Axiomatizing AECs and Applications

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Overview

1. Main results
2. Abstract elementary classes (AECs)
3. Proof idea
4. Other results
5. Open questions

Main results

Theorem (Shelah)

Let $\mathbf{K}$ *be an AEC and* $\lambda = \mathrm{LS}(\mathbf{K})$*. Then* K *is* $PC_{\lambda, 2^\lambda}$*.*

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2. $|T| \leq \lambda$;
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Theorem (4.1,4.10)

Let $\mathbf{K}$ *be an AEC and* $\lambda = \mathrm{LS}(\mathbf{K})$*. Then there is* χ *depending on* $\mathbf{K}$ *such that* $\lambda \leq \chi \leq 2^\lambda$ *and* K *is* $PC_{\chi,\chi}$*.*

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Theorem (4.1,4.10)

Let $\mathbf{K}$ be an AEC and $\lambda = \mathrm{LS}(\mathbf{K})$. Then there is χ depending on $\mathbf{K}$ such that $\lambda \leq \chi \leq 2^\lambda$ and K is $PC_{\chi,\chi}$.
Moreover, under $2^\lambda < 2^{\lambda^+}$, if $\mathbf{K}$ is categorical in λ, λ^+ and stable in λ, then $\mathbf{K}$ is $PC_{\lambda,\lambda}$.

Fact (Shelah)

Let **K** be an AEC, $\theta \geq \text{LS}(\mathbf{K})$. Suppose the following hold:

1. $K, K^{<}$ are both $PC_{\theta,\theta}$;
2. **K** is categorical in both θ and θ^{+};
3. $\delta(\theta, 1) = \theta^{+}$. (Threshold cardinal for an infinite decreasing chain to exist in a $PC_{\theta,1}$-class.)

Then $K_{\theta^{++}} \neq \emptyset$.

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Corollary (4.8)

(1) is true for $\theta \geq \chi$. Moreover, under $2^{\lambda} < 2^{\lambda^{+}}$ and stability in λ, (2) already implies (1) for $\theta = \lambda$.

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Definition

Let L be a finitary language. An abstract elementary class $\mathbf{K} = \langle K, \leq_{\mathbf{K}} \rangle$ in $L = \mathrm{L}(\mathbf{K})$ satisfies the following axioms:

1. K is a class of L-structures and $\leq_{\mathbf{K}}$ is a partial order on $K \times K$.
2. For $M_1, M_2 \in K$, $M_1 \leq_{\mathbf{K}} M_2$ implies $M_1 \subseteq M_2$ (as L-substructures).

Abstract elementary classes (AECs)

Definition (Continued)

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   a. If $M \in K$, N is an L-structure, $M \cong N$, then $N \in K$.
   b. Let $M_1, M_2, N_1, N_2 \in K$. If $f : M_1 \cong M_2$, $g : N_1 \cong N_2$, $g \supseteq f$ and $M_1 \leq_{\mathbf{K}} N_1$, then $M_2 \leq_{\mathbf{K}} N_2$.

$$\begin{array}{ccc} N_1 & \xrightarrow{g} & N_2 \\ \leq_{\mathbf{K}} \uparrow & & \leq_{\mathbf{K}} \uparrow \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

Abstract elementary classes (AECs)

Definition (Continued)

4. Coherence: Let $M_1, M_2, M_3 \in K$. If $M_1 \leq_{\mathbf{K}} M_3$, $M_2 \leq_{\mathbf{K}} M_3$ and $M_1 \subseteq M_2$, then $M_1 \leq_{\mathbf{K}} M_2$.

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5. Löwenheim-Skolem axiom: There exists an infinite cardinal $\lambda \geq |\,\mathrm{L}(\mathbf{K})|$ such that: for any $M \in K$, $A \subseteq |M|$, there is some $N \in K$ with $A \subseteq |N|$, $N \leq_{\mathbf{K}} M$ and $\|N\| \leq \lambda + |A|$. We call the minimum such λ the Löwenheim-Skolem number $\mathrm{LS}(\mathbf{K})$.

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6. Chain axioms: Let α be an ordinal and $\langle M_i : i < \alpha \rangle \subseteq K$ such that for $i < j < \alpha$, $M_i \leq_{\mathbf{K}} M_j$.
   1. Then $M = \bigcup_{i<\alpha} M_i$ is in K and for all $i < \alpha$, $M_i \leq_{\mathbf{K}} M$.
   2. Let $N \in K$. If in addition for all $i < \alpha$, $M_i \leq_{\mathbf{K}} N$, then $M \leq_{\mathbf{K}} N$.

Abstract elementary classes (AECs)

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$$I_2(\lambda, \mathbf{K}) = |\{(M, N)/_{\cong} : M \leq_{\mathbf{K}} N \text{ both in } K_\lambda\}|$$

where $(M_1, N_1) \cong (M_2, N_2)$ iff $M_1 \leq_{\mathbf{K}} N_1$, $M_2 \leq_{\mathbf{K}} N_2$ and there is $g : N_1 \cong N_2$ such that $g \restriction M_1 : M_1 \cong M_2$.

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Fact

$I_2(\lambda, \mathbf{K}) \leq 2^\lambda$.

In the main results, we had $\chi = \lambda + I_2(\lambda, \mathbf{K})$, and bypassed I_2 under more assumptions.

Proof idea

Definition

Let I be an index set. A directed system $\langle M_i : i \in I \rangle \subseteq K$ indexed by I satisfies the following: for any $i, j \in I$, there is $k \in I$ such that $M_i \leq_{\mathbf{K}} M_k$ and $M_j \leq_{\mathbf{K}} M_k$.

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Given $M \in K$, we index the system by finite tuples of elements in M. Namely, $I = |M|^{<\omega}$ (ordered by inclusion).

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Given $M \in K$, we index the system by finite tuples of elements in M. Namely, $I = |M|^{<\omega}$ (ordered by inclusion). Conversely,

Fact

Let $\langle M_i : i \in I \rangle \subseteq K$ be a directed system. Then

1. $M = \bigcup_{i \in I} M_i \in K$;
2. For all $i \in I$, $M_i \leq_{\mathbf{K}} M$;
3. Let $N \in K$. If in addition for all $i \in I$, $M_i \leq_{\mathbf{K}} N$, then $M \leq_{\mathbf{K}} N$.

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Fact

Let $\mathbf{K}$ and $\mathbf{K}'$ be two AECs with $L(\mathbf{K}) = L(\mathbf{K}')$. If there exists a cardinal λ such that

1. $\lambda \geq LS(\mathbf{K}) + LS(\mathbf{K}')$;
2. $\mathbf{K}_\lambda = \mathbf{K}'_\lambda$ (both the models and the ordering),

then $\mathbf{K}_{\geq\lambda} = \mathbf{K}'_{\geq\lambda}$.

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Hence, given an AEC $\mathbf{K}$, it suffices to encode the models and the ordering in $\mathbf{K}_{LS(\mathbf{K})}$.

Proof idea

Let M be an L($\mathbf{K}$)-structure, $\lambda = \mathrm{LS}(\mathbf{K})$.

1. Expand L($\mathbf{K}$) by adding λ-many functions. They map a finite tuple to a $\mathbf{K}$-structure containing it:

$$a \in |M|^{<\omega} \mapsto \{f_i(a) : i < \lambda\} = |M_a| \text{ with } M_a \in K.$$

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2. Stipulate that $\langle M_a : a \in |M|^{<\omega} \rangle$ forms a directed system:
If $a \cup b \subseteq c$ then $M_a \leq_{\mathbf{K}} M_c$ and $M_b \leq_{\mathbf{K}} M_c$.

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   If $a \cup b \subseteq c$ then $M_a \leq_{\mathbf{K}} M_c$ and $M_b \leq_{\mathbf{K}} M_c$.
   - "$\leq_{\mathbf{K}}$" is by listing all the isomorphism types of pairs $N_0 \leq_{\mathbf{K}} N_1$ in K_λ, there are $I_2(\lambda, \mathbf{K})$-many.

Proof idea

We have axiomatized **K** in an $L'_{\chi^+,\omega}$-sentence σ where L' contains extra functions than L(**K**) and $\chi = \lambda + I_2(\lambda, \mathbf{K})$. To convert this to a PC-class, we use the following fact:

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Chang's presentation theorem

Let θ be an infinite cardinal, L be a language of size $\leq \theta$, T be an $L_{\theta^+,\omega}$-theory contained in a fragment of size $\leq \theta$, then the models of T is a $PC_{\theta,\theta}$-class.

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Taking $T = \{\sigma\}$, we obtain **K** as a $PC_{\chi,\chi}$-class.

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Theorem

Let $\mathbf{K}$ *be an AEC and* $\lambda = \mathrm{LS}(\mathbf{K})$*. Under* $2^\lambda < 2^{\lambda^+}$*, if* $\mathbf{K}$ *is categorical in* λ, λ^+ *and stable in* λ*, then* $\mathbf{K}$ *is* $PC_{\lambda,\lambda}$*.*

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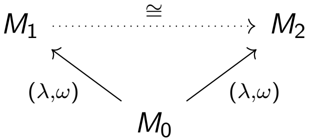
By categoricity in λ, we avoid listing (individual) models indexed by $I(\lambda, \mathbf{K})$. The assumptions also imply that we can build limit models that are isomorphic over the same base:



(In other words, if $I(\lambda, \mathbf{K}) = 1$ and $\leq_{\mathbf{K}}$ is replaced by "(λ, ω)-limit", then $I_2(\lambda, \mathbf{K}) = 1$!)

Proof idea

Instead of encoding $M_a \leq_{\mathbf{K}} M_b$, we use coherence axiom and encode $M_a \subseteq M_b$ and the existence of a common limit model over them.

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$$\begin{array}{c} M_b \\ \leq_{\mathbf{K}} \big\uparrow \\ M_a \end{array} \quad \text{is the same as} \quad \begin{array}{ccc} M_b & \xrightarrow{(\lambda,\omega)} & M^* \\ \subseteq \big\uparrow & \nearrow_{(\lambda,\omega)} & \\ M_a & & \end{array}$$

Other results

One can also axiomatize an AEC $\mathbf{K}$ within the same language $\mathrm{L}(\mathbf{K})$.

Fact (Shelah-Villaveces)

Let $\mathbf{K}$ be an AEC, $L = \mathrm{L}(\mathbf{K})$ and $\lambda = \mathrm{LS}(\mathbf{K})$. Then $\mathbf{K}$ can be axiomatized by a sentence in $L_{(2^{2^{\lambda^+}})^{+++}}$.

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Theorem (3.7)

Let $\mathbf{K}$ be an AEC, $L = \mathrm{L}(\mathbf{K})$, $\lambda = \mathrm{LS}(\mathbf{K})$ and $\chi = \lambda + I_2(\lambda, \mathbf{K})$. Then $\mathbf{K}$ can be axiomatized by a sentence in $L_{\chi^+,\lambda^+}(\omega \cdot \omega)$ (game quantification of $\omega \cdot \omega$ steps).

Other results

Let $\mu \geq \aleph_0$. Define PC^{μ} as before except that the underlying languages L, L' are $(< \mu)$-ary.

K is	**K** is axiomatizable in	**K** is
An AEC	$L_{\chi^+,\lambda^+}(\omega \cdot \omega)$	$PC_{\chi,\chi}$
A μ-AEC	$L_{\chi^+,\lambda^+}(\mu \cdot \mu)$	$PC^{\mu}_{\chi,\chi}$

Open questions

Question

Let $\mathbf{K}$ be an AEC and $\lambda = \mathrm{LS}(\mathbf{K})$.

- Under extra assumptions (categoricity, stability, etc), is it possible to bound $I_2(\lambda, \mathbf{K})$ strictly below 2^λ ?
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  - This might remove the game quantification in our axiomatization.
- Does the Hanf number exist for μ-AECs?
  - Hanf number is the threshold cardinal for arbitrarily large models. In AECs, the Hanf number is $\beth_{(2^\lambda)^+}$.