First Stability Cardinals of AECs

2023 North American Annual Meeting

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March 28, 2023

Overview

Main results

Fact

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Open question

Can we lower the bound of (2) to $2^{\mathrm{LS}(\mathbf{K})}$? Or are there counterexamples?

Main results

Theorem (Proposition 4.1)

Let λ be an infinite cardinal and α be an ordinal with $\lambda \leq \alpha < (2^\lambda)^+$. Then there is a stable AEC $\mathbf{K}$ such that $\mathrm{LS}(\mathbf{K}) = \lambda$ and its first stability cardinal is $\beth_\alpha(\lambda)$. Moreover, $\mathbf{K}$ is tame but fails AP.

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First stability cardinal	Tame+AP	Tame+$(\neg AP)$
Upper bound	$\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ (Vasey)	? (Open)
Can go up to	? (Open)	$\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ (4.1)

Abstract elementary classes (AECs)

Shelah developed an axiomatic framework to contain certain classes of models, including models of first-order theories.

Definition

Let L be a finitary language. An abstract elementary class $\mathbf{K} = \langle K, \leq_{\mathbf{K}} \rangle$ in $L = \mathrm{L}(\mathbf{K})$ satisfies the following axioms:

1. K is a class of L-structures and $\leq_{\mathbf{K}}$ is a partial order on K.
2. For $M_1, M_2 \in K$, $M_1 \leq_{\mathbf{K}} M_2$ implies $M_1 \subseteq M_2$ (as L-substructures).

Abstract elementary classes (AECs)

Definition (Continued)

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   a. If $M \in K$, N is an L-structure, $M \cong N$, then $N \in K$.
   b. Let $M_1, M_2, N_1, N_2 \in K$. If $f : M_1 \cong M_2$, $g : N_1 \cong N_2$, $g \supseteq f$ and $M_1 \leq_{\mathbf{K}} N_1$, then $M_2 \leq_{\mathbf{K}} N_2$.

$$\begin{array}{ccc} N_1 & \xrightarrow{g} & N_2 \\ \leq_{\mathbf{K}} \uparrow & & \leq_{\mathbf{K}} \uparrow \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

Abstract elementary classes (AECs)

Definition (Continued)

4. Coherence: Let $M_1, M_2, M_3 \in K$. If $M_1 \leq_{\mathbf{K}} M_3$, $M_2 \leq_{\mathbf{K}} M_3$ and $M_1 \subseteq M_2$, then $M_1 \leq_{\mathbf{K}} M_2$.

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5. Löwenheim-Skolem axiom: There exists an infinite cardinal $\lambda \geq |\mathrm{L}(\mathbf{K})|$ such that: for any $M \in K$, $A \subseteq |M|$, there is some $N \in K$ with $A \subseteq |N|$, $N \leq_{\mathbf{K}} M$ and $\|N\| \leq \lambda + |A|$. We call the minimum such λ the Löwenheim-Skolem number $\mathrm{LS}(\mathbf{K})$.

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6. Chain axioms: Let α be an ordinal and $\langle M_i : i < \alpha \rangle \subseteq K$ such that for $i < j < \alpha$, $M_i \leq_{\mathbf{K}} M_j$.
   1. Then $M = \bigcup_{i<\alpha} M_i$ is in K and for all $i < \alpha$, $M_i \leq_{\mathbf{K}} M$.
   2. Let $N \in K$. If in addition for all $i < \alpha$, $M_i \leq_{\mathbf{K}} N$, then $M \leq_{\mathbf{K}} N$.

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Definition

$\mathbf{K}$ has the *amalgamation property* (AP) if for any $M_0, M_1, M_2 \in K$ with $M_0 \leq_{\mathbf{K}} M_1$ and $M_0 \leq_{\mathbf{K}} M_2$, then there exist $M_3 \in K$, $f_1 : M_1 \underset{M_0}{\longrightarrow} M_3$ and $f_2 : M_2 \underset{M_0}{\longrightarrow} M_3$ such that $M_2 \leq_{\mathbf{K}} M_3$.

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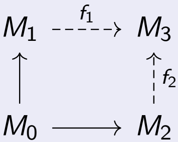

Abstract elementary classes (AECs)

Definition (Galois types)

Let $a_i \in N_i$ and $M_i \leq_{\mathbf{K}} N_i$ for $i = 1, 2$. We define $(a_1, M_1, N_1) \sim (a_2, M_2, N_2)$ when $M_1 = M_2$ and there are $N \in K$, $f_i : N_i \xrightarrow[M_1]{} N$ such that $f_1(a_1) = f_2(a_2)$.

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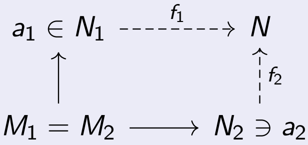

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$$
\begin{array}{ccc}
a_1 \in N_1 & \overset{f_1}{\dashrightarrow} & N \\
\uparrow & & \uparrow f_2 \\
M_1 = M_2 & \longrightarrow & N_2 \ni a_2
\end{array}
$$

Take the transitive closure of $\sim$ to $\equiv$. We define $\mathrm{gtp}(a_1/M_1; N_1) = (a_1, M_1, N_1)/\equiv$. The *Galois types over M* is written as $\mathrm{gS}(M) = \{(a, M, N)/\equiv :\ a \in N, M \leq_{\mathbf{K}} N\}$.

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Definition (Tameness)

- Let $p = \mathrm{gtp}(a/M; N)$, $M_0 \leq M$ and $a \in N$. $p \restriction M_0 = \mathrm{gtp}(a/M_0; N)$.

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- Let κ be a cardinal. $\mathbf{K}$ is *κ-tame* if for any Galois types $p \neq q$ both in $\mathrm{gS}(M)$, there is $M_0 \leq M$, $\|M_0\| \leq \kappa$ such that $p \restriction M_0 \neq q \restriction M_0$.

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First-order theories are $(< \aleph_0)$-tame!

Proof idea

Theorem (Proposition 4.1)

Let λ be an infinite cardinal and α be an ordinal with $\lambda \leq \alpha < (2^\lambda)^+$. Then there is a stable AEC $\mathbf{K}$ such that $\mathrm{LS}(\mathbf{K}) = \lambda$ and its first stability cardinal is $\beth_\alpha(\lambda)$. Moreover, $\mathbf{K}$ is tame but fails AP.

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Idea of the construction:

- Encode $\alpha < (2^\lambda)^+$ with $\mathrm{LS}(\mathbf{K}) = \lambda$;
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- Check instability below $\beth_\alpha(\lambda)$ and stability at $\beth_\alpha(\lambda)$.

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    $\implies$ This ruins *AP*!

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1. Refine our examples (e.g. change the substructure relation);
2. Lower the bound of the first stability cardinal below $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$:
   1. Find a substitute of Galois types?
   2. Investigate the notion of "order property".

Definition

Let μ be a cardinal. $\mathbf{K}$ has the *order property of length* μ if there exist $\langle a_i : i < \mu \rangle$, $M \leq_{\mathbf{K}} N$ such that for $i_0 < i_1$ and $j_0 < j_1$, we have $\mathrm{gtp}(a_{i_0} a_{i_1} / M; N) \neq \mathrm{gtp}(a_{j_1} a_{j_0} / M; N)$.

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