Summary of Day 16

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June 11, 2014

1 Objectives

- Use determinants to calculate eigenvalues, eigenvectors, find eigenspaces.
- Define and begin to explore algebraic and geometric multiplicity.
- Begin the discussion of similarity and diagonalization.

2 Summary

- We can also now find eigenvalues more efficiently.

  Theorem λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$
- Now, let's look back at the calculation of eigenvalues and eigenvectors. We will usually find the eigenvalues in the following way:
  (a) Calculate the **characteristic polynomial**: $\det(A - \lambda I)$.
  (b) The eigenvalues correspond to the zeros of this polynomial.
  (c) Calculate the null space of $A - \lambda I$ for each eigenvalue. These are the corresponding eigenvectors for each eigenvalue.
  (d) Because there are ∞-many eigenvectors associated with each eigenvalue, we will always find a basis for the eigenspace in order to express it.

  Example Find the eigenvalues and a basis for each eigenspace of:

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 6 \end{pmatrix}$$

  Example Find the eigenvalues and a basis for each eigenspace of:

$$B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

- It's easy to see now that there are at most n eigenvalues of a $n \times n$ matrix since a degree n polynomial could not have more than n roots. There are some conditions that would make $< n$ eigenvalues. There is a very important theorem called the *Fundamental Theorem of Algebra* which says that any degree n polynomial has n roots *over the complex numbers*. Over the real numbers, polynomials have no such guarantee. For example, if the characteristic polynomials was $\lambda^2 + 1$ it would have no *real* eigenvalues.

- Apart from that though, the fact that a degree n polynomial has n roots counts *multiplicity*. For example, $(\lambda - 1)^2$ has only one distinct root ($\lambda = 1$) but that roots has multiplicity two.

- We define the **alegbraic multiplicity** of an eigenvalue to be it's multiplicity in the characteristic polynomial. It's **geometric multiplicity** is the dimension of it's eigenspace. We will compare these two notions soon, but they are not in general the same.

  Example Determine the algebraic and geometric multiplicity for the two matrices in the above examples.

- Question: What does it mean to have an eigenvalue of 0?

- We can read of the eigenvalues of some matrices that come from simple operations on others from the eigenvalues of the others (that wasn't the best way to say this). You'll see what I mean:

  Theorem If A is a $n \times n$ matrix with eigenvalue λ with corresponding eigenvector $\mathbf{x}$ then

  (a) λ^m is an eigenvalue of A^m with corresponding eigenvector $\mathbf{x}$.

  (b) $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{x}$ (assuming A is invertible)

- A useful application for eigenvalues is the aid in doing some calculations that would be otherwise infeasible to do. We'll revisit this kind of application more when we talk about diagonalization, but here's a taste:

Example Compute:

$$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}^{10} \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

- For now though, we will explore another equivalence relation on matrices. We have explored one already: having the same reduced row echelon form. This preserves lots of nice properties, like invertibility, rank, nullity, and even the row space and the null space (but *not* the column space, although it will preserve the dimension of this space, which is exactly the rank).

  The bad thing of this equivalence relation is it does *not* preserve most spectral properties. That is, elementary row operations in general do change the spectrum or eigenvectors (although it does preserve some properties of the spectrum; for example elementary row operations will never introduce/eliminate a 0 eigenvalue). It also does not preserve the determinant (although it does preserve the nonzero-ness of it).

- We say a matrix B is **similar to** (or **conjugate to**) matrix A if there is some invertible matrix P such that:

  $$B = P^{-1}AP$$

  We notate this as $A \sim B$.

  Theorem For any A, B square matrices:

  - $A \sim A$.
  - If $A \sim B$ then $B \sim A$.
  - If $A \sim B$ and $B \sim C$ then $A \sim C$.

- Like the equivalence relation of row equivalence, similarity preserves lots of properties of a matrix. Let's write them down:

  Theorem If A and B are $n \times n$ matrices where $A \sim B$ then:

  - $\det(A) = \det(B)$
  - A is invertible if and only if B is.
  - A and B have the same rank.
  - A and B have the same characteristic polynomial.
  - A and B have the same eigenvalues.

  Proof.

- This theorem gives us some easy ways to determine that matrices are not similar.

  Example The following matrices are not similar: DO IT

- We have seen that having upper triangular and diagonal matrices helps a lot with computations. To that end we make the following definition: a matrix A is **diagonalizable** if there is some diagonal matrix B such that $A \sim B$.

- This might seem artificial, but it is of computational significance. For example, suppose A was diagonalizable, and we wanted to calculate A^{100} (which is actually not the most uncommon thing to do. We might even want to look at $\lim A^n$ as $n \to \infty$!). Well, if A is diagonalizable by B then there is P such that $A = P^{-1}BP$. Then:

$$A^{100} = \underbrace{(P^{-1}BP)(P^{-1}BP)\cdots(P^{-1}BP)}_{100}$$

  But, this is just:

$$A^{100} = P^{-1}B^{100}P$$

  But raising a diagonal matrix to the 100 power is just raising the diagonal entries by that power!

- If A is diagonalizable, then the matrix B must have the eigenvalues of A in its diagonal entries. Can you see why?

Theorem If a matrix A is diagonalizable then it's determinant is the product of its eigenvalues.

- We will explore when matrices are diagonalizable.