

Partial Fractions

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Goals:

- Learn/Memorize $\int \frac{dx}{x^2+a^2}$
- Learn how to do partial fractions decomposition of rational function.
- Review polynomial long division.

1 Quick Formula

Yesterday, we did trigonometric substitution. A particular integral will be helpful. It will be given here, but not proven, although it's an easy exercise to do:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

2 Motivation

When you add together two rational functions, you find a common denominator. For example:

$$\frac{1}{x^2+1} + \frac{1}{x-1} = \frac{x-1+x^2+1}{x^3-x^2+x-1} = \frac{x^2+x}{x^3-x^2+x-1}$$

Therefore, we have

$$\begin{aligned} \int \frac{x^2+x}{x^3-x^2+x-1} dx &= \int \frac{1}{x^2+1} dx + \int \frac{1}{x-1} dx \\ &= \arctan(x) + \ln|x-1| + C \end{aligned}$$

This actually turned out to be quite a simple integral, because we were able to *decompose* the fraction into the sum of easier to handle fractions.

As it turns out, such decomposition is always possible. We will now investigate how to decompose it. The general idea is that we guess the form that the decomposition will take place, and then we find the coefficients that make that form actually the decomposition. The only trick is to work with a sufficiently general form so that our guess can actually capture the decomposition of the rational function, but not too general so that it is too difficult and time consuming to actually solve it.

We are interested in integrating $\frac{M(x)}{N(x)}$ where M and N are polynomials. If the degree of the top, M , is greater than or equal to the degree of the bottom, N , then we can do polynomial long division, and write $\frac{M(x)}{N(x)} = Q(x) + \frac{R(x)}{N(x)}$ where $Q(x)$ is the quotient and $R(x)$ is the remainder, whose degree is strictly less than degree of $N(x)$.

Example 1. In class, we will do the following example:

$$\int \frac{x^3}{x^2+1} dx$$

Warning: For the partial fraction procedure that we will do in the rest of the notes, we are going to assume that the degree of the numerator is always strictly less than the degree of the denominator. If this is not the case, then you must do polynomial long division first to reduce it!

Seemingly, it is clear that if the $N(x)$, the denominator, factors into, say, two polynomials, $N_1(x)N_2(x)$ then it should be possible to write:

$$\frac{M(x)}{N_1(x)N_2(x)} = \frac{A(x)}{N_1(x)} + \frac{B(x)}{N_2(x)}$$

where A and B are some polynomials. This is the idea of partial fractions: we will undo the step of finding a common divisor and adding, by factoring and then splitting the fractions into small fractions where the denominators are the factors. The problem, however, is how to find A and B .

Definition 1. We say that a polynomial is *irreducible* if it has no real roots. For example, $x^2 + 1$ is irreducible. Otherwise, we say a polynomial is reducible. For example, $x^2 - 1$ is reducible because 1 is a root.

Example 2. For a quadratic, $ax^2 + bx + c$, the quadratic formula tells us that it is irreducible if and only if the discriminant is non-negative, ie $b^2 - 4ac \geq 0$.

All odd degree polynomials with real coefficients are reducible. The reason why is because all polynomials of degree n have n roots, but some may be complex numbers. In order for the coefficients to be real numbers, the roots must all ‘cancel’ each other, and therefore must be in conjugate pairs (e.g. if $2+i$ is a root then $2-i$ is as well). Therefore, if there is an odd degree polynomial, there must be at least one real root, otherwise there would be an unpaired complex root.

We will completely factor our denominator into irreducible factors. We say that a factor is repeated if it appears more than once. For instance, in $(x+1)^2(x+2)$, the factor $(x+1)$ is repeated. Its multiplicity is the number of times it is repeated. In the example, the multiplicity of $(x+1)$ is 2, but the multiplicity of $(x+2)$ is only 1 since it is not repeated.

Rules:

- When decomposing a fraction where one of the factors on the denominator has degree n , we must allow for the corresponding decomposed fractions with that denominator to be of degree $n - 1$.

Example: If a factor is $x^3 + 2$ you need to allow the numerator of the fractions with that factor to have the form $Ax^2 + Bx + C$.

- When decomposing a fraction where one of the factors has multiplicity n , we must allow for $n - 1$ corresponding decomposed fractions, with each power of the factor present all the way up to n .

Example: If a factor $x^2 + 1$ has multiplicity 3, then we will need to allow for three different fractions, one with denominator $x^2 + 1$, one with denominator $(x^2 + 1)^2$, and the last with denominator $(x^2 + 1)^3$; all of these will have numerators of the form $Ax + B$ by the previous rule.

Example 3.

$$\frac{x+2}{x^6 + 2x^4 + x^2}$$

We first begin by factoring the denominator into $x^2(x^4 + 2x^2 + 1) = x^2(x^2 + 1)^2$. Notice now the two factors, x , and $x^2 + 1$, are both irreducible, and both occur with multiplicity 2. The first rule above says the decomposed fractions with denominator containing the factor x must be degree 0, ie, be constants, and there will be two such fractions: one with denominator x and the other with denominator x^2 . Similarly, the fractions with denominator containing the factor $x^2 + 1$ must be of degree 1, i.e., be linear functions, and there will be two such fractions: one with denominator $x^2 + 1$ and the other with denominator $(x^2 + 1)^2$. Therefore, to do the decomposition, we need to have:

$$\frac{x+2}{x^6 + 2x^4 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} + \frac{Ex + F}{(x^2 + 1)^2}$$

Note: Even though the degree of the denominator of $\frac{B}{x^2}$ has degree 2, the top still only needs degree 0 since it's not the degree of the denominator that matters, but the degree of the factor that's in the denominator.

3 Problems

1. $\int \frac{x^2 + 2x - 1}{x^3 - x} dx$

2. $\int \frac{x^2 - x + 6}{x^3 + 3x} dx$

3. $\int \frac{ax}{x^2 - bx} dx$

4. $\int \frac{x^2 + 1}{(x + 3)(x + 2)} dx$

5. $\int \frac{\sqrt{x}}{x - 4} dx$ Ask for hint if you can't get this one.