Day 1

May 21, 2012

1 Introduction

Welcome to concepts of mathematics. This course is intended to be an introductory course to mathematical reasoning. After this course, you will be prepared to take other courses in mathematics (eg. discrete math, logic, analysis) and computer science.

1.1 Me

My name is William Gunther. I'm a third year graduate student here at CMU. I'm studying mathematical logic. I have TA'd this course three times in the past, this is my first time teaching it. I'm very excited about it!

1.2 These Notes

These notes will be provided by me for each day. They will give all the broad details of each days lecture. Many bits will be omitted however, so reading these notes in lieu of lecture is a terrible idea.

Likewise, the notes will have exercises sprinkled in them that you should do on your own time. So, ideally you will read these notes right after lecture and work through some of the exercises.

1.3 Course Website

If you found these notes, presumably you found the course website. The address for this website is `http://math.cmu.edu/~wgunther/127m12/index.html`. Here, you can see the course calendar where all homework is posted, as well as where all these course notes are posted.

1.4 Syllabus

The syllabus is on the course website. I'll spend time going through it, especially the course policies and homework and exams. Homeworks and Exams in this class may be different than what you might be used to. Please read the syllabus.

2 Puzzles

Today, we will begin with some puzzles. These puzzles will hopefully get our brain warmed up, and give ourselves an idea of how to think and argue.

2.1 Knights and Knaves

We begin with a very classic "genre" of puzzles called "Knights and Knaves." Imagine we are a visitor on Smullyan Island where everyone is either a Knight or a Knave. A Knight always tells the truth, but a Knave always lies. On the exterior, there is nothing differentiating a Knight and a Knave, so as a visitor you are having a difficult time navigating the island.

Problem 1. You meet your first resident of the island! He said, "Hello! I am a Knight!" What can you deduce from this?

Solution 1. If they were a Knight, they would would tell you they're a Knight.

If they were a Knave, they weould lie and tell you they're a Knight.

Therefore, you can't conclude anything!

Problem 2. You see a pair of people walking down. Before you can say anything, one says, "We are both Knaves!" What do you know?

Solution 2. If they were both Knaves, they would be telling the truth, and therefore they would not be a Knave since Knave's lie. Therefore, they are not both Knaves. But then they are lying! Therefore, they are a Knave, and the other is a Knight.

Problem 3. You encounter two residents of the island, Alice and Bob. Alice says, "Bob is a Knave!" and Bob says, "Exactly one of us is a Knight." What can do you deduce?

Solution 3. If Alice were telling the truth, then Alice would be a Knight, and therefore Bob is a Knave. But, then "Exactly one of us a Knight" is true, so Bob wouldn't say that, because he's a Knave!

Therefore, Alice is a Knave, ans so Bob must be a Knight to make Alice's statement false. Bob's statement is consistent with the fact he is a Knight.

Problem 4. Now that you have identified Bob as a Knight, you ask him "How do you get to Raymondville?" He says, "I'm not sure, but there are two folks down the road who know. Be careful though, one is a Knight and the other is a Knave!"

You walk down the road, and find the two people along a fork. You want to ask a question to determine which is the correct way...but what do you ask!

Solution 4. Let's call the people April and Bradly. You have no idea which is a Knight or a Knave, so you just ask April, "If I asked Bradly 'Should I take the left path?' what would he say?"

If April is the Knight and the left path is correct, she will tell you the truth, which is that Bradly would say "No." If the left path is incorrect, she would say "Yes"

If April is the Knave and the left path is correct, she will lie; since Bradly is then the Knight, Bradly would say yes to the question, and therefore April would say "No." If the left path is incorrect, she will say "Yes."

Regardless of April's affiliation, she will say "No" if the path is correct, and "Yes" if the path is incorrect!

2.2 Chessboards

Problem 5. Can one tile an 8x8 chessboard with the opposite corners removed with dominoes, so that every square is covered?

Solution 5. Notice that a domino covers exactly one black and one white square. The opposite corners are the same color, say black. Therefore, after being removed there are 32 black squares and 30 white squares, which means that you cannot cover the board!

Problem 6. There is a 6 by 3 board, each containing a bean. Every move, you must take two beans and move them independently vertically or horizontally. Is it possible to gather all the beans in one square?

Solution 6. No. Imagine you checkered the board. Then there are 9 beans on white squares, and 9 beans on black squares. If every move, you change the bean's color. If you chose two beans on different colored squares, then the number on white and black remain the same. Otherwise, one gets reduced by 2. Since it is not possible to reduce the number of beans on a color by an odd number and they each begin with an odd number of beans, one will always have beans remaining on both colors! Therefore it is certainly impossible to get them all on one square.

3 Numbers

We finish off the day by reminding ourselves about the different kind of numbers.

3.1 Natural Numbers

The **natural numbers**, which we shall denote with $\mathbb{N}$, are all the numbers we use to count things. They are the numbers

$$0, 1, 2, 3, \ldots$$

3.2 Integers

The **integers**, which we shall denote with $\mathbb{Z}$, are the positive and negative counting numbers. They are the numbers

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

Definition 1 (Dichotomy of Parity)**.** We say that a number is **even** if it can be evenly divided by 2. This means the even numbers are

$$\ldots, -4, -2, 0, 2, 4, \ldots$$

A number is **odd** if it is not even.

By definition, every number is either even or odd. Such demarcations where everything falls into one of two categories is called a **dichotomy**. The property of being either even or odd is often call **parity**, thus this definition explains the dichotomy of parity.

Remark 1. It is a little contentious in the mathematical community whether 0 is a natural number. We will take the stance of logicians and computer scientists that 0 is indeed a natural number. It is not an issue which will concern us much.

For the the positive integers, which are of course the naturals excepting 0, we use the symbol $\mathbb{Z}^+$.

3.3 Rational Numbers

The **rational numbers**, which we shall denote $\mathbb{Q}$, are the fractions. That is they are the numbers of the form

$$\frac{a}{b}$$

where a and b are integers, and b is nonzero.

3.4 Real Numbers

The **real numbers**, which we shall denote $\mathbb{R}$, are all the decimals.

It is not immediately obvious that all decimals cannot be represented by fractions. Later, we will prove a result which was first proved by a group of ancient Greek philosophers known as the Pythagoreans

Theorem 1. *The diagonal of a 1 by 1 square cannot be measured with a fraction.*

Notice, I called the above thing a theorem. We will talk soon about what exactly this means. In general, a theorem is a mathematical statement which can be proved true.