Distance Sequences in Locally Infinite Vertex-Transitive Digraphs

Wesley Pegden*

July 7, 2004

Abstract

We prove that the out-distance sequence $\{f^+(k)\}$ of a vertex-transitive digraph of finite or infinite degree satisfies $f^+(k+1) \leq f^+(k)^2$ for $k \geq 1$, where $f^+(k)$ denotes the number of vertices at directed distance k from a given vertex. As a corollary, we prove that for a connected vertex-transitive undirected graph of infinite degree d, we have $f(k) = d$ for all k, $1 \leq k < \mathrm{diam}(G)$. This answers a question by L. Babai.

1 Introduction

L. Babai has pointed out that results on local expansion of vertex-transitive graphs [BS] limit the possible pathologies (*i.e.*, hills and valleys) in the distance sequences of locally finite undirected vertex-transitive graphs, and has asked if "valleys" can occur in the locally infinite case [B]. In this note we prove that no valleys can occur in the locally infinite case for undirected graphs and give a full characterization of possible distance sequences.

For directed graphs (digraphs), we show that the infinite terms of the out-distance sequences of vertex-transitive digraphs are nonincreasing, and additionally that any nonincreasing sequence of infinite cardinals is the out-distance sequence of some vertex-transitive digraph. Additionally, we offer constructions to show that any out-distance sequence of a vertex-transitive digraph of finite degree can be 'mimicked' by the tail of the out-distance sequence of a vertex-transitive digraph of infinite out-degree (in particular, valleys *can* occur in the directed case).

Definition 1.1. The positive n–sphere, $S^+(k, x)$, about a vertex x in a digraph is the set of vertices at directed distance k from x.

Definition 1.2. The diameter $\mathrm{diam}(G)$ is the supremum of the pairwise distances of vertices of G. This is either an integer or ∞.

*Department of Mathematics, University of Chicago, Chicago IL 60637-1504, Email: wes@cs.uchicago.edu

Definition 1.3. A digraph G is *vertex-transitive* if every pair of vertices in G is equivalent under the automorphism group of G.

Let $f^+(k)$ denote the cardinality of $S^+(k,x)$ for some vertex x in the digraph G. Because G is vertex-transitive, $f^+(k)$ does not depend on the choice of the vertex x. We call the sequence $f^+(0), f^+(1), f^+(2), \dots$ the 'out-distance sequence' of G. For $k > \operatorname{diam}(G)$ we set $f^+(k) = 0$. For undirected graphs, we denote the n-sphere and distance sequence by $S(k,x)$ and $\{f(k)\}$, respectively.

Our main results are the following.

Theorem 1.4. *Let G be an infinite, vertex-transitive, undirected graph of infinite degree d. Then, for $0 < k < \operatorname{diam}(G)$ we have $f(k) = d$.*

Noting that for infinite cardinals we have $d \cdot d = d$ and therefore $f(k) \leq d$ for all k, Theorem 1.4 implies the following options for the distance sequence of a locally infinite vertex-transitive undirected graph.

Corollary 1.5. *Let G be an infinite, vertex-transitive, undirected graph of infinite degree d. If G has infinite diameter then its distance sequence is $1, d, d, \dots$. If G has finite diameter then its distance sequence is $1, d, \dots, d, e$, where $e \leq d$.*

In Section 3 we shall prove that all possibilities permitted by Corollary 1.5 can actually occur, so Corollary 1.5 gives a full characterization of the distance sequences of locally infinite vertex-transitive undirected graphs.

For directed graphs, we prove the following.

Theorem 1.6. *Let G be an infinite, vertex-transitive digraph of infinite out-degree d. Then, for all $k > 0$ for which $f^+(k+1)$ is infinite, we have $f^+(k) \geq f^+(k+1)$.*

Theorems 1.4 and 1.6 follow from the following lemma.

Lemma 1.7. *For a vertex-transitive digraph of finite or infinite out-degree, the out-distance sequence $\{f^+(k)\}$ satisfies $f^+(k+1) \leq f^+(k)^2$, for $k \geq 1$.*

Note. A sequence $a_0, a_1, a_2, \dots$ is *log-concave* if $(\forall i)(a_i^2 \geq a_{i-1} \cdot a_{i+1})$. For positive integers, log-concavity implies unimodality (the sequence increases up to a point, then decreases). For a sequence of infinite cardinals, log-concavity implies that the sequence is constant except for its first and last terms. Therefore Corollary 1.5 can be restated as follows:

If G is a vertex-transitive undirected graph of infinite degree then its distance sequence is log-concave.

In Section 4 we will give constructions of directed graphs of infinite out-degree whose distance sequences are not log-concave, or even unimodal. For comments on the analogous question in the locally finite case for undirected graphs, see Section 5.

For information about combinatorial parameters of vertex-transitive graphs, we refer to the survey [B1].

Acknowledgment

I'd like to thank László Babai for posing the questions I consider here, and for helping me polish the proofs.

2 The Proofs

Theorem 1.6 follows immediately from Lemma 1.7. To infer Theorem 1.4 from Lemma 1.7, we need one more observation.

Observation 2.1. *For the distance sequence $\{f(k)\}$ of an undirected vertex-transitive graph of infinite degree d, if $1 \leq f(k+1) < d$ then $f(k) = d$ or $f(k+2) = d$.*

Proof. For undirected graphs, the number of neighbors of a vertex in $S(k+1, x)$ cannot be greater than $f(k) + f(k+1) + f(k+2)$. □

We are now ready to prove Theorem 1.4. Let d be an infinite cardinal.

Let $k \geq 1$ be the smallest k such that $f(k) < d$. Then by Lemma 1.7, $f(k+1) < d$ and therefore $f(k+2) < d$. By Observation 2.1, we conclude that $f(k+1) = 0$ so $\mathrm{diam}(G) \leq k$. □

We now prove Lemma 1.7. With an eye on possible applications in the locally finite case, we prove a more general statement.

Lemma 2.2. *For a vertex-transitive digraph of finite or infinite out-degree, the out-distance sequence $\{f^+(k)\}$ satisfies $f^+(k) \leq f^+(\ell) \cdot f^+(m)$ for all k, ℓ, m such that $1 \leq \ell, m \leq k$, $m + \ell \geq k$. In particular, $f^+(k+1) \leq f^+(k)^2$ for all $k \geq 1$.*

For the proof of Lemma 2.2, we will need the following observation.

Observation 2.3. *If a shortest directed path from vertex x to vertex w passes through a vertex q then the portion between x and q can be replaced by any shortest directed path from x to q and the resulting directed path from x to w will still be shortest.*

We are now ready to prove Lemma 2.2.

Let x be some vertex in G. Let $D = S^+(k, x)$. Then $|D| = f^+(k)$. For any number $1 \leq \ell \leq k$, every directed path from x to a vertex in D must pass through some vertex in $S^+(\ell, x)$, so

$$D = \bigcup_{q \in S^+(\ell, x)} D_q$$

where D_q consists of those vertices $w \in D$ for which there exists a shortest directed path from w to x which passes through $q \in S^+(\ell, x)$. Therefore,

$$|D| \leq \sum_{q \in S^+(\ell, x)} |D_q|.$$

Claim. $(\forall q \in S^+(\ell, x))(\forall m)(k - \ell \leq m \leq k \implies |D_q| \leq f^+(m))$.

Consequently $f^+(k) = |D| \leq f^+(\ell) \cdot f^+(m)$. □

Proof of the Claim.

Let $\mathcal{P}$ be a shortest directed path from x to q and let z be the (unique) vertex lying on $\mathcal{P}$ for which the vertices in D_q are at distance m from z (so $m+\ell \geq k$). By Observation 2.3, $D_q \subseteq S^+(m,z)$, so $|D_q| \leq |S^+(m,z)| = f^+(m)$. $\square$

3 Constructions: Undirected Graphs

To demonstrate that Corollary 1.5 completely describes the restrictions on the distance sequences of locally infinite vertex-transitive undirected graphs, we offer constructions for each distance sequence permitted. All graphs in this section are undirected.

3.1 $1, d, d, d, \ldots$

For a vertex-transitive graph with the distance sequence $\{f(k)\}$ such that $f(k) = d$ for all $k > 1$ we need only consider an infinite tree of degree d.

3.2 $1, d, e$ $(1 \leq e \leq d)$

Define $L(d,e)$ to be the complement of the union of d disjoint copies of the complete graphs K_{e+1} on $e+1$ vertices. $L(d,e)$ is vertex-transitive, has diameter 2, and distance sequence $1, d, e$.

3.3 $1, d, \ldots, d, e$ $(1 \leq e \leq d)$

Let $n \geq 1$ be an integer, d an infinite cardinal, and $1 \leq e \leq d$. We now consider the case where the diameter is $n \geq 3$.

We construct a vertex-transitive graph G of degree d, diameter n and distance sequence $f(k) = d$ $(1 \leq k \leq n-1)$, $f(n) = e$. If a is a cardinal then we use $[a]$ to denote a (standard) set of cardinality a.

Consider the graph $G = C_{2n-4} \times L(d,e)$ where $L(d,e)$ is the graph from Construction 3.2, C_{2n-4} is the $(2n-4)$-cycle, and $\times$ refers to the Cartesian product. The Cartesian product $G = H_1 \times H_2$ of the graphs H_1 and H_2 is given by $V(G) = V(H_1) \times V(H_2)$, with $(h_1, h_2) \sim (h_1', h_2')$ in G if either $h_1 = h_1'$ and $h_2 \sim h_2'$, or $h_2 = h_2'$ and $h_1 \sim h_1'$, where $\sim$ refers to adjacency in the appropriate graph (cf. [B1, p. 1463]). Note that if H_1 and H_2 are vertex-transitive then so is G.

Note that $L(d,e)$ is the complement of $\overline{K_d} \times K_{e+1}$, so $V(L(d,e)) = [d] \times [e+1]$ and $V(G) = [2n-4] \times [d] \times [e+1]$. There is an edge between (i,j,k) and (i',j',k') in G if either $i = i'$ and $j \neq j'$, or $i \equiv i' + 1 \mod (2n-4)$, $j = j'$, and $k = k'$.

G has degree d, so its distance sequence begins with $1, d, \ldots, d$. Considering some vertex $v = (i,j,k)$, we see that the sphere $S(t,v)$ is the set of vertices $w = (i',j',k')$ for which one of the following holds:

1. $t \leq n-2$, $i' = i \pm t \mod (2n-4)$, $j' = j$, and $k' = k$;

2. $1 \le t \le n-1$, $i' = i \pm (t-1) \mod (2n-4)$ and $j' \ne j$;

3. $2 \le t \le n$, $i' = i \pm (t-2) \mod (2n-4), j' = j,$ and $k' \ne k$.

So $S(n,v)$ is the set of vertices described by condition (3), and $|S(n,v)| = f(n) = e$.

4 Constructions: Directed Graphs

Theorem 4.1. *Let $\{d(k)\}$ be any nonincreasing sequence of infinite cardinals. Then $1, d(1), d(2), \ldots$ is the out-distance sequence of some vertex-transitive digraph. Furthermore, let $n > 0$ and let $\{g^+(k)\}$ be the out-distance sequence of some vertex-transitive digraph of finite out-degree. Then*

$$1, d(1), \ldots, d(n), g^+(n+1), g^+(n+2), \ldots$$

is the out-distance sequence of some vertex-transitive digraph.

In particular then, Theorem 4.1 implies that the out-distance sequences of vertex-transitive digraphs with infinite out-degree are not always unimodal. Theorem 4.1 follows from the following constructions.

4.1 $1, d, \ldots, d$ and $1, d, d, \ldots$

Let d be an infinite cardinal, and let $n > 0$. We construct $G_{n,d}$, which has the vertex set $\mathbb{Z} \times [d]$. There is an edge from (z_1, α_1) to (z_2, α_2) in $G_{n,d}$ when $z_2 - z_1 = 1$ or $z_2 - z_1 > n$. So $G_{n,d}$ is acyclic and has out-distance sequence $\{g^+(k)\}$ satisfying $g^+(k) = d$ for all $0 < k \le n$, and $g^+(n+1) = 0$.

We also define $G_{\infty,d}$ on the same vertex set. There is an edge from (z_1, α_1) to (z_2, α_2) in $G_{\infty,d}$ if and only if $z_2 - z_1 = 1$. Thus $G_{\infty,d}$ has out-distance sequence $1, d, d, \ldots$.

4.2 $1, d(1), d(2), \ldots$

Definition 4.2. Let G_1 and G_2 be digraphs with vertex sets V_1 and V_2, respectively. Then the lexicographic product $G_1 \bullet G_2$ (cf. [B1, p. 1463]) has vertex set $V_1 \times V_2$; there is an edge from the vertex (v_1, v_2) to the vertex (v_1', v_2') in $G_1 \bullet G_2$ when either (v_1, v_1') is an edge in G_1, or $v_1 = v_1'$ and (v_2, v_2') is an edge in G_2. On more than two graphs, we define the product recursively: $G_1 \bullet \cdots \bullet G_n = (G_1 \bullet \cdots \bullet G_{n-1}) \bullet G_n$. (This product is associative.)

Note that if G and H are vertex-transitive, $G \bullet H$ is vertex-transitive.

Observation 4.3. *If G and H are acyclic, $G \bullet H$ is acyclic.*

Observation 4.4. *Let $\{d(k)\}$ be a nonincreasing sequence of infinite cardinals. Let G_1 be a vertex-transitive digraph with out-distance sequence $1, d(1), \ldots, d(m)$,* *and let G_2 be a vertex-transitive digraph with $|V(G_2)| \leq d(m)$ and with out-distance sequence $1, g^+(1), g^+(2), \ldots$. If G_1 is acyclic, then $G_1 \bullet G_2$ has out-distance sequence*

$$1, d(1), \ldots, d(m), g^+(m+1), g^+(m+2), \ldots.$$

Let $\{d(k)\}$ be a nonincreasing sequence of infinite cardinals; so the sequence must be eventually constant. Thus there exists an m such that $d(i) = d(m)$ for all $i \geq m$. Then by Observations 4.3 and 4.4,

$$(G_{1,d(1)} \bullet \cdots \bullet G_{m,d(m)}) \bullet G_{\infty,d(m+1)}$$

(with $G_{n,d}$ as constructed in 4.1) is vertex-transitive and has out-distance sequence $1, d(1), d(2), \ldots$. This proves the first part of Theorem 4.1.

4.3 $1, d(1), \ldots, d(n), g^+(n+1), g^+(n+2), \ldots$

Let $\{d(k)\}$ as above and let G be a vertex-transitive digraph of finite out-degree with out-distance sequence $\{g^+(k)\}$. From Observations 4.3 and 4.4, then,

$$(G_{1,d(1)} \bullet G_{2,d(2)} \bullet \cdots \bullet G_{n,d(n)}) \bullet G$$

is a vertex-transitive digraph with out-distance sequence

$$1, d(1), \ldots, d(n), g^+(n+1), g^+(n+2), \ldots.$$

This proves the second part of Theorem 4.1. □

5 The Locally Finite Case (Undirected)

Watkins and Shearer [WS] note that the distance sequences of (undirected) locally finite vertex-transitive graphs may not be log-concave; in fact, even unimodality does not necessarily hold. Log-concavity does hold in the locally finite case, however, under the much more restrictive condition of *distance transitivity* (or, more generally, *distance regularity*) [TL] (cf. [BCN, p. 167]).

Watkins and Shearer provide examples of families of vertex-transitive locally finite graphs, both finite and infinite, whose distance sequences are not unimodal. Of particular interest are their examples of infinite locally finite graphs with infinitely many valleys in their distance sequences. One such example (T_1 in their paper) is the archimedean tessellation of the plane by regular hexagons and equilateral triangles, of a common sidelength, such that every edge separates a hexagon and a triangle. The distance sequence $\{f(k)\}$ of this graph is given by $f(0) = 1$, $f(1) = 4$, and for $k > 1$,

$$f(k) = \begin{cases} 5k - 2 & \text{if } k \text{ is even} \\ 4k + 2 & \text{if } k \text{ is odd} \end{cases}$$

thus the triple $f(k)$, $f(k+1)$, $f(k+2)$ is a valley in the distance sequence of the graph for all even $k \geq 10$.

Examples such as these motivate the study of pathologies of the distance sequence. See Question 4, Section 6.

6 Open Questions

Definition 6.1. A graph G is *vertex-primitive* if the automorphism group of G is primitive, i. e., the vertex set of G has no nontrivial partitions invariant under all automorphisms of G.

1. Do all options for the distance sequences of locally infinite vertex-transitive undirected graphs permitted by Theorem 1.4 occur for vertex-primitive graphs?

2. Let X be a finite subset of the vertices of an undirected graph G of infinite degree d. Define $f_X(k)$ as the number of vertices at distance k from the set X. Let m be smallest such that $f_X(m) = 0$. Is it true that we have $f_X(k) = d$, for all $0 < k < m - 1$?

3. Let X be a finite subset of the vertices of a vertex-transitive digraph G, and let $\{f^+_X(k)\}$ be the out-distance sequence from X in G. Is it true that for $f^+_X(k+1)$ infinite and $k > 0$, we have $f^+_X(k) \geq f^+_X(k+1)$? (From Observation 2.1, an affirmative answer to this question implies an affirmative answer to Question 2.)

Note. A digraph G is *growth-regular* if for all vertices x and y in G and for all $k \geq 0$, $|S^+(k,x)| = |S^+(k,y)|$. This condition of growth-regularity, weaker than vertex-transitivity, is sufficient for our proofs in this note. However, it is not difficult to construct a growth-regular graph, choose a finite subset X of the vertices, and give a $k > 0$ such that $f_X(k+1)$ is infinite, yet $f_X(k) < f_X(k+1)$.

4. A little-studied area appears to be the pathologies (hills and valleys) of the distance sequences of locally finite vertex-transitive (undirected) graphs. Results by Babai et al. [BS] bound the depth of possible valleys ($f(k) > f(k+1) < f(k+2)$) in the locally finite case up to $k < \frac{\text{diam}(G)}{2}$. Their Theorems 3.2 and 4.2 imply that for $2k+1 \leq \text{diam}(G)$, we have

$$f(k+1) \geq \frac{\sum\limits_{i=0}^{k} f(i)}{2k+1} \tag{6.1}$$

(see also [L]). Our Lemmas 1.7 and 2.2 provide another restriction, without extra conditions such as $2k+1 \leq \text{diam}(G)$. Another result that puts some limitation on these pathologies is Gromov's Lemma [G], which states that the volume sequence of a vertex-transitive graph, given by

$$g(k) = \sum_{i=0}^{k} f(i),$$

obeys $g(k)g(5k) \le (g(4k))^2$ (cf. [B1, p. 1477]). These results, however, permit deeper valleys than those found by Watkins and Shearer.

References

[B] László Babai. Discrete Math Problem Set #9, Problem #16. REU, University of Chicago. 2003.

[B1] László Babai. Automorphism Groups, Isomorphism, Reconstruction, in *Handbook of Combinatorics* (eds: R.L. Graham et al.), Elsevier Science B.V., Amsterdam, 1995, 1447-1540.

[BS] László Babai and Mario Szegedy. Local Expansion of Symmetrical Graphs. *Combinatorics, Probability and Computing* **1** (1992) 1-11.

[BCN] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance Regular Graphs.* SpringerVerlag.

[G] M. Gromov. Groups of polynomial growth and expanding maps, in *Publ. Math. IHES* **53**, 53-73.

[L] András Lukács. On Local Expansion of Vertex-Transitive Graphs. *Combinatorics, Probability and Computing* **7** (1998) 205-209.

[TL] D. E. Taylor and R. Levingston, Distance-Regular Graphs, pp. 313-323 in *Combinatorial Mathematics, Proc. Canberra 1977*, Lecture Notes in Math, 686 (D. A. Holton and J. Severr, eds.), Springer, Berlin, 1978.

[WS] Mark E. Watkins and James B. Shearer. Counterexamples to Two Conjectures about Distance Sequences. *Discrete Mathematics* **66** (1987) pp 289-298.