Model Theory Seminar

Superstable Fields and Groups

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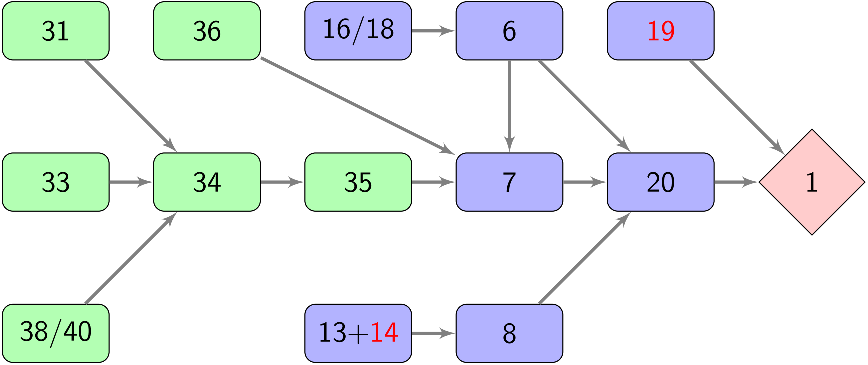
Cherlin and Shelah (1980)

Main goal: Theorem 1

Any infinite superstable field is algebraically closed.



13+14→8

Definition

$$\lambda\text{-rank}(S) = R[\phi_S, L, \lambda^+]$$

$$\infty\text{-rank}(S) = \lim_{\lambda} \lambda\text{-rank}(S)$$

Fact

T is superstable iff $R[x = x, L, (2^{|T|})^{++}] < |T|^+$

The λ-rank function is total, elementary and satisfies the λ-splitting condition: for any definable subset $S \subset |M|$ with λ-rank$(S) < \infty$, any $\{S_\alpha : \alpha < \lambda\}$ disjoint definable subsets of S, there is $\alpha < \lambda$ such that λ-rank$(S_\alpha) < \lambda$-rank(S).

13+14→8

Lemma (13)

Let H be a definable subgroup of G. Suppose G is superstable, then

$$\infty\text{-}rank(H) < \infty\text{-}rank(G) \quad \Leftrightarrow \quad [G : H] \geq \aleph_0.$$

Proof.

Cosets of H have the same ∞-rank. □

Lemma (14)

Let M be superstable, E be a definable equivalence relation on M having finite equivalence classes of bounded size, then

$$\infty\text{-}rank(M) = \infty\text{-}rank(M/E).$$

13+14→8

Lemma (13)

Let H be a definable subgroup of G. Suppose G is superstable, then

$$\infty\text{-}rank(H) < \infty\text{-}rank(G) \quad \Leftrightarrow \quad [G : H] \geq \aleph_0.$$

Proof.

Let $\lambda = (2^{|T|})^+$. Then $R[p, \Delta, \lambda^+] = R[p, \Delta, \infty]$ for any type p.
Let $\phi(x; \bar{h})$ define H in G, where $\bar{h} \in G$. For any $a \in G$, $\phi(xa^{-1}; \bar{h})$ define the coset Ha. We show that λ-rank$(H) = \lambda$-rank(Ha).

By induction, we prove λ-rank$(H) \geq \alpha$ iff λ-rank$(Ha) \geq \alpha$.

For $\alpha = 0$, λ-rank$(H) \geq 0$ iff H is nonempty iff Ha is nonempty iff λ-rank$(Ha) \geq 0$. For limit ordinal γ, λ-rank$(H) \geq \gamma$ iff λ-rank$(H) \geq \alpha$ for all $\alpha < \gamma$ iff λ-rank$(Ha) \geq \alpha$ for all $\alpha < \gamma$ by I.H. iff λ-rank$(Ha) \geq \gamma$.

13+14→8

Proof continued.

If λ-rank$(H) \geq \alpha + 1$, then there are $\{S_i : i < \lambda\}$ disjoint definable subsets of H such that λ-rank$(S_i) \geq \alpha$ for all $i < \lambda$. Let $\psi_i(x; \bar{a}_i)$ define S_i. Then $\psi_i(xa^{-1}; \bar{a}_i)$ define $S_i a$ and $\{S_i a : i < \lambda\}$ are disjoint definable subsets of Ha. By I.H., λ-rank$(S_i a) \geq \alpha$. Hence λ-rank$(Ha) \geq \alpha + 1$.

$\Rightarrow$: Suppose $[G : H] < \aleph_0$. List all the distinct cosets $\{Ha_i : i < n\}$ of H in G where $n < \omega$. Then λ-rank$(H) = \lambda$-rank(Ha_i) for all $i < n$. By the ultrametric property of λ-rank,

$$\lambda\text{-rank}(G) = \lambda\text{-rank}\Big(\bigcup_i Ha_i\Big) = \max_i \lambda\text{-rank}(Ha_i) = \lambda\text{-rank}(H),$$

$$\infty\text{-rank}(G) = \lambda\text{-rank}(G) = \lambda\text{-rank}(H) = \infty\text{-rank}(H).$$

13+14→8

Proof continued.

$\Leftarrow$: Suppose $[G : H] \geq \aleph_0$. For $n < \omega$,

$$G \vDash \exists y_1 \ldots \exists y_n \forall x \bigwedge_{1 \leq i < j \leq n} \left(\phi(xy_i^{-1}; \bar{h}) \leftrightarrow \neg\phi(xy_j^{-1}; \bar{h}) \right)$$

By compactness, there is an elementary extension G' of G such that $[G' : H'] \geq \lambda$, where H' is defined by $\phi(x; \bar{h})$ in G'. By the λ-splitting condition, there is some coset $H'a$ of H' such that

$$\lambda\text{-rank}(G') > \lambda\text{-rank}(H'a) = \lambda\text{-rank}(H') = \lambda\text{-rank}(H).$$

The last equality holds because H, H' are defined by the same formula. But then $\infty\text{-rank}(G) = \infty\text{-rank}(G') > \infty\text{-rank}(H)$. □

13+14→8

Definition

A group G is connected iff there is no proper definable subgroup of finite index.

Theorem (8)(Surjectivity Theorem)

Let G be a connected superstable group, $h : G \to G$ be a definable endomorphism. If $|\ker(h)| < \aleph_0$, then h is surjective.

Proof.

$\ker(h)$ induces a definable equivalence relation having equivalence classes of size $|\ker(h)|$. Let $H = h[G] = G/\ker(h)$.
By lemma 14, ∞-rank$(G) = \infty$-rank(H). By lemma 13, $[G : H] < \aleph_0$.
But G is connected, thus $G = H$. □

$16/18 \to 6$

Definition

Let $\mathscr{G} = \{H_\alpha\}$ be a family of definable subgroups of G.
$\mathscr{G}$ is *uniformly definable* iff there is a formula $\phi(x; \bar{y})$, and some $\bar{g_\alpha} \in G$ such that $\phi(x; \bar{g_\alpha})$ defines H_α.
G satisfies the $\mathscr{G}$-*chain condition* iff $\mathscr{G}$ does not contain an infinite decreasing chain by inclusion.
G satisfies the *stable chain condition* iff for every uniformly definable $\mathscr{G}_0$, let $\mathscr{G}$ be its closure under arbitrary intersections, then G satisfies the $\mathscr{G}$-chain condition.

Lemma (16)

If G is a stable group, then G satisfies the stable chain condition.

$16/18 \to 6$

Lemma (16)

If G is a stable group, then G satisfies the stable chain condition.

Proof.

Let $\mathscr{G}_0$ be uniformly definable by $\phi(x;\bar{y})$. We first show that G satisfies the $\mathscr{G}_0$-chain condition. Suppose there is an infinite decreasing chain $\{H_n : n < \omega\}$ with H_n defined by $\phi(x;\bar{h}_n)$. For each n pick $b_n \in H_n \setminus H_{n+1}$.

$$G \vDash \phi[b_n;\bar{h}_n] \wedge \neg\phi[b_n;\bar{h}_{n+1}] \wedge \forall x\big(\phi(x;\bar{h}_{n+1}) \to \phi(x;\bar{h}_n)\big)$$

The formula $\psi(\bar{y}_1,\bar{y}_2) \equiv \forall x\big(\phi(x;\bar{y}_1) \to \phi(x;\bar{y}_2)\big)$ has the order property, contradicting the stability of G.

$16/18 \to 6$

Proof continued.

Let $\mathscr{G}$ be closure of $\mathscr{G}_0 = \{H_\alpha\}$ under arbitrary intersections. Suppose there is an infinite decreasing chain $\{K_n : n < \omega\}$ in $\mathscr{G}$. For each $n < \omega$, write $K_n = \bigcap_{A_n} H_\alpha$ for some index set A_n. Without loss of generality, we may assume A_n is increasing. Fix $a_0 \in A_0$. Since $K_n \supsetneq K_{n+1}$, there is $a_{n+1} \in A_{n+1} \setminus A_n$ and some $b_n \in K_n \setminus H_{a_{n+1}}$. Thus we may replace A_{n+1} by $A_n \cup \{a_{n+1}\}$, and write

$$K_n = \bigcap_{0 \leq i \leq n} H_i \equiv \bigcap_{0 \leq i \leq n} H_{a_i}.$$

Each K_n is defined by finitely many formulas. To use the previous case, it suffices to show that any finite intersection of H_i reduces to an N-intersection for some $N < \omega$.

16/18→6

Proof continued.

Otherwise, for each $n < \omega$, there is a finite $I \subset \omega$ such that $|I| \geq n+1$ and for each $j \in I$, $\bigcap_{i \in I} H_i \subsetneq \bigcap_{i \in I \setminus \{j\}} H_i$. We may assume $I = n+1$ and for each $j < n+1$ pick c_j witnessing the proper inclusion, i.e.

$$c_j \in H_i \text{ for } i \neq j \text{ and } c_j \notin H_j.$$

For any $J \subset I$, $c_J = \prod_{j \in J} c_j \in H_i$ iff $i \notin J$. Let $\phi(x; \bar{h}_i)$ define H_i, then

$$i \in J \text{ iff } G \vDash \neg\phi[c_J; \bar{h}_i]$$

showing independence property, contradicting the stability of G. □

16/18→6

Lemma (16)

If G is a stable group, then G satisfies the stable chain condition.

is equivalent to

Lemma (18)

Suppose G is a stable group, $\mathscr{G}_0$ is uniformly definable in G. Then

- *G satisfies the $\mathscr{G}_0$-chain condition.*
- *There is an integer $n < \omega$ such that any arbitrary intersection in $\mathscr{G}_0$ equals to an n-intersection.*

Proof.

18→16: Let $\phi(x; \bar{y})$ define $\mathscr{G}_0$, $\mathscr{G}$ be the closure of $\mathscr{G}_0$ under intersections, then $\bigwedge_{i=1}^{n} \phi(x; \bar{y}_i)$ uniformly defines $\mathscr{G}$. □

$16/18 \to 6$

Theorem (6)

If D is an infinite stable division ring, then the additive group of D is connected.

Proof.

Let A be a definable additive subgroup of $(D, +)$ with $[D : A] < \aleph_0$, we need to show that $A = (D, +)$.
Let $\phi(x; \bar{a})$ define A for some $\bar{a} \in D$. For each $d \in D\backslash\{0\}$, dA is also definable by $\phi(d^{-1}x; \bar{a})$ and $[D : dA] < \aleph_0$. Hence $\mathscr{G}_0 = \{dA : d \in D\backslash\{0\}\}$ is uniformly definable.
Let $\mathscr{G}$ be its closure under arbitrary intersections. By lemma 18, there is $n < \omega$ such that

$$\mathscr{G} = \left\{ \bigcap_{i=1}^{n} d_i A : d_1, \ldots, .d_n \in D\backslash\{0\} \right\}.$$

$16/18 \to 6$

Theorem (6)

If D is an infinite stable division ring, then the additive group of D is connected.

Proof (continued).

In particular, $\bigcap \mathscr{G}_0 = \bigcap_{i=1}^n d_i A$ for some $d_1, \ldots, d_n \in D \backslash \{0\}$. Since

$$\left[D : \bigcap_{i=1}^n d_i A \right] \leq \prod_{i=1}^n [D : d_i A] < \aleph_0,$$

$\bigcap_{i=1}^n d_i A$ is infinite. Pick $g \in \bigcap \mathscr{G}_0 \backslash \{0\}$. For any $f \in D \backslash \{0\}$,

$$f = (fg^{-1})g \in (fg^{-1}) \cdot \bigcap \{dA : d \in D \backslash \{0\}\} = \bigcap \mathscr{G}_0.$$

Notice that $A \in \mathscr{G}_0$ so $\bigcap \mathscr{G}_0 \subset A$ and $f \in \bigcap \mathscr{G}_0 \subset A$, $f \in A$. Also, $0 \in A$, therefore $D = A$. □

$6 \leftrightarrow 7$

We have proved:

Theorem (8)(Surjectivity Theorem)

Let G be a connected superstable group, $h : G \to G$ be a definable endomorphism. If $|\ker(h)| < \aleph_0$, then h is surjective.

Theorem (6)

If D is an infinite stable division ring, then the additive group of D is connected.

In the next seminar, we will prove the equivalence of Theorem 6 and Theorem 7.

Theorem (7)

If D is an infinite stable division ring, then the multiplicative group of D is connected.

6+7+8→20

Lemma (19)

Let F be a field of characteristic p, and K be a Galois extension of F, with $[K : F] = q$ prime and $x^q - 1$ splits in F.
(Artin-Schreier extension) If $p = q$, then K is generated over F together with a solution of $x^p - x = a$.
(Kummer extension) If $p \neq q$, then K is generated over F together with a solution of $x^q = a$.

Lemma (20)

A superstable field F is perfect and has no Artin-Schreier/Kummer extension.

6+7+8→20

Lemma (20)

A superstable field F is perfect and has no Artin-Schreier/Kummer extension.

Proof.

Let p be the characteristic of F. Consider the following maps:

$$h : x \mapsto x^p - x \qquad p \neq 0$$
$$k : x \mapsto x^q \qquad x \neq 0, q \geq 1$$

h and k are definable endomorphisms of $(F, +)$ and $(F, \cdot)$ respectively.
Their kernels are finite.
By Theorems 6 and 7, $(F, +)$ and $(F, \cdot)$ are connected.
By Theorem 8, both h and k are surjective. □

19+20→1

Theorem (1)

Any infinite superstable field is algebraically closed.

Proof.

Suppose there exists an infinite superstable field F_0 that is not algebraically closed. We say $P(K, F)$ whenever K is a Galois extension of F of finite degree greater than 1, F is infinite and superstable.
By assumption, P is nonempty and we pick a pair $(K, F) \in P$ of minimal degree q. We show that q is prime and $x^q - 1$ splits in F.

If q is not prime, pick a proper prime factor r of q. Let F_1 be the fixed field of an element of order r in $\mathrm{Gal}(K/F)$. Then F_1 is superstable, $P(K, F_1)$ and $[K : F_1] < [K : F]$. If $x^q - 1$ does not split in F, then the splitting extension of $x^q - 1$ over F has degree $q - 1 < q$.
By Lemma 19, K is an Artin-Schreier/Kummer extension of F. But F is superstable so it contradicts Lemma 20. □

$35+36 \to (6 \leftrightarrow 7)$

We will complete the proof of Theorem 1 by establishing the equivalence of

Theorem (6)

If D is an infinite stable division ring, then the additive group of D is connected.

and

Theorem (7)

If D is an infinite stable division ring, then the multiplicative group of D is connected.

Δ-rank

Let $M \models T$, $\Delta \subset \mathrm{Fml}(L(T))$. Denote $\Delta(M)$ to be the boolean algebra of subsets of M definable by $\phi(x; \bar{a})$ for some $\phi(x; \bar{y}) \in \Delta$, $\bar{a} \in M$.

Definition

Let $S \in \Delta(M)$, $\mathscr{S} = \{S_\alpha\}$ be an infinite family of subsets of S. $\mathscr{S}$ Δ-splits S iff

- S_α are pairwise disjoint, and
- $S_\alpha = S \cap D_\alpha$ for some $D_\alpha \in \Delta(M)$, for all α.

Definition

$$\Delta\text{-rank}(S) = R[S, \Delta, \aleph_0]$$

Δ-rank is the least elementary rank function with the Δ-splitting condition: for any $S \in \Delta(M)$, Δ-rank$(S) < \infty$ and $\mathscr{S} = \{S_\alpha\}$ Δ-splits S then there is some α such that Δ-rank$(S_\alpha) < \Delta$-rank(S).

Δ-rank

In the following, we assume $\Delta\text{-rank}(M) < \infty$. Let $S, X, Y \in \Delta(M)$.

Definition

$$S \text{ is } \Delta\text{-small iff } \Delta\text{-rank}(S) < \Delta\text{-rank}(M).$$

$$X \equiv_\Delta Y \text{ iff } X \triangle Y \text{ is } \Delta\text{-small.}$$

Fact

T is stable iff for every finite $\Delta \subset \mathrm{Fml}(L(T))$, Δ-rank is total.
Let $S_1, S_2 \in \Delta(M)$.

$$\Delta\text{-rank}(S_1 \cup S_2) = \max\left(\Delta\text{-rank}(S_1), \Delta\text{-rank}(S_2)\right)$$

Δ-rank

Let $I = \{S \in \Delta(M) : S \text{ is } \Delta\text{-small}\}$.

- I is an ideal of $\Delta(M)$ and $\Delta(M)/I$ is a finite Boolean algebra.
- We call the number of atoms in $\Delta(M)/I$ the Δ-*multiplicity* of M.
- Let M have Δ-multiplicity $m < \omega$. There are disjoint $\{M_i : 1 \leq i \leq m\} \subset \Delta(M)$ such that $\Delta\text{-rank}(M_i) = \Delta\text{-rank}(M)$, $M = \bigcup_{i=1}^{m} M_i$ and the M_i are unique up to $\equiv_\Delta$.
- For any $S \in \Delta(M)$, there is a unique $I \subset \{1, \ldots, m\}$ such that $S \equiv_\Delta \bigcup_{i \in I} M_i$. We call $|I|$ the Δ-*multiplicity* of S.
- $S \in \Delta(M)$ is Δ-*indecomposable* iff S has Δ-multiplicity is 1.

31+33→34

Definition

Let $T \supset T_{\text{groups}}$, $\Delta \subset \mathrm{Fml}(L(T))$.
Δ is *right-invariant* iff $\forall \phi(x; \bar{y}) \in \Delta$, $\forall G \models T$, $\forall \bar{a}, g \in G$

$$\phi(xg; \bar{a}) \text{ is } G\text{-equivalent to an instance of a formula in } \Delta.$$

Δ-rank is *right-invariant* iff for any $S \in \Delta(G)$ with $\Delta\text{-rank}(S) < \infty$, any $g \in G$, $\Delta\text{-rank}(Sg) = \Delta\text{-rank}(S)$.
Similarly for left invariance and (bi-)invariance.

Lemma (31)

Let $T \supset T_{groups}$, $\Delta \subset \mathrm{Fml}(L(T))$.
If Δ is invariant, then Δ-rank is invariant.

31+33→34

Lemma (33)

Let $T \supset T_{groups}$, $\Delta \subset \mathrm{Fml}(L(T))$. *For* $\phi(x;\bar{y}) \in \Delta$, *let*

$$\tilde{\phi}(x;\bar{y},z_1,z_2) \equiv \phi(z_1\,x\,z_2;\bar{y})$$

Then $\tilde{\Delta} = \{\phi, \tilde{\phi} : \phi \in \Delta\}$ *is invariant.*

Theorem (34)(The Indecomposability Theorem)

Let G *be a stable group. The following are equivalent:*

1. G *is connected.*
2. G *is* Δ*-indecomposable for any finite invariant* Δ.
3. *For any finite* Δ_0, *there is a finite* $\Delta \supset \Delta_0$ *such that* Δ *is invariant and* G *is* Δ*-indecomposable.*

31+33→34

Theorem (34)(The Indecomposability Theorem)

1. *G is connected.*
2. *G is Δ-indecomposable for any finite invariant Δ.*
3. *For any finite Δ_0, there is a finite $\Delta \supset \Delta_0$ such that Δ is invariant and G is Δ-indecomposable.*

Proof.

We will prove (2)⇒(3)⇒(1) and leave (1)⇒(2) for later.
(2)⇒(3): Given any finite Δ_0, by Lemma 33 there is an invariant $\Delta \supset \Delta_0$. $|\Delta| \leq 2|\Delta_0| < \aleph_0$. By assumption (2), G is Δ-indecomposable.
(3)⇒(1): Let $H \leq G$ of finite index be definable by $\phi(x; \bar{a})$ for some $\bar{a} \in G$. Set $\Delta_0 = \{\phi(x; \bar{a})\}$ to obtain Δ. Since Δ is invariant, by Lemma 31, Δ-rank is invariant. So all cosets of H have the same Δ-rank, which must be the same as Δ-rank(G). By indecomposability, $[G : H] = 1$. □

38/40→34

Lemma (40)

Let G be a group and $K \leq G$ of finite index. Suppose G is κ^+-saturated and K is the intersection of κ-many definable subsets of G, then K is definable in G.

Proof.

Let $k = [G : K]$ and $g_1, \ldots, g_k \in G$ be such that $G = \bigcup_{i=1}^{k} Kg_i$. Assume $k > 1$ and $g_1 = 1$. Let $K = \bigcap_{\alpha<\kappa} S_\alpha$ with $S_\alpha \in \Delta(G)$ for $\alpha < \kappa$. Assume $\{S_\alpha : \alpha < \kappa\}$ is closed under finite intersections.
Fix $i \in [2, k]$. Consider the following type $p(x)$ with κ constants:

$$x \in K \cap Kg_i = \bigcap_{\alpha<\kappa} \left(S_\alpha \cap S_\alpha g_i\right)$$

Since G is κ^+-saturated and does not realize p, p is inconsistent.

$38/40 \to 34$

Proof continued.

By compactness, there is $\alpha_i < \kappa$ such that $S_{\alpha_i} \cap S_{\alpha_i} g_i = \emptyset$.
Let $S = \bigcap_{i=2}^k S_{\alpha_i}$. Since

$$S \cap \bigcup_{i=2}^{k} Kg_i \subset S \cap \bigcup_{i=2}^{k} Sg_i = \bigcup_{i=2}^{k} (S \cap Sg_i) = \emptyset,$$

$S \subset K$ and $K = S \in \{S_\alpha : \alpha < \kappa\}$, hence K is definable. □

Let G be a stable group and Δ be a finite invariant set of formulas. Then Δ-rank$(G) < \omega$ and we can decompose $G = \bigcup_{i=1}^m A_i$ for some disjoint indecomposable $A_i \in \Delta(G), 1 \leq i \leq m$.
By uniqueness (up to $\equiv_\Delta$), the right multiplication by $g \in G$ induces a permutation ρ_g of the indices i. Thus we can define a group homomorphism $\rho : g \mapsto \rho_g$. Let $K = \ker(\rho)$.

38/40→34

Corollary (38)

If G is $\aleph_1$-saturated, then K is a definable subgroup of G.

Proof.

Observe that for any $g \in G$,
$g \in K \Leftrightarrow A_i g \equiv_\Delta A_i$ for all $i \Leftrightarrow \Delta$-rank$(A_i g \cap A_i) = \Delta$-rank$(G)$ for all i.
Since Δ-rank$(G) < \omega$, Δ-rank$(G) = n$ for some $n < \omega$.
For each i, Δ-rank$(A_i g \cap A_i) = n$ is equivalent to the consistency of some countable theory, so is $g \in K$. By compactness, it suffices to check countably many finite subtheories. Hence K is a countable intersection of definable subsets of G.
Since $G/K \simeq \rho[G]$ is finite, $[G : K] < \aleph_0$. Also, G is $\aleph_1$-saturated by assumption. Therefore, by Lemma 40, K is definable in G. □

$38/40 \to 34$

Theorem (34)(The Indecomposability Theorem)

Let G be a stable group. The following are equivalent:

1. *G is connected.*
2. *G is Δ-indecomposable for any finite invariant Δ.*
3. *For any finite Δ_0, there is a finite $\Delta \supset \Delta_0$ such that Δ is invariant and G is Δ-indecomposable.*

Proof.

We have proved (2)⇒(3)⇒(1). Now we proceed to prove (1)⇒(2). Assume G is connected, we can further assume that G is $\aleph_1$-saturated. By Corollary 38, K is a definable subgroup of G, so $K = G$ by connectedness. For all $g \in G$, $i \in [1, m]$, $A_i g \equiv_\Delta A_i$. For any finite $F \subset G$, $\bigcap_{g \in F} A_i g \equiv_\Delta A_i$ so $\bigcap_{g \in F} A_i g \neq \emptyset$. By compactness, the theory $T^* = CD(G) \cup \{cc_g \in A_1 : g \in G\}$ is consistent, with a new constant c.

38/40→34

Theorem (34)(The Indecomposability Theorem)

Let G be a stable group. The following are equivalent:

1. *G is connected.*
2. *G is Δ-indecomposable for any finite invariant Δ.*
3. *For any finite Δ_0, there is a finite $\Delta \supset \Delta_0$ such that Δ is invariant and G is Δ-indecomposable.*

Proof continued.

Let $G^* \vDash T^*$ with $G \leq G^* \restriction L(T)$, $a^* = c^{G^*}$ and A_i^* defined in G^* by the same formula as A_i. Then $a^* G \subset A_1^*$.
$a^* A_i \subset A_1^* \cap a^* A_i^* \subset A_1^*$. Since $a^* A_i^*$ are disjoint, so are $a^* A_i$.
$\Delta\text{-rank}(a^* A_i) = \Delta\text{-rank}(A_i)$ because Δ is invariant (Lemma 31).
Also, $\Delta\text{-rank}(A_i) = \Delta\text{-rank}(A_i^*)$ by elementarity $= \Delta\text{-rank}(A_1^*)$.
However A_1^* is Δ-indecomposable, so $i = 1$. □

34→35

Theorem (34)(The Indecomposability Theorem)

Let G be a stable group. The following are equivalent:

1. *G is connected.*
2. *G is Δ-indecomposable for any finite invariant Δ.*
3. *For any finite Δ_0, there is a finite $\Delta \supset \Delta_0$ such that Δ is invariant and G is Δ-indecomposable.*

Theorem (35)

Let $\cdot$ and $+$ be two binary operations of a stable structure M; X, Y be definable in M such that

- *$(M \backslash X, +)$ and $(M \backslash Y, \cdot)$ are groups;*
- *For every finite Δ_0, there is $\Delta \supset \Delta_0$ finite and invariant with respect to both $\cdot$ and $+$ such that X, Y are Δ-small.*

Then $(M \backslash X, +)$ is connected iff $(M \backslash Y, \cdot)$ is connected.

34→35

Proof.

We establish the following equivalences:

a. $(M \backslash X, +)$ is connected.
b. For any finite Δ_0, there is a finite $(\cdot, +)$ -invariant $\Delta \supset \Delta_0$ such that $M \backslash X$ and $M \backslash Y$ are both Δ-indecomposable.
c. $(M \backslash Y, \cdot)$ is connected.

(b)⇒(a),(c): $(M \backslash X, +)$ and $(M \backslash Y, \cdot)$ are stable groups, so we can directly use Theorem 34(3)⇒(1).
(a)⇒(b): For any finite Δ_0, by the second assumption of the theorem, there is a finite $(\cdot, +)$-invariant $\Delta \supset \Delta_0$ such that X, Y are Δ-small. By Theorem 34(1)⇒(2), $(M \backslash X, +)$ is Δ-indecomposable. As X, Y are small, $(M \backslash Y, \cdot)$ is also Δ-indecomposable. (c)⇒ (b) is similar. □

$35+36 \to (6 \leftrightarrow 7)$

Lemma (36)

Let $T \supset T_{rings}$, Δ *be a finite subset of* $\mathrm{Fml}(L(T))$. *For any* $\phi(x;\bar{y}) \in \Delta$, *define*

$$\tilde{\phi}(x;\bar{y},z_1,z_2,z_3) = \phi(z_1\,x\,z_2 + z_3;\bar{y})$$

Then $\tilde{\Delta} = \{\phi, \tilde{\phi} : \phi \in \Delta\} \supset \Delta$ *is finite and invariant with respect to both* $\cdot$ *and* $+$.

Theorem ($6 \leftrightarrow 7$)

If D is an infinite stable division ring, then the additive group of D is connected iff the multiplicative group of D is connected.

Proof.

Take $X = \emptyset$ and $Y = \{0\}$ in Theorem 35. The first assumption is satisfied. The second assumption follows from Lemma 36 and the fact that X, Y are finite, hence Δ-small. □

Cherlin and Shelah (1980)

Main goal: Theorem 1

Any infinite superstable field is algebraically closed.