iii) $\dim \mathcal{U}_1 + \dim \mathcal{U}_2 = \dim \mathcal{V}$.

Proposition 6: *For any two subspaces* $\mathcal{U}_1, \mathcal{U}_2$ *of* $\mathcal{V}$*, we have*

$$\dim \mathcal{U}_1 + \dim \mathcal{U}_2 = \dim(\mathcal{U}_1 + \mathcal{U}_2) + \dim(\mathcal{U}_1 \cap \mathcal{U}_2). \tag{17.3}$$

The following Theorem is perhaps the single most useful fact about linear mappings between finite-dimensional spaces:

Theorem on Dimensions of Range and Nullspace: *If* $\mathbf{L}$ *is a linear mapping whose domain is finite-dimensional, then* Rng $\mathbf{L}$ *is finite dimensional and*

$$\dim(\text{Null}\ \mathbf{L}) + \dim(\text{Rng}\ \mathbf{L}) = \dim(\text{Dom}\ \mathbf{L}). \tag{17.4}$$

Proof: Put $\mathcal{V} := \text{Dom}\ \mathbf{L}$. By Prop. 3 we may choose a supplement $\mathcal{U}$ of Null $\mathbf{L}$ in $\mathcal{V}$. By Prop. 5 (iii), we have $\dim(\text{Null}\ \mathbf{L}) + \dim \mathcal{U} = \dim \mathcal{V}$. By Prop. 5 of Sect. 13, $\mathbf{L}|_{\mathcal{U}}^{\text{Rng}\mathbf{L}}$ is invertible and hence a linear isomorphism. Since linear isomorphisms obviously preserve dimension, we infer that $\dim \mathcal{U} = \dim(\text{Rng}\ \mathbf{L})$ and hence that (17.4) holds. ■

The following result is an immediate consequence of the Theorem just stated and of Prop. 4 of Sect. 13. Its name comes from its analogy to the Pigeonhole Principle stated in Sect. 05.

Pigeonhole Principle for Linear Mappings: *Let* $\mathbf{L}$ *be a linear mapping with finite-dimensional domain and codomain. If* $\mathbf{L}$ *is injective, then* $\dim(\text{Dom}\ \mathbf{L}) \leq \dim(\text{Cod}\ \mathbf{L})$*. If* $\mathbf{L}$ *is surjective, then* $\dim(\text{Cod}\ \mathbf{L}) \leq \dim(\text{Dom}\ \mathbf{L})$*. If* $\dim(\text{Dom}\ \mathbf{L}) = \dim(\text{Cod}\ \mathbf{L})$*, then the following are equivalent:*

(i) $\mathbf{L}$ *is invertible,*

(ii) $\mathbf{L}$ *is surjective,*

(iii) $\mathbf{L}$ *is injective,*

(iv) Null $\mathbf{L} = \{\mathbf{0}\}$.

Let I be a finite index set. Since the standard basis δ^I of $\mathbb{F}^I$ (see Sect. 16) has $\#I$ terms, it follows from Cor. 1 to the Characterization of Dimension that

$$\dim(\mathbb{F}^I) = \#I. \tag{17.5}$$

If I is replaced by a set product $J \times I$ of finite sets J and I, (17.5) yields

$$\dim(\mathrm{Lin}(\mathbb{F}^I, \mathbb{F}^J)) = \dim(\mathbb{F}^{J\times I}) = (\#I)(\#J). \tag{17.6}$$

Propositon 7: *If $\mathcal{V}$ and $\mathcal{W}$ are finite-dimensional linear spaces, so is* $\mathrm{Lin}(\mathcal{V}, \mathcal{W})$, *and*

$$\dim(\mathrm{Lin}(\mathcal{V}, \mathcal{W})) = (\dim \mathcal{V})(\dim \mathcal{W}). \tag{17.7}$$

Proof: In view of Cor. 1, we may choose a basis $(\mathbf{b_i} \mid \mathbf{i} \in I)$ of $\mathcal{V}$ and a basis $(\mathbf{c_j} \mid m \in J)$ of $\mathcal{W}$, so that dim $\mathcal{V} = \#I$ and dim $\mathcal{W} = \#J$. The desired result follows from (17.6) and the fact that $\mathrm{Lin}(\mathbb{F}^I, \mathbb{F}^J)$ is linearly isomorphic to $\mathrm{Lin}(\mathcal{V}, \mathcal{W})$ by virtue of (16.10). ■

Notes 17

(1) The dimensions of the nullspace and of the range of a linear mapping **L** are often called the "nullity" and "rank" of **L**, respectively. I believe that this terminology burdens the memory unnecessarily.

18 Lineons

Let $\mathcal{V}$ be a linear space. A linear mapping from $\mathcal{V}$ to *itself* will be called a **lineon** on $\mathcal{V}$ and the space of all lineons on $\mathcal{V}$ will be denoted by

$$\mathrm{Lin}\ \mathcal{V} := \mathrm{Lin}(\mathcal{V}, \mathcal{V}).$$

The composite of two lineons on $\mathcal{V}$ is again a lineon on $\mathcal{V}$ (see Prop.1 of Sect. 13). Composition in Lin $\mathcal{V}$ plays a role analogous to multiplication in $\mathbb{F}$. For this reason, the composite of two lineons is also called their **product**, and composition in Lin $\mathcal{V}$ is also referred to as **multiplication**. The lineon $\mathbf{0}$ is the analogue of the number 0 in $\mathbb{F}$ and the identity-lineon $\mathbf{1}_\mathcal{V}$ is the analogue of the number 1 in $\mathbb{F}$. It follows from Props. 1 and 3 of Sect. 14 that the multiplication and the addition in Lin $\mathcal{V}$ are related by distributive laws and hence that Lin $\mathcal{V}$ has the structure of a ring (see Sect. 06). Moreover, the composition-multiplication and scalar multiplication in Lin $\mathcal{V}$ are related by the associative laws

$$(\xi\mathbf{L})\mathbf{M} = \xi(\mathbf{LM}) = \mathbf{L}(\xi\mathbf{M}), \tag{18.1}$$

valid for all $\mathbf{L},\mathbf{M} \in \mathrm{Lin}\ \mathcal{V}, \xi \in \mathbb{F}$.

Since Lin $\mathcal{V}$ has, in addition to its structure as a linear space, also a structure given by the composition-multiplication, we refer to Lin $\mathcal{V}$ as the

algebra of lineons on $\mathcal{V}$. Most of the rules of elementary algebra are also valid in Lin $\mathcal{V}$. The most notable exception is the commutative law of multiplication: In general, if $\mathbf{L},\mathbf{M} \in$ Lin $\mathcal{V}$, then $\mathbf{LM}$ is not the same as $\mathbf{ML}$. If it happens that $\mathbf{LM} = \mathbf{ML}$, we say that $\mathbf{L}$ and $\mathbf{M}$ **commute**.

A subspace of Lin $\mathcal{V}$ that contains $\mathbf{1}_\mathcal{V}$ and is stable under multiplicationis called a **subalgebra** of Lin $\mathcal{V}$. For example

$$\mathbb{F}\mathbf{1}_\mathcal{V} := \{\xi\mathbf{1}_\mathcal{V} \mid \xi \in \mathbb{F}\}$$

is a subalgebra of Lin $\mathcal{V}$. A subalgebra that contains a given $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ is the set

$$\mathrm{Comm}\,\mathbf{L} := \{\mathbf{M} \in \mathrm{Lin}\ \mathcal{V} \mid \mathbf{ML} = \mathbf{LM}\} \tag{18.2}$$

of all lineons that commute with $\mathbf{L}$. It is called the **commutant-algebra** of $\mathbf{L}$.

The set of all automorphisms of $\mathcal{V}$, i.e., all linear isomorphisms from $\mathcal{V}$ to itself, is denoted by

$$\mathrm{Lis}\ \mathcal{V} := \mathrm{Lis}(\mathcal{V},\ \mathcal{V}.$$

Lis $\mathcal{V}$ is a subgroup of the group Perm $\mathcal{V}$ of all permutations of $\mathcal{V}$. We call Lis $\mathcal{V}$ the **lineon-group** of $\mathcal{V}$.

Pitfall: If $\mathcal{V}$ is not a zero-space, then Lis $\mathcal{V}$ does not contain the zero-lineon and hence cannot be a subspace of Lin $\mathcal{V}$. If $\dim\mathcal{V} > 1$, then $\mathrm{Lis}\,\mathcal{V} \neq (\mathrm{Lin}\,\mathcal{V})^\times$, i.e., there are non-zero lineons that are not invertible. ∎

Definition 1: *Let* $\mathbf{L}$ *be a lineon on the given linear space* $\mathcal{V}$. *We say that a subspace* $\mathcal{U}$ *of* $\mathcal{V}$ *is an* **L-subspace** *of* $\mathcal{V}$, *or simply an* **L-space**, *if it is* $\mathbf{L}$*-invariant, i.e., if* $\mathbf{L}_>(\mathcal{U}) \subset \mathcal{U}$ (see Sect. 03).

The zero-space $\{\mathbf{0}\}$ and $\mathcal{V}$ itself are $\mathbf{L}$-spaces for every $\mathbf{L} \in \mathrm{Lin}\,\mathcal{V}$. Also, Null $\mathbf{L}$ and Rng $\mathbf{L}$ are easily seen to be $\mathbf{L}$-spaces for every $\mathbf{L} \in \mathrm{Lin}\,\mathcal{V}$. Every subspace of $\mathcal{V}$ is an $(\lambda\mathbf{1}_\mathcal{V})$-space for every $\lambda \in \mathbb{F}$. If $\mathcal{U}$ is an $\mathbf{L}$-space, it is also an $\mathbf{L}^m$-space for every $m \in \mathbb{N}$, where $\mathbf{L}^m$, the m-th lineonic power of $\mathbf{L}$, is defined by $\mathbf{L}^m := \mathbf{L}^{\circ m}$, the m-th iterate of $\mathbf{L}$.

If $\mathcal{U}$ is an $\mathbf{L}$-subspace, then the adjustments $\mathbf{L}_{|\mathcal{U}} := \mathbf{L}\,|_\mathcal{U}^\mathcal{U}$ is linear and hence a lineon on $\mathcal{U}$. We have $(\mathbf{L}_{|\mathcal{U}})^m = (\mathbf{L}^m)_{|\mathcal{U}}$ for all $m \in \mathbb{N}$.

Let $\mathcal{V},\ \mathcal{V}'$ be linear spaces and let $\mathbf{A} : \mathcal{V} \to \mathcal{V}'$ be a linear isomorphism. The mapping

$$(\mathbf{L} \mapsto \mathbf{ALA}^{-1}) : \mathrm{Lin}\,\mathcal{V} \to \mathrm{Lin}\,\mathcal{V}'$$

is then an algebra-isomorphism. Also, we have

$$\mathrm{Rng}(\mathbf{ALA}^{-1}) = \mathbf{A}_{>}(\mathrm{Rng}\,\mathbf{L}) \tag{18.3}$$

and

$$\mathrm{Null}\,(\mathbf{ALA}^{-1}) = \mathbf{A}_{>}(\mathrm{Null}\,\mathbf{L}) \tag{18.4}$$

for all $\mathbf{L} \in \mathrm{Lin}\,\mathcal{V}$.

We assume now that $\mathcal{V}$ is finite-dimensional. By Prop. 7 of Sect. 17 we then have

$$\dim \mathrm{Lin}\,\mathcal{V} = (\dim\ \mathcal{V})^2. \tag{18.5}$$

The Linear Pigeonhole Principle of Sect. 17 has the following immediate consequences:

Proposition 1: *For a lineon* $\mathbf{L}$ *on* $\mathcal{V}$, *the following are equivalent.*

(i) $\mathbf{L}$ *is invertible*,

(ii) $\mathbf{L}$ *is surjective*,

(iii) $\mathbf{L}$ *is injective*,

(iv) $\mathrm{Null}\,\mathbf{L} = \{\,\mathbf{0}\}$.

Proposition 2: *Let* $\mathbf{L} \in \mathrm{Lin}\,\mathcal{V}$ *be given. Assume that* $\mathbf{L}$ *is left-invertible or right-invertible, i.e., that* $\mathbf{ML} = \mathbf{1}_{\mathcal{V}}$ *or* $\mathbf{LM} = \mathbf{1}_{\mathcal{V}}$ *for some* $\mathbf{M} \in \mathrm{Lin}\,\mathcal{V}$. *Then* $\mathbf{L}$ *is invertible, and* $\mathbf{M} = \mathbf{L}^{-1}$.

Let $\mathbf{b} := (\mathbf{b_i} \mid \mathbf{i} \in I)$ be a basis of $\mathcal{V}$. If $\mathbf{L} \in \mathrm{Lin}\,\mathcal{V}$, we call

$$[\mathbf{L}]_{\mathbf{b}} := \mathrm{lnc}_{\mathbf{b}}^{-1}\,\mathbf{L}\,\mathrm{lnc}_{\mathbf{b}} \tag{18.6}$$

the **matrix of L relative to the basis b**. (This is the matrix of $\mathbf{L}$, as defined in Sect. 16, when the bases $\mathbf{b}$ and $\mathbf{c}$ coincide.) This matrix is an element of $\mathrm{Lin}\,\mathbb{F}^I \cong \mathbb{F}^{I\times I}$ and $(\mathbf{L} \mapsto [\mathbf{L}]_{\mathbf{b}}) : \mathrm{Lin}\,\mathcal{V} \to \mathrm{Lin}\,\mathbb{F}^I$ is an algebra-isomorphism, i.e., a linear isomorphism that also preserves multiplication and the identity. By (16.11), the matrix $[\mathbf{L}]_{\mathbf{b}}$ is characterized by

$$\mathbf{Lb_i} = \sum_{j\in I}([\mathbf{L}]_{\mathbf{b}})_{j,i}\mathbf{b}_j, \quad \text{for all } i \in I. \tag{18.7}$$

The matrix of the identity-lineon is the **unit matrix** given by

$$[\mathbf{1}_{\mathcal{V}}]_{\mathbf{b}} = \mathbf{1}_{\mathbb{F}^I} = (\delta_{i,j} \mid (i,j) \in I \times I), \tag{18.8}$$

where $\delta_{i,j}$ is 0 or 1 depending on whether $i \neq j$ or $i = j$, respectively.

If I is a finite set, then the identification Lin $\mathbb{F}^I \cong \mathbb{F}^{I \times I}$ shows that a lineon on $\mathbb{F}^I$ may be regarded as an $I \times I$-matrix in $\mathbb{F}$. The algebra Lin $\mathbb{F}^I$ is therefore also called a **matrix-algebra**.

Notes 18

(1) The commonly accepted term for a linear mapping of a linear space to itself is "linear transformation", but "linear operator", "operator", or "tensor" are also used in some contexts. I have felt for many years that there is a crying need for a short term with no other meanings. About three years ago my colleague Victor Mizel proposed to me the use of the contraction "lineon" for "linear transformation", and his wife Phyllis pointed out that this lends itself to the formation of the adjective "lineonic", which turned out to be extremely useful.

I conjecture that the lack of a short term such as "lineon" is one of the reasons why so many mathematicians talk so often about matrices when they really mean, or should mean, lineons.

(2) Let $n \in \mathbb{N}$ be given. The group Lis $\mathbb{F}^n$ of lineons of $\mathbb{F}^n$, is often called the "group of invertible n by n matrices of $\mathbb{F}$" or the "general linear group $Gl(n, \mathbb{F})$". Sometimes, the notations $Gl(\mathcal{V})$ is used for the lineon-group Lis $\mathcal{V}$.

(3) What we call the "commutant-algebra" of $\mathbf{L}$ is often called the "centralizer" of $\mathbf{L}$.

19 Projections, Idempotents

Definition 1: *Let $\mathcal{V}$ be a linear space. A linear mapping $\mathbf{P} : \mathcal{V} \to \mathcal{U}$ to a given subspace $\mathcal{U}$ of $\mathcal{V}$ is called a* **projection** *if $\mathbf{P}|_{\mathcal{U}} = \mathbf{1}_{\mathcal{U}}$. A lineon $\mathbf{E} \in \text{Lin}\,\mathcal{V}$ is said to be* **idempotent** (*and is called* **an idempotent**) *if* $\mathbf{E}^2 = \mathbf{E}$.

Proposition 1: *A lineon $\mathbf{E} \in \text{Lin}\,\mathcal{V}$ is idempotent if and only if $\mathbf{E}|^{\text{Rng}}$ is a projection, i.e., if and only if $\mathbf{E}|_{\text{Rng}\mathbf{E} \subset \mathcal{V}} = \mathbf{1}_{\text{Rng}\mathbf{E} \subset \mathcal{V}}$.*

Proof: Put $\mathcal{U} := \text{Rng}\,\mathbf{E}$.

Assume that $\mathbf{E}$ is idempotent and let $\mathbf{u} \in \mathcal{U}$ be given. We may choose $\mathbf{v} \in \mathcal{V}$ such that $\mathbf{u} = \mathbf{E}\mathbf{v}$. Then

$$\mathbf{E}\mathbf{u} = \mathbf{E}(\mathbf{E}\mathbf{v}) = \mathbf{E}^2\mathbf{v} = \mathbf{E}\mathbf{v} = \mathbf{u}.$$

Since $\mathbf{u} \in \mathcal{U}$ was arbitrary, it follows that $\mathbf{E}|_{\mathcal{U}} = \mathbf{1}_{\mathcal{U} \subset \mathcal{V}}$.

Now assume that $\mathbf{E}|_{\mathcal{U}} = \mathbf{1}_{\mathcal{U}\subset\mathcal{V}}$ and let $\mathbf{v} \in \mathcal{V}$ be given. Then $\mathbf{Ev} \in \text{Rng}\,\mathbf{E} = \mathcal{U}$ and hence

$$\mathbf{E}^2\mathbf{v} = \mathbf{E}(\mathbf{Ev}) = \mathbf{1}_{\mathcal{U}\subset\mathcal{V}}(\mathbf{Ev}) = \mathbf{Ev}.$$

Since $\mathbf{v} \in \mathcal{V}$ was arbitrary, it follows that $\mathbf{E}^2 = \mathbf{E}$. ■

Proposition 2: *A linear mapping* $\mathbf{P} : \mathcal{V} \to \mathcal{U}$ *from a linear space* $\mathcal{V}$ *to a subspace* $\mathcal{U}$ *of* $\mathcal{V}$ *is a projection if and only if it is surjective and* $\mathbf{P}|^{\mathcal{V}} \in \text{Lin}\,\mathcal{V}$ *is idempotent.*

Proof: Apply Prop. 1 to the case when $\mathbf{E} := \mathbf{P}|^{\mathcal{V}}$. ■

Proposition 3: *Let* $\mathbf{E} \in \text{Lin}\,\mathcal{V}$ *be given. Then the following are equivalent:*

(i) $\mathbf{E}$ *is idempotent.*

(ii) $\mathbf{1}_{\mathcal{V}} - \mathbf{E}$ *is idempotent.*

(iii) $\text{Rng}(\mathbf{1}_{\mathcal{V}} - \mathbf{E}) = \text{Null}\,\mathbf{E}$.

(iv) $\text{Rng}\,\mathbf{E} = \text{Null}\,(\mathbf{1}_{\mathcal{V}} - \mathbf{E})$.

Proof: We observe that

$$\text{Null}\,\mathbf{L} \subset \text{Rng}(\mathbf{1}_{\mathcal{V}} - \mathbf{L}) \tag{19.1}$$

is valid for all $\mathbf{L} \in \text{Lin}\,\mathcal{V}$.
(i) ⇔ (iii): We have $\mathbf{E} = \mathbf{E}^2$ if and only if $(\mathbf{E} - \mathbf{E}^2)\mathbf{v} = \mathbf{E}(\mathbf{1}_{\mathcal{V}} - \mathbf{E})\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in \mathcal{V}$, which is the case if and only if $\text{Rng}(\mathbf{1}_{\mathcal{V}} - \mathbf{E}) \subset \text{Null}\,\mathbf{E}$. In view of (19.1), this is equivalent to (iii).
(ii) ⇔ (iv): This follows by applying (i) ⇔ (iii) with $\mathbf{E}$ replaced by $\mathbf{1}_{\mathcal{V}} - \mathbf{E}$.
(i) ⇔ (ii): This follows from the identity $(\mathbf{1}_{\mathcal{V}} - \mathbf{E})^2 = (\mathbf{1}_{\mathcal{V}} - \mathbf{E}) + (\mathbf{E}^2 - \mathbf{E})$, valid for all $\mathbf{E} \in \text{Lin}\,\mathcal{V}$. ■

The following proposition shows how projections and idempotents are associated with pairs of supplementary subspaces.

Proposition 4: *Let* $\mathcal{U}_1, \mathcal{U}_2$ *be subspaces of the linear space* $\mathcal{V}$. *Then the following are equivalent:*

(i) $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *are supplementary.*

(ii) *There are projections* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *onto* $\mathcal{U}_1$ *and* $\mathcal{U}_2$, *respectively, such that*

$$\mathbf{v} = \mathbf{P}_1\mathbf{v} + \mathbf{P}_2\mathbf{v} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}. \tag{19.2}$$

(iii) *There is a projection* $\mathbf{P}_1 : \mathcal{V} \to \mathcal{U}_1$ *such that* $\mathcal{U}_2 = \text{Null}\ \mathbf{P}_1$.

(iv) *There are idempotents* $\mathbf{E}_1, \mathbf{E}_2 \in \text{Lin}\ \mathcal{V}$ *such that* $\mathcal{U}_1 = \text{Rng}\ \mathbf{E}_1, \mathcal{U}_2 = \text{Rng}\ \mathbf{E}_2$ *and*

$$\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{1}_{\mathcal{V}}. \tag{19.3}$$

(v) *There is an idempotent* $\mathbf{E}_1 \in \text{Lin}\ \mathcal{V}$ *such that* $\mathcal{U}_1 = \text{Rng}\ \mathbf{E}_1$ *and* $\mathcal{U}_2 = \text{Null}\ \mathbf{E}_1$.

The projections $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *and the idempotents* $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *are uniquely determined by the subspaces* $\mathcal{U}_1, \mathcal{U}_2$.

Proof: (i) $\Leftrightarrow$ (ii): If $\mathcal{U}_1$ and $\mathcal{U}_2$ are supplementary, it follows from Prop. 4, (ii) of Sect. 12 that every $\mathbf{v} \in \mathcal{V}$ uniquely determines $\mathbf{u}_1 \in \mathcal{U}_1$ and $\mathbf{u}_2 \in \mathcal{U}_2$ such that $\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2$. This means that there are mappings $\mathbf{P}_1 : \mathcal{V} \to \mathcal{U}_1$ and $\mathbf{P}_2 : \mathcal{V} \to \mathcal{U}_2$ such that (19.2) holds. It is easily seen that $\mathbf{P}_1$ and $\mathbf{P}_2$ are projections.
(ii) $\Rightarrow$ (iii): It is clear from (19.2) that $\mathbf{v} \in \text{Null}\ \mathbf{P}_1$, i.e., $\mathbf{P}_1\ \mathbf{v} = \mathbf{0}$, holds if and only if $\mathbf{v} = \mathbf{P}_2\mathbf{v}$, which is the case if and only if $\mathbf{v} \in \mathcal{U}_2$.
(iii) $\Rightarrow$ (v): This is an immediate consequence of Prop. 2, with $\mathbf{E}_1 := \mathbf{P}_1 |^{\mathcal{V}}$.
(v) $\Rightarrow$ (iv): Put $\mathbf{E}_2 := \mathbf{1}_{\mathcal{V}} - \mathbf{E}_1$. Then $\mathbf{E}_2$ is idempotent and $\text{Rng}\ \mathbf{E}_2 = \text{Null}\ \mathbf{E}_1 = \mathcal{U}_2$ by Prop. 2.
(iv) $\Rightarrow$ (i): We observe that $\text{Null}\ \mathbf{L} \cap \text{Null}\ (\mathbf{1}_{\mathcal{V}} - \mathbf{L}) = \{\mathbf{0}\}$ and $\mathcal{V} = \text{Rng}\ \mathbf{L} + \text{Rng}(\mathbf{1}_{\mathcal{V}} - \mathbf{L})$ hold for all $\mathbf{L} \in \text{Lin}\mathcal{V}$. Using this observation when $\mathbf{L} := \mathbf{E}_1$ and hence $\mathbf{E}_2 = \mathbf{1}_{\mathcal{V}} - \mathbf{L}$ we conclude that $\mathcal{V} = \mathcal{U}_1 + \mathcal{U}_2$ and, from Prop. 3, that $\{\mathbf{0}\} = \mathcal{U}_1 \cap \mathcal{U}_2$. By Def. 2 of Sect. 12 this means that (i) holds.

The proof of uniqueness of $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{E}_1$, $\mathbf{E}_2$ is left to the reader. ∎

The next result, which is easily proved, shows how linear mappings with domain $\mathcal{V}$ are determined by their restrictions to each of two supplementary subspaces of $\mathcal{V}$.

Proposition 5: *Let* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *be supplementary subspaces of* $\mathcal{V}$. *For every linear space* $\mathcal{V}'$ *and every* $\mathbf{L}_1 \in \text{Lin}(\mathcal{U}_1, \mathcal{V}')$, $\mathbf{L}_2 \in \text{Lin}(\mathcal{U}_2, \mathcal{V}')$, *there is exactly one* $\mathbf{L} \in \text{Lin}(\mathcal{V}, \mathcal{V}')$ *such that*

$$\mathbf{L}_1 = \mathbf{L}\,|_{\mathcal{U}_1} \quad \text{and} \quad \mathbf{L}_2 = \mathbf{L}\,|_{\mathcal{U}_2}\,.$$

It is given by

$$\mathbf{L} := \mathbf{L}_1\mathbf{P}_1 + \mathbf{L}_2\mathbf{P}_2,$$

where $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are the projections of Prop. 4, (ii).*

Notes 19

(1) Some textbooks use the term "projection" in this context as a synonym for "idempotent". Although the two differ only in the choice of codomain, I believe that the distinction is useful.

110 Problems for Chapter 1

1. Let $\mathcal{V}$ and $\mathcal{V}'$ be linear spaces. Show that a given mapping $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ is linear if and only if its graph $\mathrm{Gr}(\mathbf{L})$ (defined by (03.1)) is a subspace of $\mathcal{V} \times \mathcal{V}'$.

2. Let $\mathcal{V}$ be a linear space. For each $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$, define the **left-multiplication mapping**

$$\mathbf{Le}_{\mathbf{L}} : \mathrm{Lin}\mathcal{V} \to \mathrm{Lin}\mathcal{V}$$

by

$$\mathbf{Le}_L(\mathbf{M}) := \mathbf{LM} \;\text{ for all }\; \mathbf{M} \in \mathrm{Lin}\,\mathcal{V} \qquad (P1.1)$$

   (a) Show that $\mathbf{Le}_{\mathbf{L}}$ is linear for all $\mathbf{L} \in \mathrm{Lin}\,\mathcal{V}$.

   (b) Show that $\mathbf{Le}_{\mathbf{L}}\mathbf{Le}_{\mathbf{K}} = \mathbf{Le}_{\mathbf{LK}}$ for all $\mathbf{L},\mathbf{K} \in \mathrm{Lin}\,\mathcal{V}$.

   (c) Show that $\mathbf{Le}_{\mathbf{L}}$ is invertible if and only if $\mathbf{L}$ is invertible and that $(\mathbf{Le}_{\mathbf{L}}^{-1}) = \mathbf{Le}_{\mathbf{L}^{-1}}$ if this is the case.

3. Consider

$$L := \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2^]\times 2^]} \cong \mathrm{Lin}\mathbb{R}^2.$$

   (a) Show that L is invertible and find its inverse.

(b) Determine the matrix of $\mathbf{Le_L}$, as defined by $(P1.1)$, relative to the list-basis

$$B := \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \text{ of Lin } \mathbb{R}^2.$$

(c) Determine the inverse of the matrix you found in (b).

4. Let $C^\infty(\mathbb{R})$ be the set of all functions $f \in \text{Map}(\mathbb{R}, \mathbb{R})$ such that f is k-times differentiable for all $\mathbf{k} \in \mathbb{N}$ (see Sect. 08). It is clear that $C^\infty\mathbb{R}$ is a subspace of the linear space Map $(\mathbb{R}, \mathbb{R})$ (see Sect. 13). Define $D \in \text{Lin } C^\infty(\mathbb{R})$ by
$$Df := \partial f \quad \text{for all} \quad f \in C^\infty(\mathbb{R}) \tag{$P1.2$}$$
(see (08.31)) and, for each $n \in \mathbb{N}$, the subspace $\mathcal{P}_n$ of $C^\infty(\mathbb{R})$ by
$$\mathcal{P}_n := \text{Null} D^n. \tag{$P1.3$}$$

(a) Show that the sequence
$$p := (\iota^k \mid k \in \mathbb{N}) \tag{$P1.4$}$$
is linearly independent (see (08.26) for notation). Is it a basis of $C^\infty(\mathbb{R})$?

(b) Show that, for each $n \in \mathbb{N}$,
$$p\,|_{n^[} = (\iota^k \mid k \in n^[) \tag{$P1.5$}$$
is a basis of $\mathcal{P}_n$ and hence that $\dim\mathcal{P}_n = n$.

5. Let $C^\infty(\mathbb{R})$ and $D \in \text{Lin } C^\infty(\mathbb{R})$ by defined as in Problem 4. Define $M \in \text{Lin } C^\infty(\mathbb{R})$ by $Mf := \iota f$ for all $f \in C^\infty(\mathbb{R})$ (see (08.26)). Prove that
$$DM - MD = 1_{C^\infty(\mathbb{R})}. \tag{$P1.6$}$$

6. Let $C^\infty(\mathbb{R})$ be defined as in Problem 4 and define $S \in \text{Lin } C^\infty(\mathbb{R})$ by
$$Sf := f \circ (\iota + 1) \text{ for all } f \in C^\infty(\mathbb{R}) \tag{$P1.7$}$$

(a) Show that, for each $n \in \mathbb{N}$, the subspace $\mathcal{P}_n$ of $C^\infty(\mathbb{R})$ defined by $(P1.3)$ is an S-space (see Sect. 18).

(b) Find the matrix of $S\,|_{\mathcal{P}_n} \in \mathbf{Lin}\mathcal{P}_n$ relative to the basis $p\,|_{n^[}$ of $\mathcal{P}_n$ given by $(P1.5)$.

7. Let $\mathcal{V}$ be a finite-dimensional linear space and let $\mathbf{L} \in \text{Lin}\mathcal{V}$ be given.

(i) Rng $\mathbf{L} \cap$ Null $\mathbf{L} = \{\mathbf{0}\}$,

(ii) Null $(\mathbf{L}^2) \subset$ Null $\mathbf{L}$,

(iii) Rng $\mathbf{L}$ + Null $\mathbf{L} = \mathcal{V}$.

8. Consider the subspaces $\mathcal{U}_1 := \mathbb{R}(1,1)$ and $\mathcal{U}_2 := \mathbb{R}(1,-1)$ of $\mathbb{R}^2$.

   (a) Show that $\mathcal{U}_1$ and $\mathcal{U}_2$ are supplementary in $\mathbb{R}^2$.

   (b) Find the idempotent matrices $E_1, E_2 \in \mathbb{R}^{2^] \times 2^]} \cong \text{Lin}\mathbb{R}^2$ which are determined by $\mathcal{U}_1, \mathcal{U}_2$ according to Prop. 4 of Sect. 19.

   (c) Determine a basis $b = (b_1, b_2)$ of $\mathbb{R}^2$ such that the matrices of E_1 and E_2 relative to b are $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, respectively.

9. Let a linear space $\mathcal{V}$, a lineon $\mathbf{L}$ on $\mathcal{V}$, and $m \in \mathbb{N}$ be given.

   (a) Show that $\text{Rng}(\mathbf{L}^m)$ is an $\mathbf{L}$-subspace of $\mathcal{V}$.

   (b) Show that $\text{Rng}(\mathbf{L}^{m+1}) \subset \text{Rng}(\mathbf{L}^m)$, with equality if and only if the adjustment $\mathbf{L}_{|\text{Rng}(\mathbf{L}^m)}$ is surjective.

10. Let $\mathcal{V}$ be a finite-dimensional linear space and let $\mathbf{L}$ be a **nilpotent** lineon on $\mathcal{V}$ with **nilpotency** m, which means that $\mathbf{L}^m = \mathbf{0}$ but $\mathbf{L}^k \neq \mathbf{0}$ for all $k \in m^[$ (see also Sect. 93). Prove:

    (a) If $m > 0$, then Null $\mathbf{L} \neq \{\mathbf{0}\}$.

    (b) The only $\mathbf{L}$-subspace $\mathcal{U}$ for which the adjustment $\mathbf{L}_{|\mathcal{U}}$ is invertible is the zero-space.

    (c) The nilpotency cannot exceed dim $\mathcal{V}$. (Hint: Use Problem 9.)

11. Let a linear space $\mathcal{V}$ and a lineon $\mathbf{J} \in$ Lin $\mathcal{V}$ be given.

    (a) Show that the commutant-algebray of $\mathbf{J}$ (see (18.2)) is all of Lin $\mathcal{V}$, i.e., that Comm $\mathbf{J}$ = Lin $\mathcal{V}$, if and only if $\mathbf{J} \in \mathbb{F}\mathbf{1}_{\mathcal{V}}$.

    (b) Prove: If dim $\mathcal{V} > 0$, if $\mathbf{J}^2 = -\mathbf{1}_{\mathcal{V}}$, and if the equation $?\xi \in \mathbb{F},\ \xi^2 + 1 = 0$ has no solution, then Comm $\mathbf{J} \neq$ Lin $\mathcal{V}$.

12. Let a linear space $\mathcal{V}$ and a lineon $\mathbf{J}$ on $\mathcal{V}$ satisfying $\mathbf{J}^2 = -\mathbf{1}_{\mathcal{V}}$ be given. Define $\mathbf{E} \in \text{Lin}(\text{Lin}\ \mathcal{V})$ by

$$\mathbf{EL} := \tfrac{1}{2}(\mathbf{L} + \mathbf{JLJ}) \text{ for all } \mathbf{L} \in \text{Lin}\ \mathcal{V}. \qquad (P1.8)$$

(a) Show that $\mathbf{E}$ is idempotent.

(b) Show that Null $\mathbf{E} = \operatorname{Comm}\mathbf{J}$, the commutant-algebra of $\mathbf{J}$ (see (18.2)).

(c) Show that

$$\operatorname{Rng}\mathbf{E} = \{\mathbf{L} \in \operatorname{Lin}\mathcal{V} \mid \mathbf{LJ} = -\mathbf{JL}\}, \qquad (P1.9)$$

the space of all lineons that "anticommute" with $\mathbf{J}$.

(d) Prove: If dim $\mathcal{V} > 0$ and if $\{\xi \in \mathbb{F} \mid \xi^2 + 1 = 0\} = \emptyset$, then the space ($P$1.9) is not the zero-space. (Hint: Use (b) of Problem 11 and Prop. 4 of Sect. 19.)