APOLLONIAN STRUCTURE IN THE ABELIAN SANDPILE

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Abstract. We state a conjecture relating integer-valued superharmonic functions on $\mathbb{Z}^2$ to an Apollonian circle packing of $\mathbb{R}^2$. The conjecture is motivated by the *Abelian sandpile* process, which evolves configurations of chips on the integer lattice by *toppling* any vertex with at least 4 chips, distributing one of its chips to each of its 4 neighbors. When begun from a large stack of chips, the terminal state of the sandpile has a curious fractal structure which has remained unexplained. Our conjecture implies that the *Sandpile PDE* recently shown to characterize the continuum limit of the sandpile is equivalent to the *Apollonian PDE*, and we use the special geometric structure of the latter to prove that it admits certain fractal solutions. Boundary condition evidence from finite sandpiles suggest that these solutions exactly correspond to regions of the limiting sandpile, leading to precise geometric conjectures on the Abelian sandpile's fractal behavior.

1. Introduction

1.1. **Background.** First introduced in 1987 by Bak, Tang and Wiesenfeld [1] as a model of self-organized criticality, the Abelian Sandpile is equally fascinating as a model of pattern formation. In its simplest form, the sandpile process evolves a configuration $\eta : \mathbb{Z}^2 \to \mathbb{N}$ of *chips* by iterating a simple rule: find a lattice point $x \in \mathbb{Z}^2$ with at least four chips and *topple* it, moving one chip from x to each of its four lattice neighbors.

When the initial configuration has finitely many total chips, the sandpile process always finds a *stable* configuration, where each lattice point has at most three chips. Dhar [6] observed that the resulting stable configuration does not depend on the toppling order, which is the reason for terming the process "Abelian." When the initial configuration consists of a large number of chips at the origin, the final configuration has a curious fractal structure [3, 8, 15–17] which (after rescaling) is insensitive to the number of chips. In 25 years of research (see [14] for a brief survey, and [7, 20] for more detail) this fractal structure has resisted explanation or even a precise description.

If $s_n : \mathbb{Z}^2 \to \mathbb{N}$ denotes the stabilization of n chips placed at the origin, then the rescaled configurations

$$\bar{s}_n(x) := s_n([n^{1/2}x])$$

(where $[x]$ indicates a closest lattice point to $x \in \mathbb{R}^2$) converge to a unique limit s_∞. This article presents a partial explanation for the apparent fractal structure of this limit.

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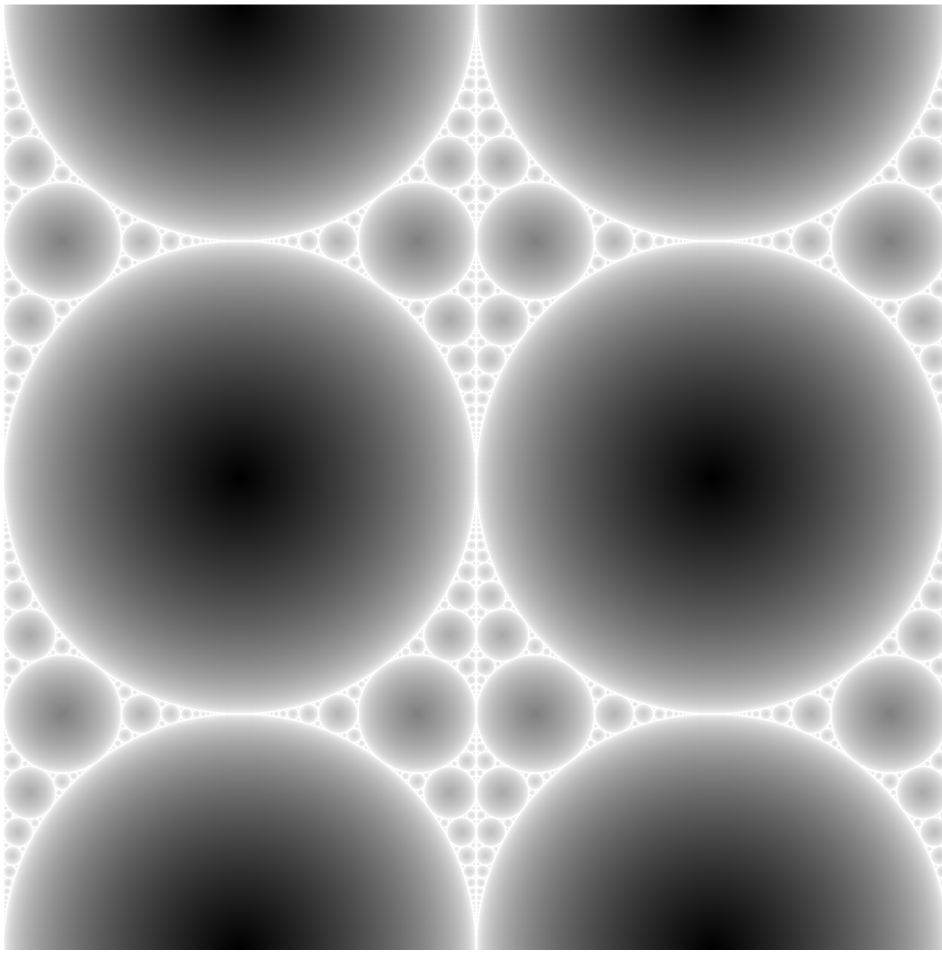

FIGURE 1. The boundary of Γ. The shade of gray at location $(a, b) \in [0, 4] \times [0, 4]$ indicates the largest $c \in [2, 3]$ such that $M(a, b, c) \in \Gamma$. White and black correspond to $c = 2$ and $c = 3$, respectively.

The convergence $\bar{s}_n \to s_\infty$ was obtained Pegden-Smart [18], who used viscosity solution theory to identify the continuum limit of the least action principle of Fey-Levine-Peres [10]. We call a 2×2 real symmetric matrix A *stabilizable* if there is a function $u : \mathbb{Z}^2 \to \mathbb{Z}$ such that

$$u(x) \geq \frac{1}{2} x^t A x \quad \text{and} \quad \Delta^1 u(x) \leq 3, \tag{1.1}$$

for all $x \in \mathbb{Z}^2$, where

$$\Delta^1 u(x) = \sum_{y \sim x} (u(y) - u(x)) \tag{1.2}$$

is the discrete Laplacian of u on $\mathbb{Z}^2$ and the sum is taken over the four lattice neighbors. (We establish a direct correspondence between stabilizable matrices and infinite stabilizable sandpile configurations in Section 3.) It turns out that the closure $\bar{\Gamma}$ of the set Γ of stabilizable matrices determines s_∞.

Theorem 1.1 (Existence Of Scaling Limit, [18]). *The rescaled configurations $\bar{s}_n$ converge weakly-$*$ in $L^\infty(\mathbb{R}^2)$ to $s_\infty = \Delta v_\infty$, where*

$$v_\infty := \min\{w \in C(\mathbb{R}^2) \mid w \geq -\Phi \text{ and } D^2(w+\Phi) \in \bar{\Gamma}\}. \tag{1.3}$$

Here $\Phi(x) := -(2\pi)^{-1}\log|x|$ is the fundamental solution of the Laplace equation $\Delta\Phi = 0$, the minimum is taken pointwise, and the differential inclusion is interpreted in the sense of viscosity.

Recall that weak-$*$ convergence simply captures convergence of the local average value of $\bar{s}_n$. The sum $u_\infty = v_\infty + \Phi$ has a natural interpretation in terms of the sandpile: it is the limit $u_\infty(x) = \lim_{n\to\infty} n^{-1}u_n([n^{1/2}x])$, where $u_n(x)$ is the number of times $x \in \mathbb{Z}^d$ topples during the formation of s_n. As explained below in Section 2.3, the function u_∞ solves the *Sandpile PDE*,

$$D^2u \in \partial\Gamma, \tag{1.4}$$

in the open set $\{u > 0\}$ where it is above the obstacle.

1.2. **Apollonian structure.** The key players in the obstacle problem (1.3) are Φ and Γ. The former encodes the initial condition (with the particular choice of $-(2\pi)^{-1}\log|x|$ corresponding to all particles starting at the origin). The set Γ is a more interesting object: it encodes the continuum limit of the sandpile stabilization rule. In Section 3, we present an algorithm to determine whether a given matrix with rational entries lies in Γ. The numerical evidence provided by this algorithm indicates that Γ is a union of downward cones based at points of a certain set $\mathcal{P}$. The elements of $\mathcal{P}$, which we call *peaks*, are visible as the locally darkest points in Figure 1.

We can state an exact conjecture characterizing the the shape of Γ in terms of Apollonian configurations of circles. Three pairwise externally tangent circles C_1, C_2, C_3 determine an *Apollonian circle packing*, as the smallest set of circles containing them that is closed under the operation of adding, for each pairwise tangent triple of circles, the two circles which are tangent to each circle in the triple. They also determine a *downward Apollonian packing*, closed under adding, for each pairwise-tangent triple, only the smaller of the two tangent circles. Lines are allowed as circles, and the *Apollonian band circle packing* is the packing $\mathcal{C}$ determined by the lines $\{x = 0\}$ and $\{x = 2\}$ and the circle $\{(x-1)^2 + y^2 = 1\}$. Its circles are all contained in the strip $[0, 2] \times \mathbb{R}$.

We put the proper circles in $\mathbb{R}^2$ (i.e., the circles that are not lines) in bijective correspondence with real symmetric 2×2 matrices of trace > 2, in the following way. To a proper circle $C = \{(x-a)^2 + (y-b)^2 = r^2\}$ in $\mathbb{R}^2$ we associate the matrix

$$m(C) := M(a, b, r+2)$$

where

$$M(a, b, c) := \frac{1}{2}\begin{bmatrix} c+a & b \\ b & c-a \end{bmatrix}. \tag{1.5}$$

We write S_2 for the set of symmetric 2×2 matrices with real entries, and, for $A, B \in S_2$ we write $B \leq A$ if $A - B$ is nonnegative definite. For a set $\mathcal{P} \subset S_2$, we define

$$\mathcal{P}^\downarrow := \{B \in S_2 \mid B \leq A \text{ for some } A \in \mathcal{P}\},$$

the order ideal generated by $\mathcal{P}$ in the matrix order.

Now let $\hat{\mathcal{C}} = \bigcup_{k\in\mathbb{Z}}(\mathcal{C} + (2k, 0))$ be the extension of the Apollonian band packing to all of $\mathbb{R}^2$ by translation. Let

$$\mathcal{P} = \{m(C) \mid C \in \hat{\mathcal{C}}\}.$$

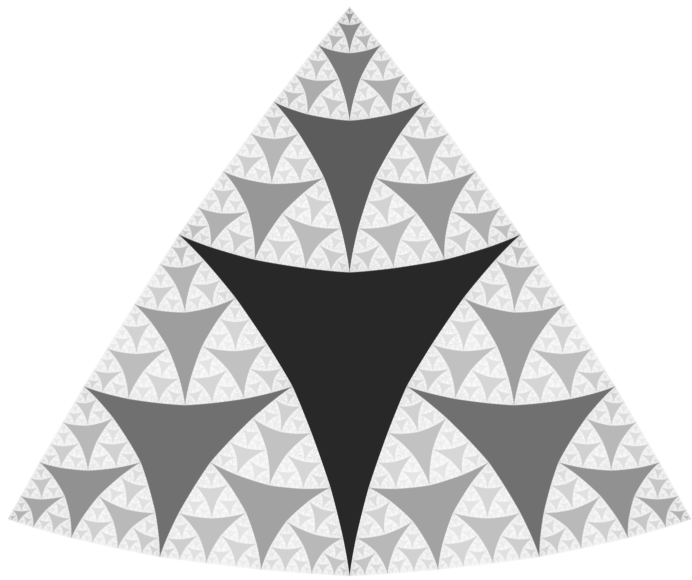

FIGURE 2. An Apollonian triangulation is a union of Apollonian triangles meeting at right angles, whose intersection structure matches the tangency structure of their corresponding circles. The solution u of Theorem 1.3 has constant Laplacian on each Apollonian triangle, as indicated by the shading (darker regions are where Δu is larger).

Conjecture 1.2 (Circle Packing Conjecture)**.** $\bar{\Gamma} = \mathcal{P}^{\downarrow}$.

Note that replacing the condition $\Delta^1 v \leq 3$ with $\Delta^1 v \leq 0$ in (1.1) results only in a translation of the set Γ. Indeed, on can simply subtract the function $v(x) = \frac{3}{2}x_1(x_1+1)$ from u and subtract $M(3,0,3)$ from A. Thus this conjecture can be interpreted as a statement on the geometry of the set of integer-valued superharmonic functions of quadratic growth on the lattice $\mathbb{Z}^2$. The set of 4-tuples of pairwise tangent circles in an Apollonian circle packing has a transitive action by a discrete subgroup of the Lorenz group $O(3,1)$, and group theory has enabled recent breakthroughs in proving deep arithmetic properties of Apollonian packings [2, 11, 19]. Conjecture 1.2 has so far resisted any attempts to bring these tools to bear.

1.3. **The Apollonian PDE.** Conjecture 1.2 asserts that the Sandpile PDE (1.4) is the same as the *Apollonian PDE*

$$D^2 u \in \partial \mathcal{P}^{\downarrow}. \tag{1.6}$$

Our main result, Theorem 1.3 below, constructs a family of piecewise quadratic solutions to the Apollonian PDE. The supports of these solutions are the closures of certain fractal subsets of $\mathbb{R}^2$ which we call *Apollonian triangulations*, giving an explanation for the fractal limit $\bar{s}_\infty$.

Of course, every matrix $A \in S_2$ with $\mathrm{tr}(A) > 2$ is now associated to a unique proper circle $C = c(A) = m^{-1}(A)$ in $\mathbb{R}^2$. We say two matrices are (*externally*) *tangent* precisely if their corresponding circles are (externally) tangent. Given pairwise externally tangent matrices A_1, A_2, A_3, denote by $\mathcal{A}(A_1, A_2, A_3)$ (resp. $\mathcal{A}^{-}(A_1, A_2, A_3)$) the set of

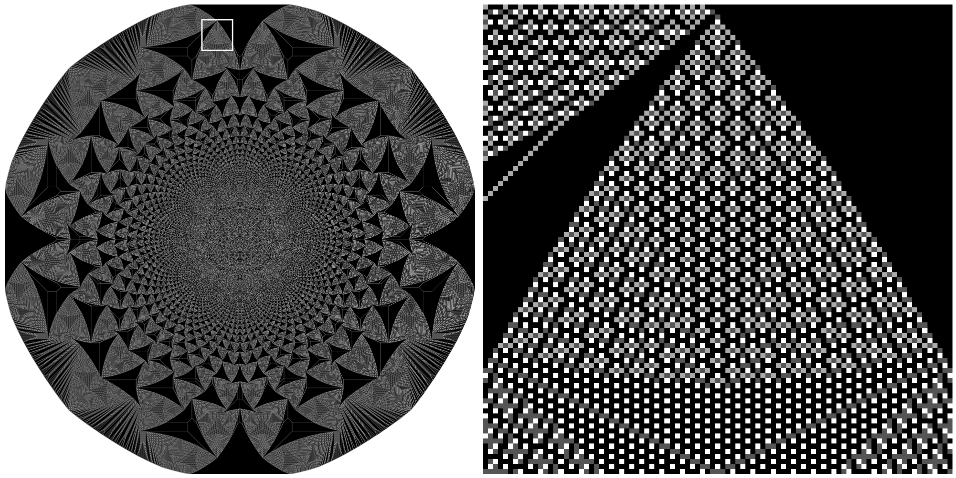

FIGURE 3. Left: The sandpile s_n for $n = 4 \cdot 10^6$. Sites with 0, 1, 2, and 3 chips are represented by four different shades of gray. Right: A zoomed view of the boxed region, one of many that we believe converges to an Apollonian triangulation in the $n \to \infty$ limit.

matrices corresponding to the Apollonian circle packing (resp. downward Apollonian packing) determined by the circles corresponding to A_1, A_2, A_3.

Theorem 1.3 (Piecewise Quadratic Solutions)**.** *For any pairwise externally tangent matrices $A_1, A_2, A_3 \in S_2$, there is a nonempty convex set $Z \subset \mathbb{R}^2$ and a function $u \in C^{1,1}(Z)$ satisfying*

$$D^2 u \in \partial \mathcal{A}(A_1, A_2, A_3)^{\downarrow}$$

in the sense of viscosity. Moreover, Z decomposes into disjoint open sets (whose closures cover Z) on each of which u is quadratic with Hessian in $\mathcal{A}^-(A_1, A_2, A_3)$.

This theorem is illustrated in Figure 2. We call the configuration of pieces where D^2u is constant an *Apollonian triangulation*. Our geometric characterization of Apollonian triangulations begins with the definition of *Apollonian curves* and *Apollonian triangles* in Section 5. We will see that three vertices in general position determine a unique Apollonian triangle with those vertices, via a purely geometric construction based on medians of triangles. We will also show that any Apollonian triangle occupies exactly 4/7 of the area of the Euclidean triangle with the same vertices.

An Apollonian triangulation, which we precisely define in Section 6, is a union of Apollonian triangles corresponding to circles in an Apollonian circle packing, where pairs of Apollonian triangles corresponding to pairs of intersecting circles meet at right angles. The existence of Apollonian triangulations is itself nontrivial and is the subject of Theorem 7.1; analagous discrete structures were constructed by Paoletti in his thesis [17]. Looking at the Apollonian fractal in Figure 2 and recalling the $SL_2(\mathbb{Z})$ symmetries of Apollonian circle packings, it is natural to wonder whether nice symmetries may relate distinct Apollonian triangulations as well. But we will see in Section 6 that Apollonian triangles are equivalent under affine transformations, precluding the possiblity of conformal equivalence for Apollonian triangulations.

If C_1, C_2, C_3 are pairwise tangent circles in the band circle packing, then letting $A_i = m(C_i)$ for $i = 1, 2, 3$, we have $\mathcal{A}^-(A_1, A_2, A_3) \subset \mathcal{P}$, so the function u in Theorem 1.3 will be a viscosity solution to the Sandpile PDE assuming Conjecture 1.2. The uniqueness machinery for viscosity solutions gives the following corollary to Theorem 1.3, which encapsulates its relevance to the Abelian Sandpile.

Corollary 1.4. *Suppose $U_1, U_2, U_3 \subseteq \mathbb{R}^2$ are connected open sets bounding a convex region Z such that $\bar{U}_i \cap \bar{U}_j = \{x_k\}$ for $\{i, j, k\} = \{1, 2, 3\}$, where the triangle $\triangle x_1 x_2 x_3$ is acute. If u_∞ is quadratic on each of U_1, U_2, U_3 with pairwise tangent Hessians $A_1, A_2, A_3 \in \mathcal{P}$, respectively, then Conjecture 1.2 implies that u_∞ is piecewise quadratic in R and the domains of the quadratic pieces form the Apollonian triangulation determined by the vertices x_1, x_2, x_3.*

Note that $s_\infty = \Delta v_\infty$ implies $\bar{s}_\infty$ is piecewise-constant in the Apollonian triangulation.

Let us briefly remark on the consequences of this corollary for our understanding of the limit sandpile. As observed in [8, 16] and visible in Figure 3, the sandpile s_n for large n features many clearly visible patches, each with its own characteristic periodic pattern of sand (sometimes punctuated by one-dimensional 'defects' which are not relevant to the weak-* limit of the sandpile). Empirically, we observe that triples of touching regions of these kinds are always regions where the observed finite $\bar{v}_n$ correspond (away from the one-dimensional defects) exactly to minimal representatives in the sense of (1.1) of quadratic forms

$$\frac{1}{2} x^t A x + bx$$

where the A's for each region are always as required by Corollary 1.4. Thus we are confident from the numerical evidence that the conditions required for Corollary 1.4 and thus Apollonian triangulations occur—indeed, are nearly ubiquitous—in s_∞. Going beyond Corollary 1.4's dependence on local boundary knowledge would seem to require an understanding the global geometry of s_∞, which remains a considerable challenge.

1.4. **Overview.** The rest of the paper proceeds as follows. In Section 2, we review some background material on the abelian sandpile and viscosity solutions. Section 3 presents our algorithm for verifying Conjecture 1.2 to arbitrary precision. After reviewing some basic geometry of Apollonian circle packings in Section 4, we define and study *Apollonian curves*, *Apollonian triangles*, and *Apollonian triangulations* in Sections 5 and 6. The proofs of Theorem 1.3 and Corollary 1.4 come in Section 7 where we construct piecewise-quadratic solutions to the Apollonian PDE. Finally, in Section 8 we discuss new problems suggested by our results.

2. Preliminaries

The preliminaries here are largely section-specific, with Section 2.1 being necessary for Section 3 and Sections 2.2 and 2.3 being necessary for Section 7.

2.1. **The Abelian sandpile.** Given a configuration $\eta : \mathbb{Z}^2 \to \mathbb{Z}$ of chips on the integer lattice, we define a toppling sequence as a finite or infinite sequence $x_1, x_2, x_3, \ldots$ of vertices to be toppled in the sequence order, such that any vertex topples only finitely many times (thus giving a well-defined terminal configuration). A sequence is *legal* if it only topples vertices with at least 4 chips, and *stabilizing* if there are at most 3 chips at every vertex in the terminal configuration. We say that η is *stabilizable* if there exists a legal stabilizing toppling sequence.

The theory of the Abelian Sandpile begins with the following standard fact:

Proposition 2.1. *Any $x \in \mathbb{Z}^2$ topples at most as many times in any legal sequence as it does in any stabilizing sequence.* □

Proposition 2.1 implies that to any stabilizable initial configuration η, we can associate an *odometer function* $v : \mathbb{Z}^2 \to \mathbb{N}$ which counts the number of times each vertex topples in any legal stabilizing sequence of topplings. The terminal configuration of any such sequence of topplings is then given by $\eta + \Delta^1 v$. Since v and so $\Delta^1 v$ are independent of the particular legal stabilizing sequence, this shows that the sandpile process is indeed "Abelian": if we start with some stabilizable configuration $\eta \geq 0$, and topple vertices with at least 4 chips until we cannot do so any more, then the final configuration $\eta + \Delta^1 v$ is determined by η.

The discrete Laplacian is monotone, in the sense that $\Delta^1 u(x)$ is decreasing in $u(x)$ and increasing in $u(y)$ for any neighbor $y \sim x$ of x in $\mathbb{Z}^2$. An obvious consequence of monotonicity is that taking a pointwise minimum of two functions cannot increase the Laplacian at a point:

Proposition 2.2. *If $u, v : \mathbb{Z}^d \to \mathbb{Z}$, $w := \min\{u, v\}$, and $w(x) = u(x)$, then $\Delta^1 w(x) \leq \Delta^1 u(x)$.* □

In particular, given any functions u, v satisfying $\eta + \Delta^1(u) \leq 3$ and $\eta + \Delta^1(v) \leq 3$, their pointwise minimum satisfies the same constraint. The proof of Theorem 1.1 in [18] begins from the *Least Action Principle* formulated in [10], which states that the odometer of an initial configuration η is the pointwise minimum of all such functions.

Proposition 2.3 (Least Action Principle)**.** *If $\eta : \mathbb{Z}^2 \to \mathbb{N}$ and $w : \mathbb{Z}^2 \to \mathbb{N}$ satisfy $\eta + \Delta^1 w \leq 3$, then η is stabilizable, and its odometer v satisfies $v \leq w$.*

Note that the Least Action Principle can be deduced from Proposition 2.1 by associating a stabilizing sequence to w. By considering the function $u = v - 1$ for any odometer function v, the Least Action Principle implies the following proposition:

Proposition 2.4. *If $\eta : \mathbb{Z}^2 \to \mathbb{Z}$ is a stabilizable configuration, then its odometer v satisfies $v(x) = 0$ for some $x \in \mathbb{Z}^2$.*

Finally, we note that these propositions generalize in a natural way from $\mathbb{Z}^2$ to arbitrary graphs; in our case, it is sufficient to note that they hold as well on the torus

$$T_n := \mathbb{Z}^2 / n\mathbb{Z}^2 \quad \text{for } n \in \mathbb{Z}^+.$$

2.2. **Some matrix geometry.** All matrices considered in this paper are 2×2 real symmetric matrices and we parametrize the space S_2 of such matrices via $M : \mathbb{R}^3 \to S_2$ defined in (1.5). We use the usual matrix ordering: $A \leq B$ if and only if $B - A$ is nonnegative definite.

Of particular importance to us is the downward cone

$$A^{\downarrow} := \{B \in S_2 : B \leq A\}.$$

Recall that if $B \in \partial A^{\downarrow}$, then $A - B = v \otimes v = vv^t$ for some column vector v. That is, the boundary $\partial A^{\downarrow}$ consists of all downward rank-1 perturbations of A.

Our choice of parametrization M was chosen to make $A^{\downarrow}$ a cone in the usual sense. Observe that

$$M(a, b, c) \geq 0 \quad \text{if and only if} \quad c \geq (a^2 + b^2)^{1/2}.$$

Moreover:

Observation 2.5. *We have*

$$v \otimes v = M(u_1, u_2, (u_1^2 + u_2^2)^{1/2}) \tag{2.1}$$

if and only if $v^2 = u$, where v^2 denotes the complex square of v. □

Thus if $B \in \partial A^{\downarrow}$, then

$$A - B = (\bar{\rho}(B) - \bar{\rho}(A))^{1/2} \otimes (\bar{\rho}(B) - \bar{\rho}(A))^{1/2}, \tag{2.2}$$

where

$$\bar{\rho}(M(a, b, c)) := (a, b),$$

and $v^{1/2}$ denotes the complex square root of a vector $v \in \mathbb{R}^2 = \mathbb{C}$.

Denoting by I the 2×2 identity matrix, we write

$$A^- = A - 2(\mathrm{tr}(A) - 2)I$$

for the reflection of A across the trace-2 plane; and

$$A^0 = \frac{A + A^-}{2}$$

for the projection of A on the trace-2 plane. Since the line $\{A + t(v \otimes v) \mid t \in \mathbb{R}\}$ is tangent to the downward cone $A^{\downarrow}$ for every nonzero vector v and matrix A, we see that matrices A_1, A_2, both with trace greater than 2, are externally tangent if and only if $A_1 - A_2^-$ has rank 1 and internally tangent if and only if $A_1 - A_2$ has rank 1. This gives the following Observation:

Observation 2.6. *Suppose the matrices A_i, A_j, A_k are mutually externally tangent and have traces > 2. Then there are at most two matrices B whose difference $A_s - B$ is rank 1 for each $s = i, j, k$: $B = A_m^-$ is a solution for any matrix A_m externally tangent to A_i, A_j, A_k, and $B = A_m$ is a solution for any A_m internally tangent to A_i, A_j, A_k.* □

Note that the case of fewer than two solutions occurs when the triple of trace-2 circles of the down-set cones of the A_i are tangent to a common line, leaving only one proper circle tangent to the triple.

2.3. **Viscosity Solutions.** We would like to interpret the Sandpile PDE $D^2 u \in \partial\Gamma$ in the classical sense, but the nonlinear structure of $\partial\Gamma$ makes this impractical. Instead, we must adopt a suitable notion of weak solution, which for us is the *viscosity* solution. The theory of viscosity solutions is quite rich and we refer the interested reader to [4,5] for an introduction. Here we simply give the basic definitions. We remark that these definitions and results make sense for any subset $\Gamma \subseteq S_2$ that is downward closed and whose boundary has bounded trace (see Facts 3.2, 3.5, and 3.6 below).

If $\Omega \subseteq \mathbb{R}^2$ is an open set and $u \in C(\Omega)$, we say that u satisfies the differential inclusion

$$D^2 u \in \bar{\Gamma} \quad \text{in } \Omega, \tag{2.3}$$

if $D^2\varphi(x) \in \bar{\Gamma}$ whenever $\varphi \in C^\infty(\Omega)$ touches u from below at $x \in \Omega$. Letting Γ^c denote the closure of the complement of Γ, we say that u satisfies

$$D^2 u \in \Gamma^c \quad \text{in } \Omega, \tag{2.4}$$

if $D^2\psi(x) \in \Gamma^c$ whenever $\psi \in C^\infty(\Omega)$ touches u from above at $x \in \Omega$. Finally, we say that u satisfies

$$D^2 u \in \partial\Gamma \quad \text{in } \Omega,$$

if it satisfies both (2.3) and (2.4).

The standard machinery for viscosity solutions gives existence, uniqueness, and stability of solutions of the Sandpile PDE. For example, the minimum in (1.3) is indeed attained by some $v \in C(\mathbb{R}^2)$ and $v + \Phi$ solves the Sandpile PDE in $\{v > -\Phi\}$. Moreover, we have a comparison principle:

Proposition 2.7. *If $\Omega \subseteq \mathbb{R}^2$ is open and bounded and $u, v \in C(\bar{\Omega})$ satisfy*

$$D^2 u \in \bar{\Gamma} \quad \text{and} \quad D^2 v \in \Gamma^c \quad \text{in } \Omega,$$

then $\sup_\Omega (v - u) = \sup_{\partial\Omega}(v - u)$. □

Recall that $C^{1,1}(U)$ is the class of differentiable functions on U with Lipshitz derivatives. In Section 7, we construct piecewise quadratic $C^{1,1}$ functions which solve the Apollonian PDE on each piece. The following standard fact guarantees that the functions we construct are, in fact, viscosity solutions of the Apollonian PDE on the whole domain (including at the interfaces of the pieces).

Proposition 2.8. *If $U \subset \mathbb{R}^2$ is open, $u \in C^{1,1}(U)$, and for Lebesgue almost every $x \in U$*

$$D^2 u(x) \text{ exists and } D^2 u(x) \in \partial\Gamma,$$

then $D^2 u \in \partial\Gamma$ holds in the viscosity sense. □

Since we are unable to find a published proof, we include one here.

Proof. Suppose $\varphi \in C^\infty(U)$ touches u from below at $x_0 \in U$. We must show $D^2\varphi(x_0) \in \bar{\Gamma}$. By approximation, we may assume that φ is a quadratic polynomial. Fix a small $\varepsilon > 0$. Let A be the set of $y \in U$ for which there exists $p \in \mathbb{R}^2$ and $q \in \mathbb{R}$ such that

$$\varphi_y(x) := \varphi(x) - \frac{1}{2}\varepsilon|x|^2 + p \cdot x + q,$$

touches u from below a y. Since $u \in C^{1,1}$, $p(y)$ is unique and that map $p : A \to \mathbb{R}^2$ is Lipschitz. Since $\varepsilon > 0$ and U is open, the image $p(A)$ contains a small ball $B_\delta(0)$. Thus we have

$$0 < |B_\delta(0)| \leq |p(A)| \leq Lip(p)|A|.$$

In particular, A has positive Lebesgue measure and we may select a point $y \in A$ such that $D^2u(y)$ exists and $D^2u(y) \in \bar{\Gamma}$. Since φ_h touches u from below at y, we have $D^2\varphi_y(y) \leq D^2u(y)$ and thus $D^2\varphi_y(y) = D^2\varphi(y) - \varepsilon I = D^2\varphi(x_0) - \varepsilon I \in \bar{\Gamma}$. Sending $\varepsilon \to 0$, we obtain $D^2\varphi(x_0) \in \bar{\Gamma}$. □

3. Algorithm to decide membership in Γ

The definition of Γ does not give a method for verifying membership in the set. In this section, we will show that matrices in Γ correspond to certain infinite stabilizable sandpiles $\mathbb{Z}^2$. If $A \in \Gamma$ has rational entries, then its associated sandpile is periodic, which yields a method for checking membership in Γ for any rational matrix, and allows us to algorithmically determine the height of the boundary of Γ at any point with arbitrary precision. If $q : \mathbb{Z}^2 \to \mathbb{R}$, write $\lceil q \rceil$ for the function $\mathbb{Z}^2 \to \mathbb{Z}$ obtained by rounding each value of q up to the nearest integer. The principal lemma is the following.

Lemma 3.1. *$A \in \Gamma$ if and only if the configuration $\Delta^1 \lceil q_A \rceil$ is stabilizable, where*

$$q_A(x) := \frac{1}{2}x^t A x$$

is the quadratic form associated to A.

Proof. If u satisfies (1.1), then the Least Action Principle applied to $w = u - \lceil q_A \rceil$ shows that $\eta = \Delta^1 \lceil q_A \rceil$ is stabilizable. On the other hand, if $\eta = \Delta^1 \lceil q_A \rceil$ is stabilizable with odometer v, then $u = v + \lceil q_A \rceil$ satisfies (1.1). □

Since $A \leq B$ implies $x^t A x \leq x^t B x$ for all $x \in \mathbb{Z}^2$, the definition of Γ implies that Γ is downward closed in the matrix order:

Fact 3.2. *If $A \leq B$ and $B \in \Gamma$, then $A \in \Gamma$.* □

It follows that the boundary of Γ is Lipschitz, and in particular, continuous; thus to determine the structure of Γ, it suffices to characterize the rational matrices in Γ. We will say that a function s on $\mathbb{Z}^2$ is *n-periodic* if $s(x+y) = s(x)$ for all $y \in n\mathbb{Z}^2$.

Lemma 3.3. *If A has entries in $\frac{1}{n}\mathbb{Z}$ for a positive integer n, then $\Delta^1 \lceil q_A \rceil$ is $2n$-periodic.*

Proof. If $y \in 2n\mathbb{Z}^2$ then $Ay \in 2\mathbb{Z}^2$, so

$$q_A(x+y) - q_A(x) = (x^t + \frac{1}{2} y^t) A y \in \mathbb{Z}.$$

Hence $\lceil q_A \rceil - q_A$ is $2n$-periodic. Writing

$$\Delta^1 \lceil q_A \rceil = \Delta^1(\lceil q_A \rceil - q_A) - \Delta^1 q_A$$

and noting that $\Delta^1 q_A$ is constant, we conclude that $\Delta^1 \lceil q_A \rceil$ is $2n$-periodic. □

Thus the following lemma will allow us to make the crucial connection between rational matrices in Γ stabilizable sandpiles on finite graphs. It can be proved by appealing to [9, Theorem 2.8] on infinite toppling procedures, but we give a self-contained proof.

Lemma 3.4. *An n-periodic configuration $\eta : \mathbb{Z}^2 \to \mathbb{Z}$ is stabilizable if and only if it is stabilizable on the torus $T_n = \mathbb{Z}^2 / n\mathbb{Z}^2$.*

Proof. Supposing η is stabilizable on the torus T_n with odometer $\bar{v}$, and extending $\bar{v}$ to an n-periodic function v on $\mathbb{Z}^2$ in the natural way, we have that $\eta + \Delta^1 v \leq 3$. Thus η is stabilizable on $\mathbb{Z}^2$ by the Least Action Principle.

Conversely, if η is stabilizable on $\mathbb{Z}^2$, then there is a function $w : \mathbb{Z}^2 \to \mathbb{N}$ such that $\eta + \Delta^1 w \leq 3$. Proposition 2.2 implies that

$$\tilde{w}(x) := \min\{w(x+y) : y \in n\mathbb{Z}^2\},$$

also satisfies $\eta + \Delta^1 \tilde{w} \leq 3$. Since $\tilde{w}$ is n-periodic, we also have $\eta + \Delta^1_{T_n} \tilde{w} \leq 3$ and thus η is stabilizable on the torus T_n. □

The preceding lemmas give us a simple prescription for checking whether a rational matrix A is in Γ: compute $s = \Delta^1 \lceil q_A \rceil$ on the appropriate torus, and check if this is a stabilizable configuration. To check that s is stabilizable on the torus, we simply topple vertices with ≥ 4 chips until either reaching a stable configuration, or until every vertex has toppled at least once, in which case Proposition 2.4 implies that s is not stabilizable.

We thus can determine the boundary of Γ to arbitrary precision algorithmically. For $(a,b) \in \mathbb{R}^2$ let us define

$$c_0(a,b) = \sup\{c \mid M(a,b,c) \in \Gamma\}.$$

By Fact 3.2, we have $M(a,b,c) \in \bar{\Gamma}$ if and only if $c \leq c_0(a,b)$. Hence the boundary $\partial\Gamma$ is completely determined by the Lipschitz function $c_0(a,b)$. In Figure 1, the shade of the pixel at (a,b) corresponds to a value c that is provably within $\frac{1}{1024}$ of $c_0(a,b)$.

(Note that because of the Lipshitz condition, for any $\varepsilon > 0$ there is a finite procedure to either prove that Γ is within ε of its conjectured shape, or disprove the conjecture.)

The above results allow for some very limited partial confirmations of Conjecture 1.2. In particular, it is easy to deduce the following two facts:

Fact 3.5. *If A is rational and $\mathrm{tr}(A) < 2$, then $A \in \Gamma$.* □

Fact 3.6. *If A is rational and $\mathrm{tr}(A) > 3$, then $A \notin \Gamma$.* □

In both cases, the relevant observation is that for rational A, $\mathrm{tr}(A)$ is exactly the average density of the corresponding configuration $\eta = \Delta^1 \lceil q_A \rceil$ on the appropriate torus. This is all that is necessary for Fact 3.6. For Fact 3.5, the additional observation needed (due to Rossin [21]) is that on any finite connected graph, a chip configuration with fewer chips than there are edges in the graph will necessarily stabilize: for unstabilizable configurations, a legal sequence toppling every vertex at least once gives an injection from the edges of the graph to the chips, mapping each edge to the last chip to travel across it.

Facts 3.5 and 3.6 along with continuity imply that $2 \leq c_0(a,b) \leq 3$ for all $(a,b) \in \mathbb{R}^2$. With some additional work, the above results can be used to show that $c_0(a,b) = 2$ for all $a \in 2\mathbb{Z}$ and $b \in \mathbb{R}$, confirming Conjecture 1.2 along the vertical lines $x = a$ for $a \in 2\mathbb{Z}$. Finally, let us remark that c_0 has the translation symmetries

$$c_0(a+2,b) = c_0(a,b) = c_0(a,b+2).$$

This follows easily from the observation that $\frac{1}{2}x(x+1) - \frac{1}{2}y(y+1)$ and xy are both integer-valued discrete harmonic functions on $\mathbb{Z}^2$.

4. Apollonian circle packings

For any three tangent circles C_1, C_2, C_3, we consider the corresponding triple of tangent closed discs D_1, D_2, D_3 with disjoint interiors. We allow lines as circles, and allow the closure of any connected component of the complement of a circle as a closed disc. Thus we allow internal tangencies, in which case one of the closed discs is actually the unbounded complement of an open bounded disc. Note that to consider C_1, C_2, C_3 pairwise tangent we must require that three pairwise intersection points of the C_i are actually distinct, or else the corresponding configuration of the D_i is not possible. In particular, there can be at most two lines among the C_i, which are considered to be tangent at infinity whenever they are parallel.

The three tangent closed discs D_1, D_2, D_3 divide the plane into exactly two regions; thus any pairwise triple of circles has two *Soddy circles*, tangent to each circle in the triple. If all tangencies are external and at most one of C_1, C_2, C_3 is a line, then exactly one of the two regions bordered by the D_i is bounded, and the Soddy circle in the bounded region is called the *successor* of the triple.

An *Apollonian circle packing*, as defined in the introduction, is a minimal set of circles containing some triple of pairwise-tangent circles and closed under adding all Soddy circles of pairwise-tangent triples. Similarly, a *downward Apollonian circle packing* is a minimal set of circles containing some triple of pairwise externally tangent circles and closed under adding all successors of pairwise-tangent triples.

For us, the crucial example of an Apollonian packing is the Apollonian band packing. This is the packing which appears in Conjecture 1.2. A famous subset is the Ford circles, the set of circles $C_{p/q}$ with center $(\frac{2p}{q}, \frac{1}{q^2})$ and radius $\frac{1}{q^2}$, where p/q is a rational number in lowest terms. A simple description of the other circles is, as far as we know, unavailable. Thus our conjecture provides a new perspective on these circles.

An important observation regarding Apollonian circle packings is that a triple of pairwise externally tangent circles is determined by its intersection points with its successor:

Proposition 4.1. *Given a circle C and points $y_1, y_2, y_3 \in C$, there is at exactly one choice of pairwise externally tangent circles C_1, C_2, C_3 which are externally tangent to C at the points y_1, y_2, y_3.* □

Proposition 4.1, together with its counterpart for the case allowing an internal tangency, allows the deduction of the following fundamental property of Apollonian circle packings.

Proposition 4.2. *Let $\mathcal{C}$ be an Apollonian circle packing. A set $\mathcal{C}'$ of circles is an Apollonian circle packing if and only if $\mathcal{C}' = \mu(\mathcal{C})$ for some Möbius transformation μ.* □

The use of Möbius transformations allows us to deduce a geometric rule based on medians of triangles concerning successor circles in Apollonian packings:

Lemma 4.3. *Suppose that circles C, C_1, C_2 are pairwise tangent, with Soddy circles C_0 and C_3, and let $z_i^2 = p_i - c$, viewed as a complex number, where c is the center of C and p_i is the intersection point of C and C_i for each i. If L_i is a line parallel to the vector z_i which passes through 0 if $i = 1, 2, 3$ and does not pass through 0 if $i = 0$, then L_3 is a median line of the triangle formed by the lines L_0, L_1, L_2.*

Proof. Without loss of generality, we assume that C is a unit circle centered at the origin, and that $z_0^2 = -1$. The Möbius transformation

$$\mu_{z_1,z_2}(z) = \frac{z_1 + z_1 z_2 - z(z_1 - z_2)}{1 + z_2 + z(z_1 - z_2)}$$

sends 0 to z_1^2, 1 to z_2^2, and ∞ to $-1 = z_0^2$. Thus, for the pairwise tangent generalized circles $C' = \{y = 0\}, C_0' = \{y = 1\}, C_1' = \{x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}\}, C_2' = \{(x-1)^2 + (y - \frac{1}{2})^2 = \frac{1}{4}\}, C_3' = \{(x - \frac{1}{2})^2 = \frac{1}{64}\}$ (these are some of the "Ford circles"), we have that μ maps the intersection point of C', C_i' to the intersection point of C, C_i for $i = 0, 1, 2$, thus it must map the intersection point of C', C_3' to the intersection point of C, C_3, giving $\mu_{z_1,z_2}(\frac{1}{2}) = z_3$. Thus it suffices to show that for

$$f(z_1, z_2) := \mu_{z_1,z_2}(1/2) = \frac{z_1 + z_2 + 2z_1 z_2}{z_1 + z_2 + 2},$$

we have that

$$f(z_1^2, z_2^2) = \frac{\left(1 + \dfrac{\mathrm{Re}(z_1)\mathrm{Im}(z_2) + \mathrm{Re}(z_2)\mathrm{Im}(z_1)}{2\mathrm{Re}(z_1)\mathrm{Re}(z_2)} i\right)^2}{1 + \left(\dfrac{\mathrm{Re}(z_1)\mathrm{Im}(z_2) + \mathrm{Re}(z_2)\mathrm{Im}(z_1)}{2\mathrm{Re}(z_1)\mathrm{Re}(z_2)}\right)^2}, \tag{4.1}$$

as the right-hand side is the square of the unit vector whose tangent is the average of the tangents of z_1 and z_2; this is the correct slope of our median line since $z_0^2 = -1$ implies

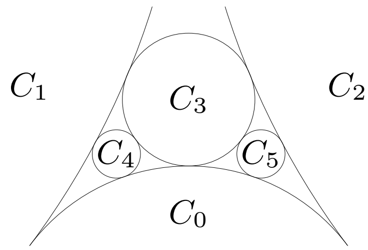

FIGURE 4. The circle arrangement from Proposition 4.5.

that L_0 is vertical. We will check (4.1) by writing $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$ to rewrite $f(z_1^2, z_2^2)$ as

$$\frac{(\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + 2(\cos\alpha + i\sin\alpha)^2(\cos\beta + i\sin\beta)^2}{(\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + 2}$$
$$= \frac{(\cos(\alpha+\beta) + i\sin(\alpha+\beta))(\cos(\alpha-\beta) + \cos(\alpha+\beta) + i\sin(\alpha+\beta))}{\cos(\alpha-\beta)(\cos(\alpha+\beta) + i\sin(\alpha+\beta)) + 1}, \quad (4.2)$$

where we have used the identity

$$(\cos x + i\sin x)^2 + (\cos y + i\sin y)^2 = 2\cos(x-y)(\cos(x+y) + i\sin(x+y)),$$

which can be seen easily geometrically. Dividing the top and bottom of the right side of (4.2) by $\cos(\alpha+\beta) + i\sin(\alpha+\beta)$ gives

$$f(z_1^2, z_2^2) = \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta) + i\sin(\alpha+\beta)}{\cos(\alpha-\beta) + \cos(\alpha+\beta) - i\sin(\alpha+\beta)}.$$

Thus to complete the proof, note that the right-hand side of (4.1) can be can simplified as

$$\frac{\left(1 + \frac{\cos\alpha\sin\beta + \cos\beta\sin\alpha}{2\cos\alpha\cos\beta} i\right)^2}{1 + \left(\frac{\cos\alpha\sin\beta + \cos\beta\sin\alpha}{2\cos\alpha\cos\beta}\right)^2} = \frac{(\cos(\alpha+\beta) + \cos(\alpha-\beta) + i\sin(\alpha+\beta))^2}{(\cos(\alpha+\beta) + \cos(\alpha-\beta))^2 + \sin^2(\alpha+\beta)}$$
$$= \frac{\cos(\alpha+\beta) + \cos(\alpha-\beta) + i\sin(\alpha+\beta)}{\cos(\alpha+\beta) + \cos(\alpha-\beta) - i\sin(\alpha+\beta)}$$

by multiplying the top and bottom by $(2\cos\alpha\cos\beta)^2$ and using the Euler identity consequences

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$$
$$\cos\alpha\sin\beta + \cos\beta\sin\alpha = \sin(\alpha+\beta). \qquad \square$$

Remark 4.4. By Proposition 4.2, a set of three points $\{x_1, x_2, x_3\}$ on a circle C uniquely determine three other points $\{y_1, y_2, y_3\}$ on C, as the points of intersection of C with successor circles of triples $\{C, C_i, C_j\}$, where C_1, C_2, C_3 are the unique triple of circles which are pairwise externally tangent and externally tangent to C at the points x_i. Since the median triangle of the median triangle of a triangle T is homothetic to T, Lemma 4.3 implies that this operation is an involution: the points determined by $\{y_1, y_2, y_3\}$ in this way is precisely the set $\{x_1, x_2, x_3\}$.

We close this section with a collection of simple geometric constraints on arrangements of externally tangent circles (Figure 4), whose proofs are rather straightforward:

Proposition 4.5. *Let C_0, C_1, C_2 be pairwise externally tangent proper circles with successor C_3, and let C_4 and C_5 be the successors of C_0, C_1, C_3 and C_0, C_2, C_3, respectively. Letting c_i denote the center of the circle C_i, we have the following geometric bounds:*

(1) $c_i c_3 c_j \leq \pi$ *for* $\{i, j\} \subset \{0, 1, 2\}$.
(2) $\angle c_4 c_0 c_3, \angle c_5 c_0 c_3 < \frac{\pi}{2}$.
(3) $\angle c_4 c_0 c_3 \geq \frac{1}{2} \angle c_5 c_0 c_3$ *(and vice versa).*
(4) $\angle c_4 c_3 c_5 \geq 2 \cdot \arctan(3/4)$. □

5. Apollonian triangles and triangulations

We build up to Apollonian triangles and triangulations by defining the *Apollonian curve* associated to an ordered triple of circles. This will allow us to define the *Apollonian triangle* associated to a quadruple of circles, and finally the *Apollonian triangulation* associated to a downward packing of circles. We will define these objects implicitly, and then show that they exist and are unique up to translation and homothety (i.e., any two Apollonian curves γ, γ' associated to the same triple satisfy $\gamma' = a\gamma + \mathbf{b}$ for some $a \in \mathbb{R}$ and $b \in \mathbb{R}^2$). In Section 6, we give a recursive description of the Apollonian curves which characterizes these objects without reference to circle packings.

Fix a circle C_0 with center c_0 and let C and C' be tangent circles tangent to C_0 at x and x', and have centers c and c', respectively. We define $s(C, C')$ to be the successor of the triple (C_0, C, C') and $\alpha(C)$ to be the angle of the vector $v(C) := c - c_0$ with the positive x-axis. Let $v^{1/2}(C)$ to be a complex square root of $v(C)$, and let $\ell^{1/2}(C) = \mathbb{R}v^{1/2}(C)$ be the real line it spans. (We will actually only use $\ell^{1/2}(C)$, so the choice of square root is immaterial.) Note that all of these functions depend on the circle C_0; we will specify which circle the functions are defined with respect to when it is not clear from context.

Now fix circles C_1 and C_2 such that C_0, C_1, C_2 are pairwise externally tangent. Let $\mathcal{C}$ denote the smallest set of circles such that $C_1, C_2 \in \mathcal{C}$ and for all tangent $C, C' \in \mathcal{C}$ we have $s(C, C') \in \mathcal{C}$. Note that all circles in $\mathcal{C}$ are tangent to C_0.

Definition 5.1. A (continuous) curve $\gamma : [\alpha(C_1), \alpha(C_2)] \to \mathbb{R}^2$ is an *Apollonian curve* associated to the triple (C_0, C_1, C_2) if for all tangent circles $C, C' \in \mathcal{C}$,

$$\gamma(\alpha(C)) - \gamma(\alpha(C')) \in \ell^{1/2}(s(C, C')).$$

We call $\gamma(\alpha(s(C_1, C_2)))$ the *splitting point* of γ. The following Observation implies, in particular, that the splitting point divides γ into two smaller Apollonian curves.

Observation 5.2. *For any two tangent circles $C, C' \in \mathcal{C}$, the restriction $\gamma|_{[\alpha(C), \alpha(C')]}$ is also an Apollonian curve.* □

To prove the existence and uniqueness of Apollonian curves, we will need the following observation, which is easy to verify from the fact that no circle lying inside the region bounded by C_0, C_1, C_2 and tangent to C_0 has interior disjoint from the family $\mathcal{C}$:

Observation 5.3. $\alpha(\mathcal{C})$ *is dense in the interval* $[\alpha(C_1), \alpha(C_2)]$. □

We can now prove the existence and uniqueness of Apollonian curves.

Theorem 5.4. *For any pairwise tangent ordered triple of circles (C_0, C_1, C_2), there is an associated Apollonian curve γ, which is unique up to translation and scaling.*

Proof. The choice of the points $\gamma(\alpha(C_1))$ and $\gamma(\alpha(C_2))$ is determined uniquely up to translation and scaling by the constraint that $\gamma(\alpha(C_1)) - \gamma(\alpha(C_2))$ is a real multiple of $v^{1/2}(s(C_1, C_2))$. This choice then determines the image $\gamma(\alpha(C))$ for all circles $C \in \mathcal{C}$ recursively: for any tangent circles $C^1, C^2 \in \mathcal{C}$ with $C^3 := s(C^1, C^2)$ the constraints

$$\gamma(\alpha(C^1)) - \gamma(\alpha(C^3)) \in \ell^{1/2}(s(C^1, C^3))$$
$$\gamma(\alpha(C^2)) - \gamma(\alpha(C^3)) \in \ell^{1/2}(s(C^2, C^3))$$

determine $\gamma(\alpha(C^3))$ uniquely given $\gamma(\alpha(C^1))$ and $\gamma(\alpha(C^2))$. To show that there is a unique and well-defined curve γ, by Observation 5.3 it is enough to show that γ is a continuous function on the set $\alpha(\mathcal{C})$. For this it suffices to find an absolute constant $\beta < 1$ for which

$$\left|\gamma(\alpha(C^1)) - \gamma(\alpha(s(C^1, C^2)))\right| \leq \beta \left|\gamma(\alpha(C^1)) - \gamma(\alpha(C^2))\right| \tag{5.1}$$

for tangent circles $C^1, C^2 \in \mathcal{C}$, as this implies, for example, that by taking successors k times, we can find a circle $C' \in \mathcal{C}$ such that all points in $\gamma([\alpha(C^1), \alpha(C')])$ lie within $\beta^k \left|\gamma(\alpha(C^1)) - \gamma(\alpha(C^2))\right|$ of $\gamma(\alpha(C^1))$. We get the absolute constant β from an application of the law of sines to the triangle with vertices $p_1 = \gamma(\alpha(C^1)), p_2 = \gamma(\alpha(C^2)), p_3 = \gamma(\alpha(s(C^1, C^2)))$: part 3 of Proposition 4.5 implies that $\theta := \angle p_3 p_1 p_2 \geq \frac{1}{2}\angle p_3 p_2 p_1$; the Law of Sines then implies that line (5.1) holds with $\beta = \frac{\sin(2\theta)}{\sin(3\theta)}$, which is $\leq \frac{2}{3}$ always since part 2 of Proposition 4.5 implies that $\theta \leq \frac{\pi}{2}$. □

Theorem 5.5. *The image of an Apollonian curve γ corresponding to (C_0, C_1, C_2) has a unique tangent line at each point $\gamma(\alpha)$. This line is at angle $\alpha/2$ to the positive x-axis. In particular, γ is a convex curve.*

Proof. Observation 5.3 and Definition 5.1 give that for any $C \in \mathcal{C}$, there is a unique line tangent to the image of γ at $\gamma(\alpha(C))$, which is at angle $\alpha(C)/2$ to the x-axis. Together with another application of Observation 5.3 and the fact that $\frac{\alpha}{2}$ is a continuous function of α, this gives that the image γ has a unique tangent line at angle $\frac{\alpha}{2}$ to the x-axis at any point $\gamma(\alpha)$. □

Definition 5.6. The *Apollonian triangle* corresponding to an unordered triple of externally tangent circles C_1, C_2, C_3 and circle C_0 externally tangent to each of them is defined as the bounded region (unique up to translation and scaling) enclosed by the images of the Apollonian curves γ_{12}, γ_{23}, γ_{31} corresponding to the triples (C_0, C_1, C_2), (C_0, C_2, C_3), (C_0, C_3, C_1) such that $\gamma_{ij}(\alpha(C_j)) = \gamma_{jk}(\alpha(C_j))$ for each $\{i, j, k\} = \{1, 2, 3\}$.

Note that Theorem 5.4 implies that each triple $\{C_1, C_2, C_3\}$ of pairwise tangent circles corresponds to an Apollonian triangle $\mathcal{T}$ which is unique up to translation and scaling. Theorem 5.5 implies that the curves $\gamma_{12}, \gamma_{23}, \gamma_{31}$ do not intersect except at their endpoints, and that $\mathcal{T}$ is strictly contained in the triangle with vertices $\gamma_{12}(C_2), \gamma_{23}(C_3), \gamma_{31}(C_1)$. Another consequence of Theorem 5.5 is that any two sides of an Apollonian triangle have the same tangent line at their common vertex. Thus, the interior angles of an Apollonian triangle are 0.

An Apollonian triangle is *proper* if C_4 is smaller than each of C_1, C_2, C_3, i.e., if C_4 is the successor of C_1, C_2, C_3, and all Apollonian triangles appearing in our solutions to the Apollonian PDE will be proper.

We also define a degenerate version of an Apollonian triangle:

Definition 5.7. The *degenerate Apollonian triangle* corresponding to the pairwise tangent circles (C_1, C_2, C_3) is the compact region (unique up to translation and scaling) enclosed by the image of the Apollonian curve γ corresponding to (C_1, C_2, C_3), and the tangent lines to γ at its endpoints $\gamma(\alpha(C_2))$ and $\gamma(\alpha(C_3))$.

Proper Apollonian triangles (and their degenerate versions) are the building blocks of Apollonian triangulations, the fractals that support piecewise-quadratic solutions to the Apollonian PDE. Recall that $\mathcal{A}^-(C_1, C_2, C_3)$ denotes the smallest set of circles containing the circles C_1, C_2, C_3 and closed under adding successors of pairwise tangent triples. To each circle $C \in \mathcal{A}^-(C_1, C_2, C_3) \setminus \{C_1, C_2, C_3\}$ we associate an Apollonian triangle $\mathcal{T}_C$ corresponding to the unique triple $\{C^1, C^2, C^3\}$ in $\mathcal{A}^-(C_1, C_2, C_3)$ whose successor is C.

Definition 5.8. The *Apollonian triangulation* associated to a triple $\{C_1, C_2, C_3\}$ of externally tangent circles is a union of (proper) Apollonian triangles $\mathcal{T}_C$ corresponding to each circle $C \in \mathcal{A}^-(C_1, C_2, C_3) \setminus \{C_1, C_2, C_3\}$, together with degenerate Apollonian triangles $\mathcal{T}_C$ for each $C = C_1, C_2, C_3$, such that disjoint circles correspond to disjoint Apollonian triangles, and such that for tangent circles C, C' in $\mathcal{A}^-(C_1, C_2, C_3)$ where $r(C') \le r(C)$, we have that $\mathcal{T}_{C'}$ and $\mathcal{T}_C$ intersect at a vertex of $\mathcal{T}_{C'}$, and that their boundary curves meet at right angles.

Figure 2 shows an Apollonian triangulation, excluding the three degenerate Apollonian triangles on the outside.

Remark 5.9. By Theorem 5.5 and the fact that centers of tangent circles are separated by an angle π about their tangency point, the right angle requirement is equivalent to requiring that the intersection of $\mathcal{T}_{C'}$ and $\mathcal{T}_C$ occurs at the point $\gamma(\alpha(C'))$ on an Apollonian boundary curve γ of $\mathcal{T}_C$.

6. Geometry of Apollonian curves

In this section, we will give a circle-free geometric description of Apollonian curves. This will allow us to easily deduce geometric bounds necessary for our construction of piecewise-quadratic solutions to work.

Recall that by Theorem 5.5, each pair of boundary curves of an Apollonian triangle have a common tangent line where they meet. Denoting the three such tangents the *spline lines* of the Apollonian triangle, Remark 4.4, and Lemma 4.3 give us the following:

Lemma 6.1. *The spline lines of an Apollonian triangle with vertices v_1, v_2, v_3 are the median lines of the triangle $\triangle v_1 v_2 v_3$, and thus meet at a common point, which is the centroid of $\triangle v_1 v_2 v_3$* □.

More crucially, Lemma 4.3 allows us to give a circle-free description of Apollonian curves. Indeed, letting c be the intersection point of the tangent lines to the endpoints p_1, p_2 of an Apollonian curve γ, Lemma 4.3 implies (via Definition 5.1 and Theorem 5.5) that the splitting point s of γ is the intersection of the medians from p_1, p_2 of the triangle $\triangle p_1 p_2 c$, and thus the centroid of the triangle $\triangle p_1 p_2 c$. The tangent line to γ at

s is parallel p_1p_2; thus, by Observations 5.2 and 5.3, the following recursive procedure determines a dense set of points on the curve γ given the triple (p_1, p_2, c):

(1) find the splitting point s as the centroid of $\triangle p_1p_2c$.
(2) compute the intersections c_1, c_2 of the p_1c and p_2c, respectively, with the line through s parallel to p_1p_2.
(3) carry out this procedure on the triples (p_1, c_1, s) and (s, c_2, p_2).

By recalling that the centroid of a triangle lies 2/3 of the way along each median, the correctness of this procedure thus implies that the "generalized quadratic Bézier curves" with constant $\frac{1}{3}$ described by Paoletti in his thesis [17] are Apollonian curves. Combined with Lemma 6.1, this procedure also gives a way of enumerating barycentric coordinates for a dense set of points on each of the boundary curves of an Apollonian triangle, in terms of its 3 vertices. Thus, in particular, all Apollonian triangles are equivalent under affine transformations. Conversely, since Proposition 4.1 implies that any 3 vertices in general position have a corresponding Apollonian triangle, the affine image of any Apollonian triangle must also be an Apollonian triangle. In particular:

Theorem 6.2. *For any three vertices v_1, v_2, v_3 in general position, there is a unique Apollonian triangle whose vertices are v_1, v_2, v_3.* □

Another consequence of the affine equivalence of Apollonian triangles is conformal *inequivalence* of Apollonian triangulations: suppose $\varphi : \mathcal{S} \to \mathcal{S}'$ is a conformal map between Apollonian triangulations which preserves the incidence structure. Let $\mathcal{T}$ and $\mathcal{T}'$ be their central Apollonian triangles, and $\alpha : \mathcal{T} \to \mathcal{T}'$ the corresponding affine map. By Remark 5.9, the points on $\partial\mathcal{T}$ computed by the recursive procedure above are the points at which $\mathcal{T}$ is incident to other Apollonian triangles of $\mathcal{S}$; thus, $\varphi = \alpha$ on a dense subset of $\partial\mathcal{T}$, and therefore on all of $\partial\mathcal{T}$. Since the real and imaginary parts of φ and α are harmonic, the maximum principle implies that $\varphi = \alpha$ on $\mathcal{T}$, and therefore on $\mathcal{S}$ as well, giving that $\mathcal{S}$ and $\mathcal{S}'$ are equivalent under a Euclidean similarity transformation. We stress that in general, even though $\mathcal{T}$ and $\mathcal{T}'$ are affinely equivalent, nonsimilar triangulations are *not* affinely equivalent, as can be easily be verified by hand.

It is now easy to see from the right-angle requirement for Apollonian triangulations that the Apollonian triangulation associated to a particular triple of circles must be also be unique up to translation and scaling: by Remark 5.9, the initial choice of translation and scaling of the three degenerate Apollonian triangles determines the rest of the figure. (On the other hand, it is not at all obvious that Apollonian triangulations exist. This is proved in Theorem 7.1 below.) Hence by Proposition 4.1, an Apollonian triangulation is uniquely determined by the three pairwise intersection points of its three degenerate triangles:

Theorem 6.3. *For any three vertices v_1, v_2, v_3, there is at most one Apollonian triangulation for which the set of vertices of its three degenerate Apollonian triangles is $\{v_1, v_2, v_3\}$.* □

To ensure that our piecewise-quadratic constructions are well-defined on a convex set, we will need to know something about the area of Apollonian triangles. Affine equivalence implies that there is a constant C such that the area of any Apollonian triangle is equal to $C \cdot A(T)$ where T is the Euclidean triangle with the same 3 vertices. In fact we can determine this constant exactly:

Lemma 6.4. *An Apollonian triangle $\mathcal{T}$ with vertices p_1, p_2, p_3 has area $\frac{4}{7}A(T)$ where $A(T)$ is the area of the triangle $T = \triangle p_1p_2p_3$.*

Proof. Lemma 6.1 implies that the spline lines of $\mathcal{T}$ meet at the centroid c of T. It suffices to show that $A(\mathcal{T} \cap \triangle p_i p_j c) = \frac{4}{7} A(\triangle p_i p_j c)$ for each $\{i, j\} \subset \{1, 2, 3\}$; thus, without loss of generality, we will show that this holds for $i = 1, j = 2$.

Let $\mathcal{T}_3 = \mathcal{T} \cap \triangle p_1 p_2 c$, and let $\mathcal{T}_3^C = \triangle p_1 p_2 c \setminus \mathcal{T}_3$. We aim to compute the area of the complement $\mathcal{T}_3^C$ using our recursive description of Apollonian curves. Step 1 of each stage of the recursive description computes a splitting point s' relative to points p_1', p_2', c', and $\mathcal{T}_3^C$ is the union of the triangles $\triangle p_1' p_2' s'$ for all such triples of points encountered in the procedure. As the median lines of any triangle divide it into 6 regions of equal area, we have for each such triple that $A(p_1' p_2' s') = \frac{1}{3} A(p_1' p_2' c')$.

Meanwhile, step 2 each each stage of the recursive construction computes new intersection points c_1', c_2' with which to carry out the procedure recursively. The sum of the area of the two triangles $\triangle p_1', c_1', s'$ and $\triangle s' c_2' p_2'$ is

$$A(\triangle p_1', c_1', s') + A(\triangle s', c_2', p_2') = \frac{5}{9} A(\triangle p_1' p_2' s') - \frac{1}{3} A(\triangle p_1' p_2' s') = \frac{2}{9} A(\triangle p_1' p_2' s'),$$

Since $\frac{5}{9} A(\triangle p_1' p_2' s')$ is the portion of the area of the triangle $p_1' p_2' s'$ which lies between the lines $p_1 p_2$ and c_1', c_2'. Thus, the area $A(\mathcal{T}_3^C)$ is given by

$$A(\triangle p_1 p_2 c) \cdot \left(\frac{1}{3} + \left(\frac{2}{9}\right) \frac{1}{3} + \left(\frac{2}{9}\right)^2 \frac{1}{3} + \left(\frac{2}{9}\right)^3 \frac{1}{3} + \cdots \right) = \frac{3}{7} A(\triangle p_1 p_2 c). \qquad \square$$

We conclude this section with some geometric bounds on Apollonian triangles. The following Observation is easily deduced from part 4 of Proposition 4.5:

Observation 6.5. *Given a proper Apollonian triangle with vertices v_1, v_2, v_3 generated from a non-initial circle C and parent triple of circles (C_1, C_2, C_3), the angles $\angle v_i v_j v_k$ ($\{i, j, k\} = \{1, 2, 3\}$) are all $> \arctan(3/4) > \frac{\pi}{5}$ if C has smaller radius than each of C_1, C_2, C_3.* □

Recall that Theorem 5.5 implies that pairs of boundary curves of an Apollonian triangle meet their common vertex at a common angle, and that there is thus a unique line tangent to both curves through their common vertex. We call such lines L_1, L_2, L_3 for each vertex v_1, v_2, v_3 the *median lines* of the Apollonian triangle, motivated by the fact that Lemma 4.3 implies that they are median lines of the triangle $\triangle v_1 v_2 v_3$.

Observation 6.6. *The pairwise interior angles of the median lines L_1, L_2, L_3 of a proper Apollonian triangle all lie in the interval $(\frac{\pi}{2}, \frac{3\pi}{4})$.*

Proof. Part 1 of Proposition 4.5 gives that the interior angles of the median lines of the corresponding Apollonian triangle must satisfy $\alpha_i \leq \pi - \frac{\pi}{4} = \frac{3}{4}\pi$. The lower bound follows from $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$. □

7. Fractal solutions to the Apollonian PDE

Our goal now is to prove that Apollonian triangulations exist, and that they support piecewise quadratic solutions to the Apollonian PDE which have constant Hessian on each Apollonian triangle. We prove the following theorems in this section:

Theorem 7.1. *To any mutually externally tangent circles C_1, C_2, C_3 in an Apollonian circle packing $\mathcal{A}$, there exists a corresponding Apollonian triangulation $\mathcal{S}$. Moreover, the closure of $\mathcal{S}$ is convex.*

Theorem 7.2. *For any Apollonian triangulation $\mathcal{S}$ there is a piecewise quadratic $C^{1,1}$ map $u : \bar{\mathcal{S}} \to \mathbb{R}$ such that for each Apollonian triangle $\mathcal{T}_C$ comprising $\mathcal{S}$, the Hessian D^2u is constant and equal to $m(C)$ in the interior of $\mathcal{T}_C$.*

Theorem 7.2 implies Theorem 1.3 from the Introduction via Proposition 2.8, by taking $Y = \mathcal{S}$ and $Z = \bar{\mathcal{S}}$, where $\mathcal{S} = \mathcal{S}(A_1, A_2, A_3)$ is the Apollonian triangulation generated by the triple of circles $c(A_i)$ for $i = 1, 2, 3$. Using the fact that $\mathcal{S}$ has full measure in $\bar{\mathcal{S}}$, proved in Section 7.2, this theorem constructs piecewise-quadratic solutions to the Apollonian PDE via Proposition 2.8.

We will prove Theorems 7.1 and 7.2 in tandem; perhaps surprisingly, we do not see a simple geometric proof of Theorem 7.1, and instead, in the course of proving Theorem 7.2, will prove that certain piecewise-quadratic approximations to u exist and use constraints on such constructions to achieve a recursive construction of approximations to $\mathcal{S}$.

7.1. **The recursive construction.** We begin our construction of u—and, simultaneously $\mathcal{S}$, which will be the limit set of the support of the approximations to u we construct—by considering the three initial matrices $A_i = m(C_i)$ for $i = 1, 2, 3$.

Observation 2.6 implies that there are vectors v_1, v_2, v_3 such that

$$A_i = A_4^- + v_i \otimes v_i \quad \text{for each } i = 1, 2, 3.$$

We may then select distinct $p_1, p_2, p_3 \in \mathbb{R}^2$ such that $v_i \cdot (p_j - p_k) = 0$ for $\{i, j, k\} = \{1, 2, 3\}$. Observation 2.5 and Definition 5.1 imply that we can choose degenerate Apollonian triangles $\mathcal{T}_{A_i}$ corresponding to (A_i, A_j, A_k) ($\{i, j, k\} = \{1, 2, 3\}$) meeting at the points p_1, p_2, p_3. Note that the straight sides of distinct $\mathcal{T}_{A_i}$ meet only at right angles.

It is easy to build a piecewise quadratic map $u_0 \in C^{1,1}(\mathcal{T}_{A_1} \cup \mathcal{T}_{A_2} \cup \mathcal{T}_{A_3})$ whose Hessian lies in the set $\{A_1, A_2, A_3\}$: for example, we can simply define u_0 as

$$u_0(x) := \frac{1}{2} x^t A_4^- x + \frac{1}{2}(v_i \cdot (x - p_j))^2 \quad \text{for } x \in \mathcal{T}_{A_i} \text{ and } i \neq j. \tag{7.1}$$

We now extend this map to the full Apollonian triangulation by recursively choosing quadratic maps on successor Apollonian triangles that are compatible with the previous pieces. The result is a piecewise-quadratic $C^{1,1}$ map whose pieces form a full measure subset of a compact set. By a *quadratic function* on $\mathbb{R}^2$ we will mean a function of the form $\varphi(x) = x^t Ax + b^t \cdot x + c$ for some matrix $A \in S_2$, vector $b \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Letting $(1, 2, 3)^3$ denote $\{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$, the heart of the recursion is the following claim, illustrated in Figure 5.

Claim. Suppose B_0 is the successor of a triple (B_1, B_2, B_3), and that for each $(i, j, k) \in (1, 2, 3)^3$, we have that γ_i is an Apollonian curve for (B_i, B_j, B_k) from p_k to p_j, φ_i is a quadratic function with Hessian B_i, and the value and gradient of φ_i, φ_j agree at p_k for each k.

Then there is a quadratic function φ_0 with Hessian B_0 whose value and gradient agree with that of φ_i at each $q_i := \gamma_i(\alpha_i(B_0))$, and for each $(i, j, k) \in (1, 2, 3)^3$, there is an Apollonian curve γ_i' from q_j to q_k corresponding to the triple (B_0, B_j, B_k). (Here, the α_i denotes the angle function α defined with respect to B_i.)

We will first see how the claim allows the construction to work. Defining the *level* of each A_1, A_2, A_3 to be $\ell(A_i) = 0$, and recursively setting the level of a successor of a triple (A_i, A_j, A_k) as $\max(\ell(A_i), \ell(A_j), \ell(A_k)) + 1$, allows us to define a *level-k partial Apollonian triangulation* which will be the domain of our iterative constructions.

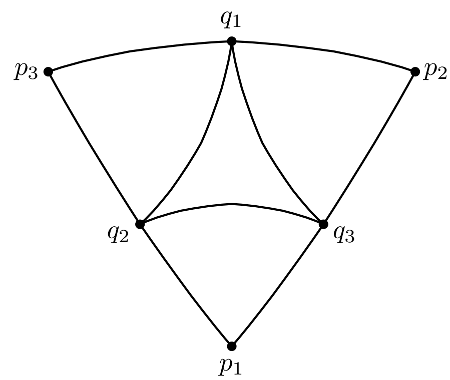

FIGURE 5. The Apollonian curves γ_i, γ_i' $(i = 1, 2, 3)$ in the claim.

Definition 7.3. A *level-k partial Apollonian triangulation* corresponding to $\{A_1, A_2, A_3\}$ is the subset $\mathcal{S}_k \subset \mathcal{S}(A_1, A_2, A_3)$ consisting of the union of the Apollonian triangles $\mathcal{T}_A \in \mathcal{S}$ for which $\ell(A) \leq k$.

Note that u_0 is defined on a level-0 partial Apollonian triangulation.

Consider now a $C^{1,1}$ piecewise-quadratic function u_{k-1} defined on the union of a level-$(k-1)$ partial Apollonian triangulation $\mathcal{S}_{k-1}$, whose Hessian on each $\mathcal{T}_{A_i} \in \mathcal{S}_{k-1}$ is the matrix A_i. Any three pairwise intersecting triangles $\mathcal{T}_{A_i}, \mathcal{T}_{A_j}, \mathcal{T}_{A_k} \in \mathcal{S}_{k-1}$ bound some region R, and, denoting by γ_s the boundary curve of each $\mathcal{T}_{A_s}$ which coincides with the boundary of R and by p_s the shared endpoint of γ_t, γ_u ($\{s, t, u\} = \{i, j, k\}$), the hypotheses of the Claim are satisfied for $(B_1, B_2, B_3) = (A_i, A_j, A_k)$, where $\varphi_1, \varphi_2, \varphi_3$ are the quadratic extensions to the whole plane of the restrictions $u_{k-1}|_{\mathcal{T}_{A_i}}, u_{k-1}|_{\mathcal{T}_{A_j}}, u_{k-1}|_{\mathcal{T}_{A_k}}$, respectively.

Noting that the three Apollonian curves given by the claim bound an Apollonian triangle corresponding to the triple (A_i, A_j, A_k), the claim allows us to extend u_{k-1} to a $C^{1,1}$ function u_k on the level-k partial fractal $\mathcal{S}_k$ by setting $u_k = \varphi_0$ on the triangle $\mathcal{T}_{A_\ell} \in \mathcal{S}_k$ for the successor A_ℓ of (A_i, A_j, A_k), for each externally tangent triple $\{A_i, A_j, A_k\}$ in $\mathcal{S}_{k-1}$. Letting U denote the topological closure of $\mathcal{S}$, we can extend the limit $\bar{u} : \mathcal{S} \to \mathbb{R}$ of the u_k to a $C^{1,1}$ function $u : U \to \mathbb{R}$; to prove Theorems 7.1 and 7.2, it remains to prove the Claim, and that $\mathcal{S}$ is a full-measure subset of its convex closure Z, so that in fact $U = Z$. We will prove that $\mathcal{S}$ is full-measure in Z in Section 7.2, and so turn our attention to proving the Claim. We make use of the following two technical lemmas for this purpose, whose proofs we postpone until Section 7.3.

Lemma 7.4. *Let $p_1, p_2, p_3 \in \mathbb{R}^2$ be in general position, and let $v_i = (p_j - p_k)^\perp$ be the perpendicular vector for which the ray $p_i + sv_i$ $(s \in \mathbb{R}^+)$ intersects the segment $\overline{p_j p_k}$, for each $\{i, j, k\} = \{1, 2, 3\}$. If $\varphi_1, \varphi_2, \varphi_3$ are quadratic functions satisfying*

$$D^2\varphi_i = A_i, \quad D\varphi_i(p_k) = D\varphi_j(p_k), \quad \text{and} \quad \varphi_i(p_k) = \varphi_j(p_k)$$

for each $\{i, j, k\} = \{1, 2, 3\}$, where $A_i = B^- + v_i \otimes v_i$ and $\operatorname{tr}(v_i \otimes v_i) > 2(\operatorname{tr}(B) - 2)$ for some matrix B and vectors v_i perpendicular to $p_j - p_k$ for each $\{i, j, k\} = \{1, 2, 3\}$, then there is a (unique) choice of $X_0 \in \mathbb{R}^2$, $y_i = X_0 + t_i v_i$ for $t_i/(v_i \cdot p_j) > 1$, and

$b \in \mathbb{R}^2, c \in \mathbb{R}$ *such that the map*

$$\varphi_0(x) := \frac{1}{2} x^t B^- x + \frac{1}{2} \operatorname{tr}(B) |x - X_0|^2 + b^t x + c \quad \text{for} \quad x \in V_4$$

satisfies $\varphi_0(y_i) = \varphi_j(y_i)$ *and* $\varphi_0'(y_i) = \varphi_j'(y_i)$ *for each* $\{i, j\} \subset \{1, 2, 3\}$.

Lemma 7.5. *Suppose the points* $p_1, p_2, p_3 \in \mathbb{R}^2$ *are in general position and the quadratic functions* $\varphi_1, \varphi_2, \varphi_3 : \mathbb{R}^2 \to \mathbb{R}$ *satisfy*

$$\varphi_i(p_k) = \varphi_j(p_k) \quad \text{and} \quad D\varphi_i(p_k) = D\varphi_j(p_k),$$

for $\{i, j, k\} = \{1, 2, 3\}$. *There is a matrix* B *and coefficients* $\alpha_i \in \mathbb{R}$ *such that*

$$D^2 \varphi_i = B^- + \alpha_i (p_j - p_k)^\perp \otimes (p_j - p_k)^\perp, \tag{7.2}$$

for $i = 1, 2, 3$.

Observe now that in the setting of the claim, the conditions of Lemma 7.4 are satisfied for $A_i := B_i$ $(i = 1, 2, 3)$, $B := B_0$ and where v_i is the vector for which $B_i - B_0 = v_i \otimes v_i$ for each $i = 1, 2, 3$; indeed Observations 2.5 and the definition of Apollonian curve ensure that v_i is perpendicular to $p_j - p_k$ for each $\{i, j, k\} = \{1, 2, 3\}$. Let now X_0, t_i, and y_i be as given in the Lemma. We wish to show that $y_i = \gamma_i(\alpha(B_0))$ for each i. Letting B_{ij} denote the successor of (B_0, B_i, B_j) for $\{i, j\} \subset \{1, 2, 3\}$, we apply Lemma 7.5 to the triples $\{p_i, p_j, y_k\}$ of points and $\{\varphi_i, \varphi_j, \varphi_0\}$ of functions for each of the three pairs $\{i, j\} \subset \{1, 2, 3\}$. In each case, we are given some matrix B for which

$$B_i = B^- + \alpha_{k,s} (p_k - y_i)^\perp \otimes (p_k - y_i)^\perp, \tag{7.3}$$

$$B_j = B^- + \alpha_{k,s} (p_k - y_j)^\perp \otimes (p_k - y_j)^\perp, \text{ and} \tag{7.4}$$

$$B_0 = B^- + \alpha_{k,0} (y_i - y_j)^\perp \otimes (y_i - y_j)^\perp \tag{7.5}$$

for real numbers $\alpha_{k,i} \in \mathbb{R}$.

Observation 2.6 now implies that either $B = B_{ij}$ or $B = B_k$; the latter possibility cannot happen, however: if we had $B = B_k$, then as $\bar\rho(B_k - B_0) = -\bar\rho(B_0 - B_k)$, Observation 2.5 would imply that $y_i - y_j$ is perpendicular to $p_i - p_j$. This is impossible since the constraint $t_s/(v_s \cdot p_t) > 1$ for $\{s, t\} = \{i, j\}$ in Lemma 7.4 implies that the segment $y_i y_j$ must intersect the segments $p_i p_k$ and $p_j p_k$, yet part 1 of Proposition 4.5 implies that $\triangle p_i p_j p_k$ is acute. So we have indeed that the matrix B given by the applications of Lemma 7.5 to the triple (B_0, B_i, B_j) is B_{ij}, for each $\{i, j\} \subset \{1, 2, 3\}$.

For each $\{i, j, k\} = \{1, 2, 3\}$, Observation 2.5, Definition 5.1, Theorem 5.4, and the constraints (7.3), (7.4) now imply that $y_i = q_i := \gamma_i(\alpha_i(B_0))$, as the point $\gamma_i(\alpha_i(B_0))$ is determined by the endpoints $\gamma_i(\alpha_k(B_j)), \gamma_i(\alpha_i(B_k))$ and the condition from Definition 5.1 that $\gamma_i(\alpha_i(B_j)) - \gamma_k(\alpha_i(B_0))$ and $\gamma_i(\alpha_i(B_k)) - \gamma_i(\alpha_i(B_0))$ are multiples of $v_i^{1/2}(s_i(B_j, B_0))$ and $v_i^{1/2}(s_i(B_k, B_0))$, respectively (and so of $p_k - y_i$ and $p_k - y_j$, respectively, by (7.3) and (7.4)).

Similarly, the constraint (7.5) implies that $q_i - q_j$ is a multiple of $v_0^{1/2}(B_{ij})$ for the function $v_0^{1/2}$ defined with respect to the circle B_0. Definition 5.1 and Theorem 5.4 now imply the existence of the curve γ_k', completing the proof of the claim.

7.2. **Full measure.** We begin by noting a simple fact about triangle geometry, easily deduced by applying a similarity transformation to the fixed case of $L = 1$:

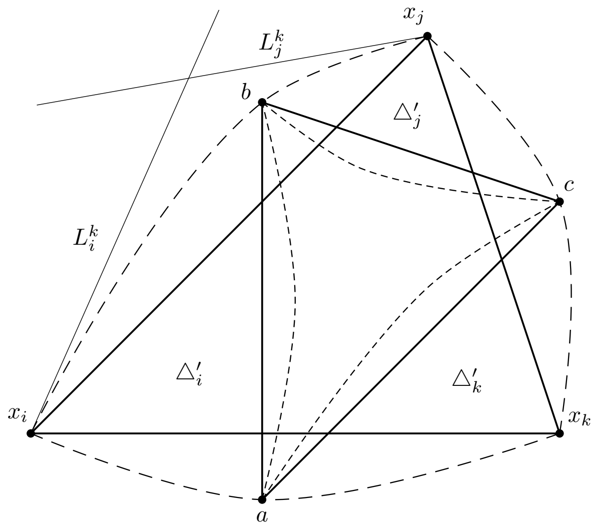

FIGURE 6. To show that $\mathcal{S}$ has full measure in Z, we show that each Apollonian triangle V_ℓ has area which is a universal positive constant fraction of the area of the region R_ℓ it subdivides. Here, the boundaries of R_ℓ and V_ℓ are shown in long- and short-dashed lines, respectively.

Proposition 7.6. *Any angle a determines constants C_a, D_a such that any triangle $\triangle$ which has an angle $\theta \geq a$ and opposite side length $\ell \leq L$ has area $A(\triangle) \leq C_a L^2$, and any triangle which has angles $\theta_1 \geq a, \theta_2 \geq a$ sharing a side of length $\ell \geq L$ has area $\geq D_a L^2$.* □

We wish to show that the interior of $\mathcal{S}$ has full measure in Z, defined as the convex closure of $\mathcal{S}$. Recall that the straight sides of each pair of incident degenerate Apollonian triangles V_i, V_j ($\{i,j\} \subset \{1,2,3\}$ intersect at right angles, so the 6 straight sides of V_1, V_2, V_3 will form a convex boundary for Z.

Letting thus $Y_t = B \setminus S_t$, we have that Y_t is a disjoint union of some open sets R_ℓ bordered by three pairwise intersecting Apollonian triangles, and X_{t+1} contains in each such region an Apollonian triangle V_ℓ dividing the region further. To prove that the interior of $\mathcal{S}$ has full measure in Z, it thus suffices to show that the area $A(V_\ell)$ is at least a universal positive constant fraction κ of the area $A(R_\ell)$ for each ℓ, giving then that $\mu(Y_t) \leq (1-\kappa)^{t-1}\mu(Y_1) \to_t 0$.

For R_ℓ bordered by Apollonian triangles V_i, V_j, V_k and letting $\triangle' = \triangle x_i x_j x_k$ be the triangle whose vertices x_s are the points of pairwise intersections V_t, V_u for each $\{s,t,u\} = \{i,j,k\}$ of the Apollonian triangles bordering R_ℓ, we will begin by noting that there is an absolute positive constant κ' such that $A(\triangle') \geq \kappa' \mu(R_\ell)$. For each $\{s,t,u\} = \{i,j,k\}$, the segment $x_s x_t$ together with the lines L_s^u and L_t^u tangent to the boundary of V_u at x_s and x_t, respectively, form a triangle $\triangle_u$ such that $R_\ell \subset \triangle' \cup \triangle_i \cup \triangle_j \cup \triangle_k$. Observations 6.5, 6.6, and 7.6 now imply that area of each $\triangle_i$ is universally bounded relative to the area of $\triangle'$, giving the existence κ' satisfying $A(\triangle') \geq \kappa' \mu(R_\ell)$.

It thus remains to show that the Apollonian triangle V_ℓ which subdivides ℓ satisfies $\mu(V_\ell) \geq \kappa'' A(\triangle')$ for some κ''. (It can in fact be shown that $\mu(V_\ell) = \frac{4}{21} A(\triangle')$ exactly, but a lower bound suffices for our purposes.) Considering the triangle $\triangle'' = \triangle abc$ whose vertices are the three vertices of V_ℓ, there are three triangular components of $\triangle'$ lying outside of $\triangle''$; denote them by $\triangle'_i, \triangle'_j, \triangle'_k$ where $\triangle'_s$ includes the vertex x_s for each $s = i, j, k$. The bound $\angle x_s x_t x_u > \frac{\pi}{4}$ for each $\{s, t, u\} = \{i, j, k\}$ together with Observation 6.5 implies there is a universal constant bounding the ratio of the area of $\triangle'_s$ to $\triangle''$ for each $s = i, j, k$. Thus we have that the area of $\triangle'$ is universally bounded by a positive constant fraction of the area of $\triangle''$, and thus via Lemma 6.4 we have that there is a universal constant κ'' such that $\mu(V_\ell) \geq \kappa'' A(\triangle')$.

Taking $\kappa = \kappa' \cdot \kappa''$ we have that $\mu(V_\ell) \geq \kappa\mu(R_\ell)$ for all ℓ, as desired, giving that the measures $\mu(Y_t)$ satisfy $\mu(Y_t) = (1-\kappa)^{t-1}\mu(Y_1) \to 0$, so that $\mu(\mathcal{S}) = \mu(Z)$.

7.3. **Proofs of two Lemmas.**

Proof of Lemma 7.4. We have

$$\varphi_i(x) = A_i x + d_i = (B^- + v_i \otimes v_i)x + d_i$$

and the agreement of $D\varphi_j, D\varphi_k$ at x_i ($\{i, j, k\} = \{1, 2, 3\}$) together with $v_i \cdot (x_j - x_k) = 0$ implies that the $D\varphi_i - B^- x$ is constant independent of i on the triangle $\triangle x_1 x_2 x_3$, giving that

$$D\varphi_i(x) = A_i x - (v_i \otimes v_i)x_j + d, \tag{7.6}$$

for a constant $d \in \mathbb{R}^2$ independent of i. Similarly, the value agreement constraints give that

$$\varphi_i(x) = \frac{1}{2}x^t A_i x - \frac{1}{2}x_j^t(v_i \otimes v_i)x_j + dx + c, \tag{7.7}$$

for a constant c independent of i. Thus, by setting $D := d$ adjusting C in the definition of u_1 as necessary, we may assume that in fact c and d are 0.

Fixing any point X_0 inside the triangle $x_1x_2x_3$, we define for each i a ray R_i emanating from X_0 coincident with the line $\{tv_i : t \in \mathbb{R}^+\}$. Our goal is now to choose X_0 such that there are points y_i on each of the rays R_i satisfying the constraints of the Lemma.

On each ray R_i, we can parametrize $\bar\varphi_i := \varphi_i(x) - \frac{1}{2}x^t B^- x$ as functions $f_i(t_i) = \frac{1}{2}a_i t^2$ ($i = 1, 2, 3$), and the function $\bar\varphi_0 := \varphi_0(x) - \frac{1}{2}x^t B^- x$ as $g_i(t_i) = \frac{1}{2}b(t + h_i)^2 + C$, where t_i is the distance from the line $\overline{x_j x_k}$ to $x \in R_i$, h_i is the distance from X_0 to the line $\overline{x_j x_k}$, and a_i and b are $\mathrm{tr}(v_i \otimes v_i)$ and $\mathrm{tr}(B)$, respectively. Moreover, since the gradients of $\bar\varphi_i$ and $\bar\varphi_0$ can both be expressed as multiple of v_i along the whole ray R_i, we have for any point x on $R_i \cap U_i$ at distance t from $\overline{x_j x_k}$ that $f'_i(t_i) = g'_i(t_i)$ implies that $D\varphi_i(x) = D\varphi_0(x)$. Thus to prove the Lemma, it suffices to show that there are X_0 and C such that for the resulting values of h_i, the systems

$$\begin{cases} f_i(t_i) = g_i(t_i) \\ f'_i(t_i) = g'_i(t_i) \end{cases} \quad \text{or, more explicitly,} \quad \begin{cases} \frac{1}{2}a_i t_i^2 = \frac{1}{2}b(t_i + h_i)^2 + C \\ a_i t_i = b(t_i + h_i) \end{cases}$$

have a solution over the real numbers for each i.

It is now easy to solve these systems in terms of C; for each i,

$$t_i = \frac{bh_i}{a_i - b} \quad \text{and} \quad h_i = \frac{\sqrt{-C}}{\sqrt{\frac{1}{2}\left(b - \frac{b^2}{a_i - b}\right)}}$$

gives the unique solution. Note that $a_i > 2b$ ensures that the denominator in the expressions for h_i (and t_i) is positive for each i. Since $\sqrt{-C}$ takes on all positive real numbers and $\mathrm{tr}(A_i) = a_i - b$, there is a (negative) value C for which the distances h_i

are the distances from the lines x_jx_k to a point X_0 inside $\triangle x_1x_2x_3$; it is the point with trilinear coordinates $\left\{\left(\frac{\mathrm{tr}(A_i)-\mathrm{tr}(B)}{\mathrm{tr}(A_i)}\right)^{-\frac{1}{2}}\right\}_{1\leq i\leq 3}$.

The Lemma is now satisfied for this choice of C and X_0 and for the points y_i on R_i at distance h_i+t_i from X_0 for $i=1,2,3$. □

Proof of Lemma 7.5. Let $q_1=p_3-p_2$, $q_2=p_1-p_3$, and $q_3=p_2-p_1$ and $A_i:=D^2\varphi_i$. Since for any individual $i=1,2,3$ we could assume without loss of generality that $\varphi_i\equiv 0$, the compatability conditions with φ_j, φ_k give

$$\begin{cases}(A_j-A_i)q_j+(A_k-A_i)q_k=0,\\ q_j^t(A_j-A_i)q_j-q_k^t(A_k-A_i)q_k=0\end{cases}\tag{7.8}$$

in each case. If we left multiply the first by q_k^t and add it to the second, we obtain

$$q_i^t(A_j-A_i)q_j=0.\tag{7.9}$$

Since $q_i\cdot q_j\neq 0$, there are unique $\alpha_{ij},\beta_{ij},\gamma_{ij}\in\mathbb{R}$ such that

$$A_j-A_i=\alpha_{ji}q_j^\perp\otimes q_j^\perp+\beta_{ji}q_i^\perp\otimes q_i^\perp+\gamma_{ij}(q_i^\perp\otimes q_j^\perp+q_j^\perp\otimes q_i^\perp),$$

where $(x,y)^\perp=(-y,x)$. Since symmetry implies $\beta_{ji}=-\alpha_{ij}$ and (7.9) implies $\gamma_{ij}=0$, we in fact have

$$A_j-A_i=\alpha_{ji}q_j^\perp\otimes q_j^\perp-\alpha_{ij}q_i^\perp\otimes q_i^\perp.$$

If we substitute this into (7.8), we obtain

$$-\alpha_{ij}(q_i^\perp\cdot q_j)q_i-\alpha_{ik}(q_i^\perp\cdot q_k)q_i=0.$$

Since $q_i^\perp\cdot q_j=-q_i^\perp\cdot q_k$, we obtain $\alpha_{ij}=\alpha_{ik}$. Thus there are $\alpha_i\in\mathbb{R}$ such that

$$A_i-A_j=\alpha_iq_i^\perp\otimes q_i^\perp-\alpha_jq_j^\perp\otimes q_j^\perp.$$

In particular, we see that $A_i-\alpha_iq_i^\perp\otimes q_i^\perp$ is constant. □

7.4. **Proof of Corollary 1.4.** This is now an easy consequence of Theorem 7.2 and the viscosity theory, via Proposition 2.7.

Proof of Corollary 1.4. Write $v=v_\infty$. Continuity of the derivative and value of v in $U_1\cup U_2\cup U_3$ imply that

$$v|_{U_i}=\frac{1}{2}x^tA_ix+Dx+C\quad\text{for}\quad i=1,2,3$$

for some $D\in\mathbb{R}^2$, $C\in\mathbb{R}$. Let β_i be the portion of the boundary of R between x_j and x_k which does not include x_i, and let v_i be the vector perpendicular to x_j-x_k such that x_i+tv_i intersects the segment x_jx_k.

We let $V_i=\beta_i+tv_i$ for $t\geq 0$. The V_i's are pairwise disjoint. Thus, by first restricting the quadratic pieces U_1,U_2,U_3 of the map v to their intersection with the respective sets V_i, and then extending the quadratic pieces to the full V_i's, we may assume that $U_i=V_i$ for each $i=1,2,3$.

We apply Lemma 7.5 to $v|_{V_1}$, $v|_{V_2}$, $v|_{V_3}$; by Observation 2.6 there are up to two possibilities for the matrix B from (7.2); the fact that $\triangle x_1x_2x_3$ is a acute, however, implies that we have that B is the successor of A_1,A_2,A_3. Thus letting S denote the Apollonian triangulation determined by x_1,x_2,x_3, Theorem 7.2 ensures the existence of a $C^{1,1}$ map u which is piecewise quadratic whose quadratic pieces have domains forming the Apollonian triangulation $\mathcal{S}$ determined by the vertices x_1,x_2,x_3. Letting U_i' denote the degenerate Apollonian triangle in $\mathcal{S}$ intersecting x_j and x_k for each

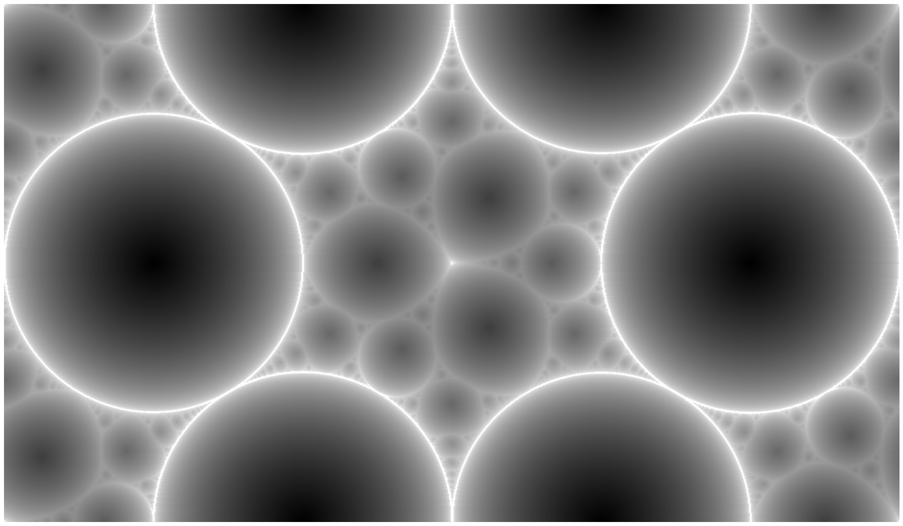

FIGURE 7. The graph of the function $c \in C(\mathbb{R}^2)$ over the rectangle $[0,6] \times [0, 6/\sqrt{3}]$, where c describes the boundary

$$\partial\Gamma_{\text{tri}} = \{\tfrac{2}{3}M(a,b,c(a,b)) : a,b \in \mathbb{R}\}$$

of Γ_{tri}. White and black correspond to $c = 3$ and $c = 4$, respectively.

$\{i,j,k\} = \{1,2,3\}$, we can extend u to a map $\bar{u}$ by extending the three degenerate pieces U_i' of $\mathcal{S}$ to sets $V_i' = \{x + tv_i : x \in U_i', t \geq 0\}$. Now we can find curves γ_i from x_j to x_k lying inside $V_i \cap V_i'$, and, letting Ω be the open region bounded by the curves $\gamma_1, \gamma_2, \gamma_3$, Proposition 2.7 implies that $\bar{u} + Dx + C$ and v are equal in Ω, as they agree on the boundary $\partial\Omega = \gamma_1 \cup \gamma_2 \cup \gamma_3$. □

8. Further Questions

Our results suggest a number of interesting questions beyond Conjecture 1.2. To highlight just a few, one direction comes from the natural extension of both the sandpile dymanics and the definition of Γ to other lattices.

Problem 1. For the triangular lattice $\mathcal{L}_{\text{tri}} \subseteq \mathbb{R}^2$ generated by $(1,0)$ and $(1/2, \sqrt{3}/2)$, we define Γ_{tri} to be the set of 2×2 real symmetric matrices A such that there exists $u : \mathcal{L}_{\text{tri}} \to \mathbb{Z}$ satisfying

$$u \geq \frac{1}{2}x^t Ax \quad \text{and} \quad \Delta^1 u \leq 5. \tag{8.1}$$

The algorithm from Section 3 can still be used in this case and we display its output in Figure 7. While the Apollonian structure of the rectangular case is missing, there does seem to be a set $\mathcal{P}_{\text{tri}}$ of isolated "peaks" such that $\bar{\Gamma}_{\text{tri}} = \mathcal{P}_{\text{tri}}^{\downarrow}$. What is the structure of these peaks? What about other lattices or graphs? To what extent does the set Γ characterize the underlying lattice/graph?

Although we have explored several aspects of the geometry of Apollonian triangulations, many natural questions remain. For example:

Problem 2. Is there a closed-form characterization of Apollonian curves?

Apollonian triangulations themsleves present some obvious targets, such as the determination of their Hausdorff dimension.

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