Day 24

June 26, 2012

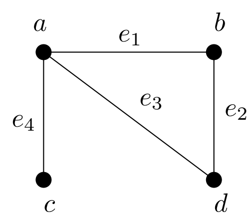
1 Graph Theory

Recall when we talked about relations. Relations were a way to talk about different objects (perhaps a set) and say that one object was related to another. Graph theory can be viewed as study of relations, especially the finite kind.

1.1 Graphs

Definition 1. A **(simple) graph** is a mathematical structure consisting of two parts: a set V called the set of **vertices**, and a relation E on the set V called the **edge relation**; we require that the edge relation is irreflexive and symmetric, but make no other restrictions.

We picture a graph by arranging vertex as dots on the plane, and connecting two of the dots if and only if they are related by the edge relation. For instance, the graph with vertices $V = \{a, b, c, d\}$, and edge relation corresponding to $\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}\}$ is pictured by:



Definition 2. Every line in the picture can represented by the double-ton $\{v_1, v_2\}$, where $v_1, v_2 \in V$, and $v_1 E v_2$. We call this connection an **edge**.

We say the **size** of a graph is the number of edges in the graph. We notate this with $|E|$ where E is the edge relation.

We say v_1 and v_2 are **adjacent** if $v_1 E v_2$.

If e is an edge and $v \in e$, then we say they are **incident**.

We define a function $\deg : V \to \mathbb{N}$ on the set of vertices. $\deg(v)$ is equal to the number of edges that contain v. We call this the **degree** of a vertex.

Example 1. In the above picture, e_1 is the edge $\{a, b\}$. vertex a is incident to edge e_1. The degree of a is 3, and the degree of c is 1.

Theorem 1. *If G is a graph with vertices V then*

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof. Every edge is counted twice in the sum of the degrees. Thus we can divide by 2 and this will count the number of edges. □

Theorem 2 (Handshaking Lemma)**.** *In any graph, there is an even number of odd degree vertices.*

Proof. Exericse. □

Given a graph, we often imagine "traversing" the graph. This leads us to make the following definitions:

Definition 3. If G is a graph with vertex set V and edge relation E, then a **walk** through the graph is a sequence

$$\langle v_1, v_2, v_3, \dots v_n \rangle$$

Where each v_i is a vertex, and v_i is adjacent to v_{i+1} for each i. A walk is closed if $v_1 = v_n$; otherwise it is open.

A **(simple) path** is a walk where no vertex is repeated. A **(simple) cycle** is a closed walk where only the first and last vertex is repeated.

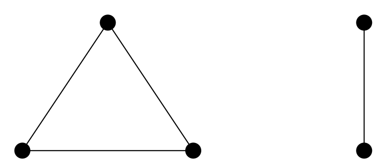
A **tail** is a walk in which no edge is repeated. A **circuit** is a closed trail.

Remark 1. Formal graph theory is actually still a rather new field, especially compared to older fields like number theory and geometry. The terminology is still sort of non-standard. Some people say "path" when really it is a walk, as we defined it. Be careful of terminology; if you look at a graph theory, make sure you know how they have defined all of these terms.

Imagine a graph represents a city. The vertices are different points of interest, and the edges are connections between them (perhaps roads, or maybe subway lines). We would like to know that the we can get anywhere in the city from any point we are at; this is obviously a very desireable property.

Definition 4. A graph is **connected** if for every distinct v_1, v_2 there is a path starting at v_1 and ending at v_2. Otherwise, we say a graph is **disconnected**.

All graphs are not connected. For example:



This, as you can see, has no paths connecting vertices on the left hand to ones on the right.

Definition 5. We define an equivalence relation $\sim$ on the set of vertices, where we say $a \sim b$ if and only if there is a path from a to b. We call the equivalence classes of this relation the **connected components** of the graph.

Example 2. If the graph is connected, there is only one connected component. However, in the graph above, there are two connected components. One is the triangle on the left, and the other the line on the right.

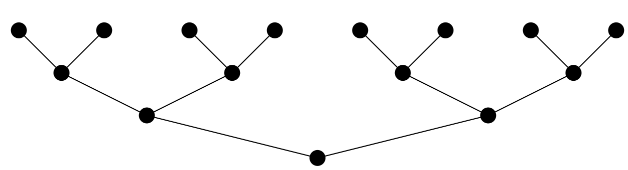
Definition 6. If G is a graph with vertex set V and edge relation E then we say G' is a **subgraph** of G, written $G' \subseteq G$, if G has vertex set V' and edge relation E' and the two conditions hold:

1. $V' \subseteq V$
2. For every a, $b \in V'$, if $aE'b$ then aEb. That is, G' makes no new edges.

1.2 Trees

Definition 7. A **tree** is a graph which is minimally connected; that is, a tree is a graph which the removal of any edge makes the graph disconnected.

Example 3. Trees are a very important notion. Here is an example of a tree:



Theorem 3. *Let T be a graph with n vertices. The following are equivalent:*

1. *T is a tree.*
2. *T is connected and has $n-1$ edges.*
3. *T is connected and has no cycles.*
4. *For every v, u vertices in T, there is a unique path from v to u*

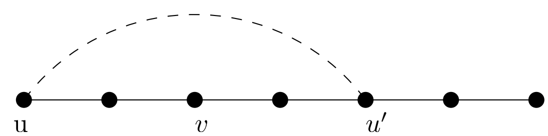
We will not prove these. You should do it as an exercise! (Seriously; this is an exam technique question). But we will prove a few helpful lemmas.

Lemma 1. *If a graph G with n vertices ($n \geq 2$) has $< n-1$ edges, then it is disconnected.*

Proof. We prove this by the method of infinite descent. We go by contradiction, assuming that it is false, and we take n to be the smallest counterexample. So, n is such that there is a connected graph on n vertices and $n-2$ edges. We claim there is a degree 1 vertex. Otherwise, all vertices are of degree 2 or more. $\left(\sum_{v \in V} \deg(v)\right) = 2|E|$. As the sum is at least $2n$, we have E is at least n. Thus there is a degree 1 vertex. Remove it from the graph; obviously the graph is still connected. This is a graph on $n-1$ vertices which is connected. □

Lemma 2. *If a connected graph has no degree one vertex then it has a cycle.*

Proof. Take G connected with no degree one vertices. We claim there is a cycle. Start at any vertex v and build a maximal path, that is build a path that you cannot make longer without repeated a vertex.



Let u be an endpoint of the path. As u is maximal, we every neighbor of v is in the path. Furthermore, as u has at least degree 2 so it has an edge not in the path. This edge connects u to some vertex u'. This makes a cycle. □

We now have a very important theorem about trees.

Theorem 4. *Every tree has a degree one vertex.*

Proof. This is from the last lemma and the theorem which says that trees are acyclic. □

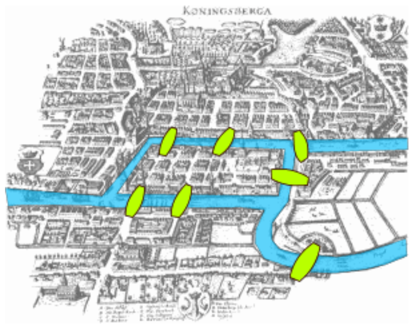
Definition 8. A vertex which has degree one is called a **leaf**

We often do induction on trees and use this property in our induction steps. An example would be (3) implies (4) above.

Theorem 5. *If T is a tree (in particular, acyclic connected) then for every u and v vertices in T there is a unique path from v to u.*

Proof. Do induction on the number of vertices of T. It is trivial in the case when $n = 1$, as there is nothing to check.

Suppose that T is a tree on $n+1$ many vertices, and for any tree on n vertices we have the property. As T is acyclic, T has a leaf, l. Remove l from T. Now we have a tree on n vertices as it must remain connected; so by the induction hypothesis, every pair of vertices has a unique path. Put l back in. Take v, w in T. If neither v or w is l, then they were on the tree with l removed, and therefore these is a unique path; l could not have created a new path as l is degree 1 so is a "dead end." If either v or w is l, then wlog it is v. Look at the vertex l is connected to, v' (there is only one as l is degree 1). Every path leaving v goes through v', and there is only one path from v' to w. □

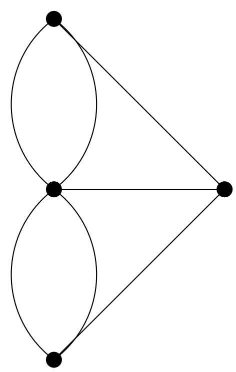


1.3 Königsberg Bridge Problem

Königsberg was a city in East Prussia (now Russia). Running through the city was the Pregel River.

As shown above, there were two main bodies of land created by the river, and a island in the center. There were 7 bridges connecting the various pieces of land (as shown above). The question posed was can you go travel across every bridge exactly once and end up where you started. Leonhard Euler proved that it was impossible (this was in 1735). The methods that he used began the field of graph theory.

Here we can represent this as the following picture.



Remark 2. Note, that this is not a simple graph as we had before. This is a **multi-graph**. A multi-graph is like a graph, but we allow from multiple edges connecting vertices. So, in reality the structure of a multigraph is a vertex set V, and a function mapping unordered pairs of vertices to $\mathbb{N}$. Then, if a pair is mapped to some number it means there is that number of edges connecting them. We will not dwell much on multi-graphs, and the following will all be able simple graphs unless otherwise stated.

Remark 3. There are two natural questions you might want to ask about a graph given the above definitions:

1. Does the graph contain a trail/circuit which uses every edge?
2. Does the graph contain a path/cycle which uses every vertex?

If the answer to the first question is yes, then we say the graph is **Eulerian**. If the answer to the second question is yes, then we say the graph is **Hamiltonian**.

We can now phrase the question Euler answered in terms of graph theory: Is the Königsberg Graph Eulerian?

Euler actually proved a much stronger theorem.

1.4 Eulerian Graph Theorem

Theorem 6. *A connected graph G is Eulerian if and only if every vertex has even degree.*

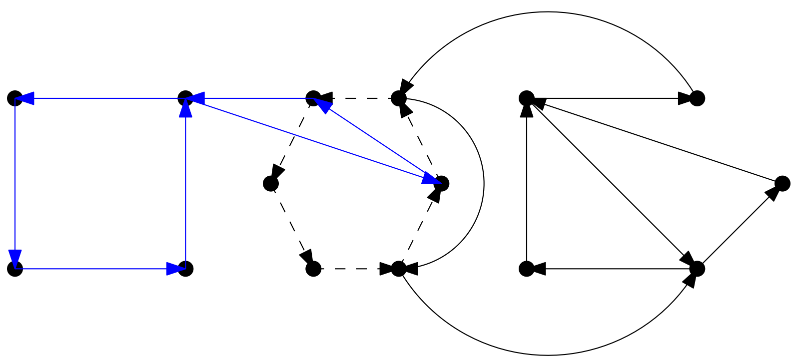
Proof. This theorem is somewhat amazing, becuase the left to right direction, as we will see, is completely trivial. So, somehow a very simple necessary condition which follows from Eulerian is also sufficent for proving that such a circuit exists.

(⇒) We prove the contrapositive. Suppose that G has a vertex v of odd degree. Then in any closed walk which spans every edge it must enter/leave the vertex at least the size of its degree as it must traverse all if its edges. Therefore, if the walk starts and ends at v, then it must have entered/left v an even number of times. Similarly, if it ends somewhere else the number of times it is entered must be equal to the number of times it left, so it is also even. But, since it has odd degree, it must be that the walk used an edge incident to v twice. Therefore, G is certainly not Eulerian.

(⇐) The direction is a bit trickier. We do induction on the size of the graph. If there are no edges, then we are done. Otherwise, there are n edges, for $n > 0$.

We claim there is a cycle. As G is connected, every vertex has at degree at least one. As every vertex has even degree, every vertex has at least degree 2. Therefore, by lemma 2, there is a cycle.

Get a graph G', subgraph of G, by eliminating the edges from a cycle from G. Perhaps we have disconnected G into different connected components. Regardless, any connected component of G now has an even number of edges (can you see why?). We can apply the induction hypothesis to each of these connected components and get a Eulerian cirucuit. Now, put back in the cycle. It's clear that we can extend the Eulerian circuits created by the induction hypothesis; start at any of the connected components of G' and continue along the circuit. When you get to a vertex in the cycle, go along the cycle until you get to a vertex in another connected component, and then follow the Eulerian circuit through thre connected component back to the cycle, and continue along it. Continue until you have traversed the cycle, and then return to the first connected component to finish that circuit.



□

Theorem 7. *The above theorem also holds for multigraphs (where the degree of a vertex of a multigraph counts the multiplicity of all of its incident edges).*

Proof. It is the same proof. □

Remark 4. This resolved the Königsberg Bridge Problem. The graph representing the bridges of Königsberg has verexes of odd degree (all of them in fact).