Day 26

June 28, 2012

1 Edge Coloring

Definition 1. If G is a graph then a **edge n-coloring** of G is an assignment of edges in G to the set $[n]$

Remember the pigeonhole principal. What it allows us to do is to say in any collection, after a certain point, you cannot avoid having some property hold. It is inevitable, for instance, if you have 5 points in a square then 2 will be closer than $\frac{1}{\sqrt{2}}$. Inevitable.

The theory of such properties is usually called **Ramsey Theory**. It proves results that guarantee regularties in large enough structures.

Theorem 1. *In any group of 6 people, there are 3 people who are either all friends or 3 people who are all not friends (here we assume that friendship is symmetric).*

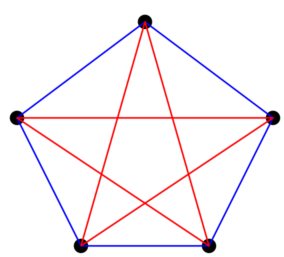
Proof. Pick some person; call him Bob. There are 5 other people in the group, and 2 categories for each relationship: friend or not. Therefore, by pigeonhole principal, 3 of those relationships are the same.

Without loss of generality, Bob is friends will all 3. Then look at their relationship with each other; if any two are friends, then we are done, as Bob is the friends with two people who are friends with each other. Otherwise, all 3 are not friends, and this completes the proofs, as it is a group of 3 people who are all not friends. □

In a graph-theoretic setting, the above theorem (and proof) can be rephrased to say:

Theorem 2. *In any edge coloring of K_6 there exists a triangle (a K_3 clique) such that every edge in the triangle has the same color (ie. is **monochromatic**).*

This is an interesting phenonomia, as it says if you increase the number of vertices of a complete graph, there is a regularity you cannot avoid: monochromatic triangles. You can avoid monochromatic triangles in K_5 with the following coloring:



So 6 is a threshold, afterwhich you will always have a monochromatic triangle.

A natural question to ask is: is it always true for any size clique you want k, is there a point for which for all large enough complete graphs and any colorings of these graphs, you will get a monochromatic k-clique? The answer is **yes**.

Theorem 3. *If there is a size for which we can guarantee we either have a k-clique of color 1 or l-clique of color 2 then we denote the smallest such size $R(k, l)$.*

Theorem 4 (Ramsey's Theorem)**.** *For any k, l, there is some N such that for any edge two coloring of K_N with colors 0 and 1 there is either a monochromatic k-clique of color 1 or a monochromatic l clique of color 2*

Proof. We prove this by induction on $n + m$.

If $k = 1$ then we are done as any graph with contain a monochromatic 1-clique, so the N we find is $N = 1$ Similarly is $l = 1$.

Therefore, suppose that we are at k, l are not 1 and assume it's true for all pairs of smaller sum. We want to prove that it is true for k and l.

Our central claim is that we can gaurentee a monochromatic k-clique of color 1 or a monochromatic l-clique of color 2 for a graph of order $R(k-1, l) + R(k, l-1)$. We will work toward this claim.

Suppose that our graph has $R(k-1, l) + R(k, l-1)$ vertices. Pick a vertex; which ever your favorite is. Call it v. Now, look at all the edges that that vertex is connnected to. There is a lot of them; namely $R(k-1, 1) + R(k, l-1) - 1$ many.

Each of these is either colored 0 or 1. Let V_0 be all the vertices that v connected to that vertex are colored 0, and V_1 the ones that it is colored red.

Okay, the graph has $R(k-1, l) + R(k, l-1)$ vertices, and some are in V_0 and some are in V_1, and then there is v which is in neither. A moments thought shows that there are either $R(k-1, l)$ vertices in V_0 or $R(l, k-1)$ in V_1 (why? well, if not then there cannot be that many vertices in the graph!).

Now we are almost done. Without loss of generality, there is $R(k-1, l)$ vertices in V_0. By definition of $R(k-1, l)$ there is either a $k-1$ clique of color 0 or a l-clique of color 1. In the latter case we are done as we have an l-clique of color 1.

Otherwise, we have a $k-1$ clique of color 0. Moreover, everything in V_0 is connected to v, and the edge connecting them is color 0. Therefore, adding v in this clique we get a monochromatic k cliquee of color 0. This completes the proof of the claim.

By the claim, letting $N = R(k-1, l) + R(k, l-1)$ we are done! □

So, because there is a natural number N is is some smallest number that works. As stated above, we call these numbers $R(k, l)$. They are called the **Ramsey Numbers**. A particular class is the **Diagonal Ramsey Numbers**, which are $R(n, n)$.

The first theory we proved today can be rephrased as $R(3, 3) = 6$. That is, we showed that in a complete graph on 6 vertices, any coloring of the edges gives a monochromatic triangle; this showed that $R(3, 3) \leq 6$. Then we showed an example of a coloring of K_5 which did not have any monochromatic triangles, which showed $R(3, 3) > 5$.

Fact 1. $R(4, 4) = 18$. No exact value for a Diagonal Ramsey number is known beyond that, although it is know that $43 \leq R(4, 4) \leq 49$ and $102 \leq R(5, 5) \leq 165$. It is also known that these numbers grow very quickly (exponentially, where the exponential base is somewhere between $\sqrt{2}$ and 4).

Paul Erdös, who was the leading combinatorist of the 20th century, and probably one of the most prolific mathematicians ever, is credited with a lot of the results in Ramsey Theory. Especially the first proofs of better bounds on the Ramsey Numbers. His proof of the lower bound from 1947 is still the best lower bound known, and was proved using a method that he popularized called the **probabilistic method**.

Erdös recognized that calculating these numbers was near impossible. He is quoted as saying that if aliens every came to Earth threatened to destroy Earth unless we give an answer to $R(5, 5)$ we should muster together all of our computational power and mathematicans to get an answer. However, if they demand $R(6, 6)$ we should muster together every person and resource to fight the aliens.