STABILITY, CATEGORICITY AND AXIOMATIZATION OF ABSTRACT ELEMENTARY CLASSES

by

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ABSTRACT

We extend current results on abstract elementary classes (AECs) in terms of stability, categoricity and axiomatization theorems. In most cases, we assume the existence of a monster model as well as tameness. The first two chapters introduce readers to the major questions and basic notions of AECs. The subsequent four chapters are the author's papers written during his PhD program, each with its own abstract, introduction and results. The last chapter uses known results to derive a new categoricity transfer.

DEDICATION

I dedicate this dissertation to Niloofar A., Razieh F., Reyhaneh P. and my students.

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CONTENTS

CHAPTER 1
INTRODUCTION

This dissertation investigates the stability theory, categoricity transfer and presentation theorems of abstract elementary classes, building mainly on the work of Shelah, Grossberg, VanDieren, Boney and Vasey. In this chapter, we go through the motivation, major open questions and the main results in the author's papers.

Model theory studies classes of mathematical structures that have the same underlying forms, which include the same logic (first-order or higher-order logics), the same language (vocabulary/similiarity types) and the same theory. For example, in first-order theories, we work in first-order logic and study the models of some first-order theory T. Since model theory does not assume a specific form (say group theory), its foundation is usually the Zermelo–Fraenkel set theory with the axiom of choice (ZFC). This allows the counting of different objects: the sizes of the models, the number of types in a model (stability), the number of nonisomorphic models (spectrum and categoricity)... Without looking into a specific class, we would like to know how one number is related to another number. These motivate a lot of developments in model theory.

For first-order theories, rich results have been produced in the past 60 years, in particular in Shelah's book on classification theory [She90]. There he solved many problems in stability and categoricity. Besides directly looking at other open questions in first-order theories, one can generalize known results to higher-order contexts. One possibility is to look at theories in $L_{\kappa,\omega}$ for some uncountable κ or even in $L_{\kappa,\lambda}$ where λ is also uncountable. Another approach due to Shelah [She87] is to provide an axiomatic framework that generalizes the first-order theories. The framework is called *abstract elementary classes* (AECs) which has fewer than ten axioms (see Chapter 2). We highlight some major obstacles of studying AECs compared to first-order theories, and state relevant assumptions to simplify the investigation.

1. Little relationship between the class and the underlying language: in a first-order theory T, there is an underlying language $L = L(T)$ which determines the Löwenheim-

Skolem number and the ordering (elementary substructure). In AECs, we only assume that the ordering is a subclass of that of the L-substructure. The size of L can be different from the Löwenheim-Skolem number. Only in the case of universal classes do we assume that the ordering is by L-substructure.

2. Lack of the compactness theorem: if a first-order theory T has an infinite model, then it has models of any size $\geq |L(T)|$. Syntactic types that are consistent can be realized in some models. In AECs, model sizes are closed downwards (by the Löwenheim-Skolem axiom) but not upwards. There are no natural syntactic types associated to AECs, and their syntactic consistency does not guarantee they are realized. If one orders the class by elementary substructure with respect to that language, they have to check it gives rise to an AEC. To remedy this, one assumes the class has the amalgamation property, the joint-embedding property, no maximal models and arbitrarily large models.

3. Lack of finite character of types: there is a notion of semantic types called the Galois types (orbital types) that generalizes first-order syntactic types. While first-order types are finitary and have good locality and continuity properties, they are not inherent in AECs. For example, when checking syntactic type equality, one only needs to compare the formulas in the types one by one; two Galois types can be different even though their smaller restrictions are the same (failure of tameness and shortness). To avoid this issue, one can assume tameness, shortness, continuity etc to handle types more effectively.

4. Lack of a set-monster model: in complete first-order theories, it is common to work in a set-monster model: types are over subsets of the monster model while models are elementary substructures of it. Even with the extra assumptions in (2) for AECs, one can at most build a weaker monster model (closed under two-dimensional amalgams). This causes difficulty in manipulating certain notions like splitting over sets and (λ, n)-uniqueness for independence relations. Certain classes allow the construction of splitting over sets [SV18a] but in general one has to make an extra assumption of

amalgamation over sets.

The above assumptions are used in our papers to investigate different problems of AECs. We give an overview of the main results and open questions related to each paper.

- In Chapter 3, we study the stability theory of AECs. There were numerous results on this area but one missing part is the first stability cardinal for stable AECs. In first-order theories, the first stability cardinal is bounded above by $2^{|L(T)|}$ (a precise bound is in [She90, Chapter V]). Vasey used Morley's method to show that under the amalgamation property, an upper bound is $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ [Vas16c]. A natural question is whether this bound could be improved, or whether amalgamation is necessary. Our paper organized known stability results on AECs and provided examples of stable AECs whose first stability cardinals and order property lengths can go up to any cardinal below $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ (see also the table after this list). However, our examples fail amalgamation so we ask the following open questions:

  Question. 1. Assuming amalgamation and tameness, can we bound the first stability cardinal lower than $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$, or even to $2^{\mathrm{LS}(\mathbf{K})}$?

  2. Assuming tameness but not amalgamation, can we bound the first stability cardinal by $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$? Or are there counterexamples?

  3. Can we say anything about the order property length (in place of the first stability cardinal) in (1) and (2)?

  On the other hand, by assuming amalgamation over sets, we gave simpler proofs of known stability results on AECs, with some using the method of Galois-Morleyization in [Vas16c]. We also showed that depending on the non-ZFC axioms, the joint-embedding property might be necessary for [Bon17, Proposition 2.7].

- In Chapter 4, we study the presentation theorems of AECs. Shelah axiomatized AECs and showed that they are $PC_{\mathrm{LS}(\mathbf{K}),2^{\mathrm{LS}(\mathbf{K})}}$, where $2^{\mathrm{LS}(\mathbf{K})}$ indicates that the PC-classes come from omitting at most $2^{\mathrm{LS}(\mathbf{K})}$-many types. The original proof told little about the actual number of omitted types, or in what cases it could be lowered. On the other hand, [SV21] had a technical proof of axiomatizing AECs in $L_{(2^{2^{\lambda^+}})^{+++},\lambda^+}$ where $\lambda = \mathrm{LS}(\mathbf{K})$. While we were not able to verify or simplify the proof there, we made use of game quantification to axiomatize AECs in $L_{\chi^+,\lambda^+}(\omega \cdot \omega)$ where $\chi = \lambda + I_2(\lambda, \mathbf{K}) \leq 2^\lambda$ (see Definition 4.3.1) and $\omega \cdot \omega$ denotes the length of the game. With extra assumptions, we could tell when $\chi = \lambda$. Also, by adapting this result, we showed that AECs are PC_χ, which has a (potentially) fewer types omitted but a bigger language size. By slightly changing the proof, we could also recover Shelah's original result. Although our focus was on AECs, we extended the results to μ-AECs (see Definition 4.5.1), showing the generality of our argument. Here are some open questions:

Question. 1. Assuming stability or categoricity, is it possible to obtain a better bound than $I_2(\mathrm{LS}(\mathbf{K}), \mathbf{K}) \leq 2^{\mathrm{LS}(\mathbf{K})}$? A positive answer provides an alternative way to lower the complexity of our presentation theorems.

2. For $\mu \geq \aleph_1$, do μ-AECs have Hanf numbers? A positive answer allows the translation of some AEC techniques to the μ-AEC context.

- In Chapter 5, we study equivalent criteria for stable as well as for superstable AECs. Throughout our paper, we assumed the existence of a monster model, tameness and continuity of nonsplitting. The last assumption, which is implied by superstability, allowed us to manipulate nonsplitting more easily while still in the strictly stable context. There were a good amount of results on superstability and [GV17] established equivalent criteria of superstability. However the criteria were eventual: one statement implies another modulo moving up the cardinal by a great interval. There were follow-up results but a clean picture was yet to be seen. We managed to organize the important superstability results in one framework, and generalized them to the strictly stable context. All but one of our equivalent criteria are local and require moving up by at most a successor cardinal. Moreover, our results have a low cardinal threshold ($\mathrm{LS}(\mathbf{K})^+$) so we can readily apply them in algebraic examples. While our results work in the strictly stable context, we still need to assume a lot of stability because we cannot control the length of the order property (see Question 3 for the discussion on Chapter 3 above). We also have the following open questions:

Question.

1. Assuming the existence of a monster model, tameness and continuity of nonsplitting, we showed in Corollary 5.6.16 that if an increasing chain of saturated models in K_{μ^+} has a saturated union, then $\mathbf{K}$ has uniqueness of limit models of size μ. Can we replace μ^+ by a general $\xi > \mu$? This will give more flexibility to the equivalent superstability criteria.

2. In AECs, one of the superstability criteria is no-long splitting chains (instead of nonsplitting over a small *set*). In the strictly stable context, the possible lengths of splitting chains in one cardinal might be different from those in another cardinal. Let λ' be the cardinal where the maximum length stabilizes and λ be the first stability cardinal. Is it possible to lower the bound of $\lambda' < \beth_{(2^\lambda)^+}$ by [Vas18c]? Our paper bypassed this problem by assuming the continuity of nonsplitting and looking at an ω-interval of cardinals.

- In Chapter 6, we study the categoricity results of AECs, assuming amalgamation over sets and tameness. Vasey had many papers on this area with gradual improvements in different aspects. This posed a problem of understanding the current progress and how different papers are related to each other. One of the goals in our paper was to organize results on the construction of an ω-successful frame. [Vas16a] built such a frame using coheir with tons of machinery, and the threshold was quite high (fixed points of the beth function). It turned out that [Vas17e] (on universal classes) had sketched a variation using nonsplitting with a better threshold, and this was used in subsequent papers. We wrote out the details and bypassed the machinery in [Vas16a, Sections 1-10]. On the other hand, [SV18b] showed that the eventual categroicity conjecture is true assuming a $(<\omega)$-extendible frame (a strengthening of an ω-successful frame) and the weak general continuum hypothesis. We adapted their proof and showed in ZFC that under amalgamation over sets (together with tameness and arbitrarily large models), we can do the same categoricity transfer: if the AEC is categorical in some $\mu > \mathrm{LS}(\mathbf{K})$, then it is categorical in all $\mu' \geq \mu$. This allows us to reprove Morley and Shelah's categoricity theorems for first-order theories, and also Vasey's categoricity theorem for homogeneous diagrams — there he used syntactic results by Shelah; our result only has AEC techniques.

Question. 1. We ended up using amalgamation over sets because the following is unknown: is it possible to build primes under the usual amalgamation and shortness? Vasey [Vas17a] constructed primes for *saturated models* but this begs the question on when it applies to non-saturated models (if they exist).

2. Adapting the examples in Chapter 3, we can construct tame AECs where the first categoricity cardinals can be higher than any $\mu < \beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$. Those examples failed amalgamation (AP) and we also ruined joint-embedding in order to guarantee arbitrarily large models (AL). As in Question 1 of the discussion on Chapter 3 above, we ask: it is possible to give similar examples that satisfy amalgamation, and perhaps also joint-embedding?

First stability cardinal / Order property length	$AP + AL$	AL only
Upper Bound	$\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ [Vas16c]	?
Counterexamples up to	?	$\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ (Ch. 3)

First categoricity cardinal	AP over sets $+AL$	AL only
Upper Bound	$\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ (Ch. 6)	?
Counterexamples up to	?	$\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ (Ch. 6)

In Chapter 7, we make use of Vasey's and Espíndola's results to derive an additional categoricity transfer. Since Espíndola's results involve heavy machinery in topos theory, we separate the new result in that chapter.

We assume the readers to be familiar with notions in first-order model theory. Knowledge of AECs would be ideal while Chapter 2 will provide the preliminaries. Readers can also refer to [Gro21] which is a comprehensive guide for basic results in first-order theories and AECs (and also a good source for historical comments).

CHAPTER 2
PRELIMINARIES

2.1 ABSTRACT ELEMENTARY CLASSES

The notion of Abstract Elementary Classes (AECs) was introduced by Shelah [She87] to generalize certain classes of models in $L_{\infty,\omega}(Q)$, which includes elementary classes (classes of models of complete first-order theories) and also $EC(T,\Gamma)$ classes (models of a complete first-order theory T omitting a set of types Γ). We will follow [Gro21, Chapter 2] in first defining abstract classes (ACs) and then adding more requirements to form AECs.

Definition 2.1.1. Fix a finitary language L. An *abstract class* $\mathbf{K} = \langle K, \leq_{\mathbf{K}} \rangle$ in L is such that

1. K is a class of L-structures.

2. $\leq_{\mathbf{K}}$ is a partial order on K. For any $M, N \in K$, if $M \leq_{\mathbf{K}} N$, then $M \subseteq N$ as substructure.

3. $\leq_{\mathbf{K}}$ respects isomorphism:

   (a) If M, N are L-structures, $M \in K$, $M \cong N$, then $N \in K$.

   (b) If $f_i : M_i \cong N_i$ for $i = 0, 1$, $M_0 \leq_{\mathbf{K}} M_1$ and $f_0 \subseteq f_1$, then $N_0 \leq_{\mathbf{K}} N_1$.

$$\begin{tikzcd} N_0 \arrow[r, dashed, "id"] & N_1 \\ M_0 \arrow[u, "f_0", "\cong"'] \arrow[r, "id"'] & M_1 \arrow[u, "f_1", "\cong"'] \end{tikzcd}$$

4. We write $\mathrm{L}(\mathbf{K}) := L$.

From now on, unless otherwise specified, we assume M, N (with superscripts or subscripts) to be in K. We do not distinguish M and its domain/universe. We also abuse notation and write K instead of $\mathbf{K}$; $\leq$ instead of $\leq_{\mathbf{K}}$ if the context is clear (In [Vas17f], more than one abstract class is considered and careful distinction would be necessary).

Notice that one might allow abstract classes in infinitary languages, but since the latter only appear in Galois-Morleyization in Chapter 3 in this dissertation, we do not generalize them here. The following requirements further exclude certain classes in infinitary languages.

Definition 2.1.2. Let K be an abstract class.

1. K is *coherent* if for any $M_0, M_1, M_2 \in K$, $M_0 \subseteq M_1$, $M_0 \leq M_2$, $M_1 \leq M_2$, then $M_0 \leq M_1$.

2. Löwenheim-Skolem axiom: we require $\mathrm{LS}(\mathbf{K})$ to be defined where $\mathrm{LS}(\mathbf{K})$ is the first infinite cardinal $\lambda \geq |\mathrm{L}(\mathbf{K})|$ such that for any $M \in K$, $A \subseteq M$, there is $N \leq M$ such that $A \subseteq N$ and $\|N\| \leq \lambda + |A|$.

$$A \longrightarrow (N) \longrightarrow M$$

3. Chain axioms:

   (a) For any ordinal α, any $\langle M_i : i < \alpha \rangle \subseteq K$ such that $M_i \leq M_j$ for $i < j < \alpha$, we have $M := \bigcup_{i<\alpha} M_i \in K$ and $M_i \leq M$ for all $i < \alpha$.

   (b) In the above clause, if we have in addition $N \in K$ and $M_i \leq N$ for all $i < \alpha$, then $M \leq N$.

4. An *abstract elementary class* (AEC) is an abstract class satisfying clauses (1) to (3).

Remark 2.1.3.

- Although AECs are our main objects of study here, some partial versions are also useful. For example coherent abstract classes appear in independence relations over saturated models because the latter might not be closed under unions.

- A typical counterexample to AECs is the class of well-orderings K ordered by initial segments: it is not even a μ-AEC (see Chapter 4 or [BGL+16, Section 2] for more discussions on μ-AECs) because $\mathrm{LS}(\mathbf{K})$ does not exist. But if we restrict the models to have order types up to a fixed ordinal α, then the class is an AEC with $\mathrm{LS}(\mathbf{K}) \leq |\alpha|$.

Sometimes it is useful to restrict models of certain cardinalities inside K.

Definition 2.1.4. Let S be a class of cardinals, λ be a cardinal.

$$K_S := \{M \in K : \|M\| \in S\}$$
$$K_\lambda := K_{\{\lambda\}}$$
$$K_{\geq\lambda} := K_{[\lambda,\infty)}$$

and similarly for $K_{>\lambda}$, $K_{\leq\lambda}$ and $K_{<\lambda}$.

Besides the $\leq_{\mathbf{K}}$ relation, it is useful to describe mappings between models that respect $\leq_{\mathbf{K}}$:

Definition 2.1.5. Let K be an AEC. A mapping $f : M \to N$ is called a *K-embedding* if $M \cong f[M] \leq N$. If f fixes some $M_0 \leq M$ pointwise, we write $f : M \underset{M_0}{\longrightarrow} N$.

Hence $M \leq N$ is equivalent to id : $M \underset{M}{\longrightarrow} N$. Throughout this paper, functions between models in K are assumed to be K-embedding, and we omit the label of an arrow if it is the identity. We now look at some nice properties that an AEC may have:

Definition 2.1.6.
1. An AEC K satisfies the *amalgamation property* (AP) if for any $M_0 \leq M_1$, $M_0 \leq M_2$ in K, there is $M_3 \in K$ such that $M_1 \leq M_3$ and there is a K-embedding $g : M_2 \underset{M_0}{\longrightarrow} M_3$.

$$\begin{tikzcd} M_2 \arrow[r, dashed, "g"] & M_3 \\ M_0 \arrow[u, "id"] \arrow[r, "id"'] & M_1 \arrow[u, dashed, "id"'] \end{tikzcd}$$

2. An AEC K satisfies the *joint embedding property* (JEP) if for any $M_0, M_1 \in K$, there are $M_2 \in K$ and $g : M_1 \to M_2$ with $M_0 \leq M_2$.

$$\begin{tikzcd} M_0 \arrow[r, dashed] & M_2 \\ & M_1 \arrow[u, dashed, "g"'] \end{tikzcd}$$

3. An AEC K has *no maximal models* (NMM) if for any $M \in K$, there is $N \in K$ with $M < N$.

4. An AEC K has *arbitrarily large models* (AL) if for any infinite cardinal λ, there is $M \in K$ such that $\|M\| > \lambda$.

Remark 2.1.7. Notice that in the definition of AP, we can equivalently require $f_i : M_0 \to M_i$ for $i = 1, 2$, using isomorphic models.

Some AECs do not satisfy NMM as in Remark 2.1.3. Some satisfy AP but not JEP, for example the class of algebraically closed fields (without specifying the characteristic). On the other hand, some satisfy JEP but not AP (adapted from [Bal09, Example 19.2]):

Example 2.1.8. Let $L = \langle P, f, d_0, d_1 \rangle$, K be the class of models ordered by substructures such that for each $M \in K$,

1. d_0 and d_1 are constants.
2. f is a unary coloring function of range $\{d_0, d_1\}$.
3. P is an equivalence relation. Each class has no more than 2 members.
4. $f(d_0) = d_0$, $f(d_1) = d_1$ and $\{d_0, d_1\}$ is one of the classes of P.

We may take $a \in M_0, b \in M_1 - M_2, c \in M_2 - M_1$ such that $P^{M_1}(a, b)$, $f^{M_1}(a) = f^{M_1}(b) = d_0$, $P^{M_2}(a, c)$, $f^{M_2}(c) = d_1$. This is also a counterexample of AP for incomplete (universal) first order theories.

Assuming JEP, AL is equivalent to NMM:

Proposition 2.1.9. *Suppose K satisfies JEP. Then it satisfies AL iff it satisfies NMM.*

Proof. Let $M \in K$. If K satisfies AL, pick $N \in K_{>\|M\|}$. By JEP there are $N' \geq M$ and $f : N \to N'$. Then $N' > M$ because $\|N'\| \geq \|N\| > \|M\|$.

If K satisfies NMM, let μ be an infinite cardinal. Applying NMM recursively, we build $\langle M_i : i \leq \mu^+ \rangle$ increasing and continuous (see Definition 2.2.15) such that $M_0 := M$. Then $\|M_{\mu^+}\| \geq \mu^+$. $\square$

2.2 GALOIS TYPES

From now on, we assume AP, JEP and NMM for a fixed K.

Unlike first-order logic, we work on semantic types more than syntactic types. A type still consists of elements realizing it, a domain (a set or a model) and an ambient model. We first state the definition in [Gro21, Chapter 5.1]:

Definition 2.2.1.
1. For $M \leq N_0$, $M \leq N_1$, $a \in N_0$, $b \in N_1$, we define $(a, M, N_0) \sim (b, M, N_1)$ iff there are $N_2 \geq N_0$, $f : N_1 \underset{M}{\longrightarrow} N_2$ such that $f(b) = a$.

$$\begin{tikzcd} a \in N_0 \arrow[r] & N_2 \\ M \arrow[u] \arrow[r] & N_1 \ni b \arrow[u, "f"'] \end{tikzcd}$$

2. We take $\equiv$ to be the transitive closure of $\sim$.

3. The *Galois type* of a over M in N_0, written as $\mathrm{gtp}(a/M, N_0)$, is the $\equiv$ equivalence class of (a, M, N_0) (symmetry is guaranteed by isomorphism axioms). We call M the *domain* of the type and N the *ambient model*.

4. A type $p = \mathrm{gtp}(a/M, N)$ is *algebraic* when $a \in M$.

5. Let $M_0 \leq M_1$, $p = \mathrm{gtp}(a/M_0, N_0)$ and $q = \mathrm{gtp}(b/M_1, N_1)$. We say q *extends* p, or $q \supseteq p$, if $p = q \restriction M := \mathrm{gtp}(b/M_0, N_1)$.

As in Remark 2.1.7, we may require in the second half of Definition 2.2.1(1) that there are N_2, $f : N_1 \underset{M}{\longrightarrow} N_2$ and $g : N_0 \underset{M}{\longrightarrow} N_2$ such that $f(b) = g(a)$. The above definition also applies to (possibly infinite) tuples a and b (in which case "$a \in M$ is replaced by "' $\mathrm{ran}(a) \subseteq M$" and similarly for b). We do not add the tuple sign if the context is clear.

Remark 2.2.2. The definition requires the second element in the triple (a, M, N_0) to be a model (K-substructure of N_0), but it still makes sense if we replace M by any set $A \subseteq N_0$. There are several advantage of doing so:

1. It allows a more general discussion of types over sets as in first-order logic. In particular, Galois types over sets coincide with syntactic types in a complete first-order theory.

2. Non-splitting of types of (finite) sets is a key property in some AECs near $\aleph_0$ when models are always infinite in size (see [SV18a, Section 5]). Also, we can better adapt the notion of *tameness* (see Definition 2.2.11).

3. It extends the definition of independence relations to include sets as domains (see [Vas16a]). For clarity, we call the relation a *nonforking relation* if the types have model domains and an *independence relation* if the types have set domains (unfortunately there has been a confusion of terminology, for example [SV18b] used *independence relations* in the multidimensional case even though the domains are models).

Moreover, under AP, the second clause of Definition 2.2.1 can be omitted. Note that we can replace M by a set $A \subseteq N_0 \cap N_1$ below:

Proposition 2.2.3. *Let* $(a, M, N_0) \equiv (b, M, N_1)$ *then* $(a, M, N_0) \sim (b, M, N_1)$.

Proof. By induction, it suffices to show that if $(a, M, N_0) \sim (c, M, N_2) \sim (b, M, N_1)$ then $(a, M, N_0) \sim (b, M, N_1)$. By the definition of $\sim$, obtain

- $N_{02} \geq N_2$, $f : N_0 \underset{M}{\longrightarrow} N_{02}$, $f(a) = c$
- $N_{12} \geq N_2$, $g : N_1 \underset{M}{\longrightarrow} N_{12}$, $g(b) = c$.

By AP, obtain $f' : N_{02} \underset{N_2}{\longrightarrow} N_{01}$, $g' : N_{12} \underset{N_2}{\longrightarrow} N_{01}$ for some N_{01}. Then $f' \circ f$ and $g' \circ g$ witness $(a, M, N_0) \sim (b, M, N_1)$ because $f'(f(a)) = f'(c) = c = g'(c) = g'(g(b))$. $\square$

$$\begin{tikzcd}
a \in N_0 \arrow[r, "f"] & N_{02} \arrow[dr, "f'"] & \\
M \arrow[u] \arrow[r] \arrow[d] & N_2 \ni c \arrow[u] \arrow[d] & N_{01} \\
b \in N_1 \arrow[r, "g"] & N_{12} \arrow[ur, "g'"] &
\end{tikzcd}$$

Under AP, types also enjoy monotonicity in the ambient model. Again we can replace M by a set $A \subseteq N$ below.

Proposition 2.2.4. *Let* $M \leq N_1 \leq N_2$ *and* $a \in N_1$. *Then* $\mathrm{gtp}(a/M, N_1) = \mathrm{gtp}(a/M, N_2)$.

Proof. Assume $\mathrm{gtp}(b/M, N) = \mathrm{gtp}(a/M, N_1)$. Then there are $N' \geq N$, $f : N_1 \underset{M}{\longrightarrow} N'$ such that $f(a) = b$. By AP, we can find $N'' \geq N'$, $\tilde{f} : N_2 \to N''$ such that the following diagram commutes.

$$\begin{tikzcd}
N_2 \arrow[r, "\tilde{f}"] & N'' \\
a \in N_1 \arrow[u] \arrow[r, "f"] & N' \arrow[u] \\
M \arrow[u] \arrow[r] & N \ni b \arrow[u]
\end{tikzcd}$$

Then $\tilde{f} \supseteq f$ which sends a to b, and so $\mathrm{gtp}(b/M, N) \subseteq \mathrm{gtp}(a/M, N_2)$.

Conversely, assume $\mathrm{gtp}(b/M, N) = \mathrm{gtp}(a/M, N_2)$. Then obtain $N' \geq N$, $f : N_2 \underset{M}{\longrightarrow} N'$ such that $f(a) = b$. Then $f \restriction N_1$ witnesses that $\mathrm{gtp}(b/M, N) \subseteq \mathrm{gtp}(a/M, N_1)$.

$$\begin{tikzcd}
N_2 \arrow[r, "f"] & N' \\
a \in N_1 \arrow[u] \arrow[r, "f \restriction N_1"] & f[N_1] \arrow[u] \\
M \arrow[u] \arrow[r] & N \ni b \arrow[u]
\end{tikzcd}$$

$\square$

Remark 2.2.5. Along with Löwenheim-Skolem axiom, when we write $\mathrm{gtp}(a/M, N)$ we may assume $\|N\| \leq \|M\| + |l(a)|$.

Now with Galois types defined, we would like to count the number of types and define stability.

Definition 2.2.6. Let α be a ordinal at least 2, λ be an infinite cardinal, $M \in K$.

1. The collection of $(< \alpha)$-ary types over M is given by

$$\mathrm{gS}^{<\alpha}(M) := \{\mathrm{gtp}(a/M, N) : a \in N, l(a) < \alpha, N \geq M\}$$

2. K is $(< \alpha)$-stable in λ if $|\mathrm{gS}^{<\alpha}(M)| = \lambda$ for all $\|M\| = \lambda$.

3. K is $(< \alpha)$ stable if there exists an infinite μ such that K is $(< \alpha)$-stable in μ.

When we omit α, we mean $\alpha = 2$.

Remark 2.2.7.
- In the first clause, we may extend the definition to types over sets. For sets of cardinality above $\mathrm{LS}(\mathbf{K})$, it does not matter because we will have a monster model (see Section 2.4) and use Löwenheim-Skolem axiom. However, it will be useful to count types over sets of cardinality below $\mathrm{LS}(\mathbf{K})$ (see [Vas16c, Remark 3.4] or Chapter 4 Section 7).

- In the second clause, we may equivalently require $|\mathrm{gS}^{<\alpha}(M)| \leq \lambda$ because there are at least λ many algebraic types.

Type counting in AECs is generally difficult because the types may not be syntactic or have finite character. Still, we have the usual upper bounds:

Proposition 2.2.8. *Let $\lambda \geq |\mathrm{L}(\mathbf{K})|$, then $I(K, \lambda) \leq 2^\lambda$. That is, there are at most 2^λ non-isomorphic models in K_λ.*

Proof. If $M \ncong N \in K_\lambda$, then for any enumerations $\bar{m} = \langle m_i : i < \lambda \rangle$ of M, $\bar{n} = \langle n_i : i < \lambda \rangle$ of N, $\bar{m} \ncong \bar{n}$. There are $\lambda^\lambda = 2^\lambda$ many enumerations.

If $\bar{m} \ncong \bar{n}$, there are two cases (constants are treated as a 0-ary function in the following):

1. There is a function $f \in \mathrm{L}(\mathbf{K})$, a finite $I \subseteq \lambda$, $j \in \lambda$ such that $f(\bar{m}^I) = m_j$ (see Definition 2.2.11(1)) but $f(\bar{n}^I) \neq n_j$, or vice versa. There are at most $|\mathrm{L}(\mathbf{K})| \leq \mathrm{LS}(\mathbf{K}) \leq \lambda$-many choices for f, $\lambda^{<\omega} = \lambda$-many for I and j. In total there are 2^λ-many choices to decide the (in)equalities of the functions.

2. There is a predicate $R \in \mathrm{L}(\mathbf{K})$, a finite $I \subseteq \lambda$ such that $R^M[\bar{m}^I]$ but $\neg R^N[\bar{n}^I]$, or vice versa (here we assume that the first case does not occur and can substitute the function values directly). There are at most $|\mathrm{L}(\mathbf{K})| \leq \mathrm{LS}(\mathbf{K}) \leq \lambda$-many choices for R, $\lambda^{<\omega} = \lambda$-many for I. In total there are 2^λ-many choices for satisfaction of predicates.

Thus there are at most 2^λ possibilities to cause $m \ncong n$.

We have showed $I(K, \lambda) \cdot 2^\lambda \leq 2^\lambda$, hence the result. $\square$

Proposition 2.2.9. *Let K be an AEC in a finitary language, α be an ordinal. For any $M \in K$,*

$$|\mathrm{gS}^{\alpha}(M)| \leq 2^{\mathrm{LS}(\mathbf{K})+\|M\|+|\alpha|}$$

Proof. By adding constants to each element in M, it suffices to show that

$$|\mathrm{gS}^{\alpha}(\emptyset)| \leq 2^{\mathrm{LS}(\mathbf{K})+|\alpha|}$$

By Remark 2.2.5 we may assume the ambient models of the types in $\mathrm{gS}^{\alpha}(\emptyset)$ to have size $\lambda := \mathrm{LS}(\mathbf{K}) + |\alpha|$. Two types are equal if the ambient models are isomorphic and an isomorphism maps the realizations of the types from one to another. Thus to give an upper bound to $|\mathrm{gS}^{\alpha}(\emptyset)|$, it suffices to count the number of non-isomorphic models ($\leq 2^{\lambda}$ by Proposition 2.2.8), and then count the number of α-sequences in a model ($\lambda^{\alpha} \leq \lambda^{\lambda} = 2^{\lambda}$). In total there are at most 2^{λ} many choices as desired. □

Remark 2.2.10.
- In general if K does not have AP, we can still repeat the above proof by more book-keeping.
- If K has a $(<\mu)$-ary language (or more generally K is a μ-AEC), then we can bound $I(K,\lambda) \leq 2^{\lambda^{<\mu}}$, $|\mathrm{gS}^{<\alpha}(M)| \leq 2^{(\mathrm{LS}(\mathbf{K})+\|M\|+\alpha)^{<\mu}}$ but we have no use of these results in this paper.

Grossberg and VanDieren isolated a nice property of types, called *tameness*, in [GV06b, Section 3] and later Boney defined a dual version called *shortness* in [Bon14b, Definition 3.3]. These notions bring an AEC closer to a complete first-order theory.

Definition 2.2.11. Let κ be an infinite cardinal.

1. Let $p = \mathrm{gtp}(a/M, N)$ where $a = \langle a_i : i < \alpha \rangle$ may be infinite, $I \subseteq \alpha$, $M_0 \leq M$. We write $p \restriction M_0 := \mathrm{gtp}(a/M_0, N)$, $a^I = \langle a_i : i \in I \rangle$ and $p^I := \mathrm{gtp}(a^I/M, N)$.
2. K is $(<\kappa)$-*tame for* $(<\alpha)$-*types* if for any $M \in K$, any $p \neq q \in \mathrm{gS}^{<\alpha}(M)$, there is $N \leq M$, $\|N\| < \kappa$ with $p \restriction N \neq q \restriction N$. We omit $(<\alpha)$ if $\alpha = 2$.
3. K is $(<\kappa)$-*short* if for any $\alpha \geq 2$, $M \in K$, $p \neq q \in \mathrm{gS}^{<\alpha}(M)$, there is $I \subseteq \alpha$, $|I| < \kappa$ with $p^I \neq q^I$.

4. κ*-tame* means $(<\kappa^+)$-tame. Similarly for shortness.

Remark 2.2.12.

- We should check that the first clause is well-defined (using the fact that $f[M] = M$ implies $f[M_0] = M_0$; $f(a) = b$ implies $f(a^I) = b^I$).

- As usual, we can require tameness and shortness to apply to types over sets.

- It is possible to define finer versions of tameness and shortness (see [Bal09, Chapter 11]), but we have no use of them here.

- $(<\kappa)$-short implies $(<\kappa)$-tame for $(<\kappa)$-types $\big($using the fact that $\mathrm{gtp}(a/M, N_0) \neq \mathrm{gtp}(b/M, N_1)$ implies $\mathrm{gtp}(aM/\emptyset, N_0) \neq \mathrm{gtp}(aM/\emptyset, N_1)\big)$.

- Fortunately the natural examples of AECs all have tameness (even $\aleph_0$ or $\aleph_1$-tame). Complete first-order theories are $<\aleph_0$-tame. Readers may consult [BV15b, Section 3.2] for more examples (and counterexamples).

A notion that is related to stability is the order property: there are several versions and one has to be careful with the quantifiers involved.

Definition 2.2.13. Let μ be an infinite cardinal, $\alpha \geq 2$ and $\beta \geq 1$ be ordinals.

1. K has β*-order property of length* μ if there exists some $\langle a_i : i < \mu \rangle \subseteq M \in K$ such that $l(a_i) = \beta$, and for $i_0 < i_1 < \mu$, $j_0 < j_1 < \mu$, $\mathrm{gtp}(a_{i_0} a_{i_1}/\emptyset, M) \neq \mathrm{gtp}(a_{j_1} a_{j_0}/\emptyset, M)$.

2. K has $(<\alpha)$*-order property of length* μ if there is a $\beta < \alpha$ witnessing (1).

3. K has $(<\alpha)$*-order property* if for all μ, K has $(<\alpha)$-order property of length μ. In other words, if we fix μ, we can find a suitable β_μ witnessing (1).

4. K satisfies *no* $(<\alpha)$*-order property* ($N(<\alpha)$-OP) if (3) fails. In other words, for each $\beta < \alpha$, there is an upper bound to the length of β-order property. We omit $(<\alpha)$ if $\alpha = 2$.

Remark 2.2.14. If K has $(<\alpha)$-order property, we can fix $\beta < \alpha$ such that K has β-order property.

Definition 2.2.15. K has *weak order property of length* κ if there are $M \in K_{<\kappa}$, $N \geq M$, $\langle a_i, b_i : i < \kappa \rangle \subseteq N$, $p \neq q \in \mathrm{gS}^{<\kappa}(M)$ such that

1. If $i \leq j < \kappa$, then $\mathrm{gtp}(a_i b_j / M, N) = p$.

2. If $j < i < \kappa$, then $\mathrm{gtp}(a_i b_j / M, N) = q$.

Notice that we fix the sequences to be of length κ and the type-lengths and the domain to be $< \kappa$. The reason is that such property (along with other hypotheses) is already enough to deduce symmetry in nonforking relations (see Chapter 5 Section 5).

2.3 SATURATED AND UNIVERSAL MODELS

From this section onwards, we will frequently use *resolutions* and chain arguments.

Definition 2.3.1. Let $M \in K$, $\|M\| = \lambda \geq \mathrm{LS}(\mathbf{K})$. A *resolution* of M is a chain of models $\langle M_i : i < \lambda \rangle$ such that $M = \bigcup_{i<\lambda} M_i$, $M_i \leq M$, $\|M_i\| \leq |i| + \mathrm{LS}(\mathbf{K})$. Unless otherwise specified, we assume the chain is

- *increasing*: For all $i < \lambda$, $M_i < M_{i+1}$; and
- *continuous*: for any limit ordinal $\delta < \lambda$, $\bigcup_{i<\delta} M_i = M_\delta$.

As in first-order logic, a (Galois-)saturated model is a model which realizes all (Galois) types over small domains.

Definition 2.3.2. Let $\lambda > \mathrm{LS}(\mathbf{K})$. $M \in K$ is λ-*saturated* if for any $N \leq M$, $|N| < \lambda$, $p \in S(N)$, then M realizes p. M is *saturated* if it is $\|M\|$-saturated.

Related properties of saturated include universal and model homogeneous.

Definition 2.3.3. Let $\lambda > \mathrm{LS}(\mathbf{K})$, $N \leq M \in K$.

1. M is λ-*universal* if for any $M' \in K_{<\lambda}$, there is $f : M' \to M$. M is *universal* if it is $\|M\|^+$-universal.

2. M is *universal* over N if for any $N' \geq N$, $\|N'\| \leq \|M\|$, there is $f : N' \underset{N}{\to} M$.

3. M is λ-*model homogeneous* if it is universal over any $M' \leq M$ with $\|M'\| < \lambda$. M is *model homogeneous* if it is $\|M\|$-model homogeneous.

Notice that when we omit the parameter λ in (1) and (2), we allow models of the same size. For (3), we consider models of smaller sizes. We can weaken the definition of model homogeneous by restricting $\|N'\| < \|M\|$:

Proposition 2.3.4. *The following are equivalent:*

- *M is model homogeneous.*
- *For any $N \leq M$ with $\|N\| < \|M\|$, any $N' \geq N$ with $\|N'\| < \|M\|$, there is $f : N' \underset{N}{\longrightarrow} M$.*

Of course we cannot simply replace $\|M\|$ by $\|M\|^+$ in the original definition of model homogeneous, because we may take $M' = M$ and some $M'' > M'$ that cannot be embedded to M fixing M'.

Proof. We prove the backward direction. Let $\|M\| = \lambda > \mathrm{LS}(\mathbf{K})$. We only need to consider the case where $\|N'\| = \|M\|$. Take a resolution of $N' = \langle N_i : i < \lambda \rangle$ with $N_0 = N$. We build an increasing continuous chain of embeddings $\langle f_i : N_i \underset{N}{\longrightarrow} M \mid i \leq \lambda \rangle$. The desired embedding will be f_λ.

Suppose $f_i : N_i \underset{N}{\longrightarrow} M$ is defined, we build f_{i+1}. Let P_i be the image of N_i, and P_{i+1} be the isomorphic copy of N_{i+1} witnessed by $f_i' \supseteq f_i$, which fixes N. By universality, there is $g : P_{i+1} \underset{N}{\longrightarrow} M$. Define $f_{i+1} := g \circ f_i' : N_{i+1} \underset{N}{\longrightarrow} M$.

$$
\begin{tikzcd}
N' & & & M \\
& N_{i+1} \arrow[ul] \arrow[r, "f_i'", "\cong"'] & P_{i+1} \arrow[ur, "g"] & \\
& & N_i \arrow[ul] \arrow[r, "f_i", "\cong"'] & P_i \arrow[ul] \arrow[uu] \\
& & & N \arrow[ul] \arrow[u]
\end{tikzcd}
$$

□

At first glance, the difference between λ-universal and λ-model homogeneous is that the latter always specifies a common model smaller than λ. We show that under JEP, λ-model homogeneous is a stronger condition.

Proposition 2.3.5. *If $M \in K$ is λ-model homogeneous, then M is also λ-universal.*

Proof. Let $N \in K_{<\lambda}$. Pick $M_0 \leq M$ with $\|M_0\| = \|N\|$. By JEP, there are $N' \geq M_0$ and $f : N \to N'$ such that $\|N'\| = \|N\| < \lambda$. By model homogeneity, there is $g : N' \underset{M_0}{\longrightarrow} M$. Hence $g \circ f : N \to M$ as desired.

$$\begin{tikzcd}
 & & M \\
 & N' \arrow[ur, "g"] & \\
N \arrow[ur, "f"] & & M_0 \arrow[ul] \arrow[uu]
\end{tikzcd}$$

$\square$

Remark 2.3.6. The converse does not hold, namely we can consider the first-order theory of dense linear orders. Since countable dense linear orders are isomorphic, $\mathbb{Q} \times \omega_1$ is $\aleph_1$-universal. But there is no embedding from $\mathbb{Q} \times \{-1, 0\}$ fixing $\mathbb{Q} \times \{0\}$. Therefore one has to be careful of embeddings with/without fixing a substructure. On the other hand, there are two notions of uniqueness of limit models (Definition 2.3.9), where one fixes a base while the other does not. They turn out to be equivalent under a monster model and tameness (see Chapter 5 Section 8).

We now show a classical result of Shelah [She09a, II 1.14] that a model is saturated iff it is model homogeneous. We adapt the proof in [Gro21, Chapter 5.2] which is more diagrammatic. It only uses AP.

Proposition 2.3.7. *Let $\lambda > \mathrm{LS}(\mathbf{K})$, $M \in K$. M is λ-saturated iff it is λ-model homogeneous.*

Proof. $\Leftarrow$: Let $N \leq M$ with $\|N\| < \|M\|$. Let $p = \mathrm{gtp}(a/N, N') \in S(N)$. By Löwenheim-Skolem axiom, we may assume $\|N'\| = \|N\|$. By model homogeneity, there is $f : N' \underset{N}{\to} M$. Then $f(a) \in M$ realizes p.

$\Rightarrow$: Let $N \leq M$, $N \leq N'$ both of size $\mu < \lambda$. Enumerate $N' = \langle a_i : i < \mu \rangle$. We define

- $\langle N_0^i \leq N_1^i \in K_\mu, f_i : N_0^i \xrightarrow[N]{} M : i \leq \mu \rangle$ increasing and continuous
- $N_0^0 := N$, $N_1^0 := N'$, $f_0 := \mathrm{id}_N$
- $a_i \in N_0^{i+1}$

If done, $f_\mu : N_0^\mu \xrightarrow[N]{} M$ with $N' \leq N_0^\mu$. We explain the construction of the successor stage:

1. Let $M_0^i := f_i[N_0^i] \leq M$. Since $N_0^i \leq N_1^i$, we can extend f_i to $g : N_1^i \cong_N M_1^i \geq M_0^i$. By λ-saturation of M, there is $b_i \in M$ realizing $\mathrm{gtp}(a_i/M_0^i, M_1^i)$. By Löwenheim-Skolem axiom, obtain $M_0^{i+1} \leq M$ such that $\|M_0^{i+1}\| = \mu$ and $\{b\} \cup M_0^i \subseteq M_0^{i+1}$.

$$\begin{tikzcd}
a_i \in N_1^i \arrow[r, "g", "\cong"'] & M_1^i & & M \\
& & M_0^{i+1} \ni b_i \arrow[ur] & \\
N_0^i \arrow[uu] \arrow[r, "f_i", "\cong"'] & M_0^i \arrow[uu] \arrow[ur] & &
\end{tikzcd}$$

2. Since b_i realizes $\mathrm{gtp}(a_i/M_0^i, M_1^i)$ (and AP), there are $M_1^{i+1} \in K_\mu$ and $h : M_1^i \xrightarrow[M_0^i]{} M_1^{i+1} \geq M_0^{i+1}$ with $h(g(a_i)) = b$.

$$\begin{tikzcd}
& & M_1^{i+1} & \\
a_i \in N_1^i \arrow[r, "g", "\cong"'] & M_1^i \arrow[ur, "h"] & & M \\
& & M_0^{i+1} \ni b_i \arrow[uu] \arrow[ur] & \\
N_0^i \arrow[uu] \arrow[r, "f_i", "\cong"'] & M_0^i \arrow[uu] \arrow[ur] & &
\end{tikzcd}$$

3. Since $h \circ g : N_2^i \xrightarrow[N]{} M_1^{i+1}$, we can extend it to an isomorphism $\widehat{h \circ g} : N_2^{i+1} \cong_N M_1^{i+1}$ where $N_2^{i+1} \geq N_1^{i+1}$. Define f_{i+1} to be the restriction of $\widehat{h \circ g}$ with codomain M_0^{i+1}. Call its domain N_0^{i+1} which contains a_i because $b \in M_0^{i+1}$.

$$
\begin{tikzcd}
& N_1^{i+1} \arrow[rr, "\widehat{h \circ g}", "\cong"'] & & M_1^{i+1} & \\
a_i \in N_1^i \arrow[ur] \arrow[rr, "g", "\cong"'] & & M_1^i \arrow[ur, "h"] & & M \\
& a_i \in N_0^{i+1} \arrow[uu, dotted] \arrow[rr, dotted, "f_{i+1}", "\cong"'] & & M_0^{i+1} \ni b_i \arrow[uu] \arrow[ur] & \\
N_0^i \arrow[uu] \arrow[ur] \arrow[rr, "f_i", "\cong"'] & & M_0^i \arrow[uu] \arrow[ur] & &
\end{tikzcd}
$$

□

Remark 2.3.8. In the proof we fixed N and built $\langle M_0^i : i < \mu \rangle$ inside M such that M_0^{i+1} realizes a specific type over M_0^i. Therefore, the same construction can take place if $M = \bigcup_{i<\mu} M^i$ such that $M^0 := N$ and M^{i+1} realizes *all* types of M^i.

A weaker version of saturated models is called limit models.

Definition 2.3.9. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$, $N \leq M \in K_\lambda$, $\delta < \lambda^+$ be a limit ordinal. M is *(λ, δ)-limit over* N if there is $\langle M_i \in K_\lambda : i \leq \delta \rangle$, $M_0 = N$, $M_\delta = M$ and M_{i+1} is universal over M_i for all i.

Sometimes we may not be able to construct a saturated model in a specific cardinality, but under stability assumption, limit models exist.

Proposition 2.3.10. *Let K be λ-stable, $N \in K_\lambda$, $\delta < \lambda^+$. There is a (λ, δ)-limit model M over N.*

Proof. It suffices to show that there is a universal model M over N in K_λ. We define $\langle M_i \in K_\lambda : i < \lambda \rangle$, with $M_0 = N$, M_{i+1} realizing all types over M_i (which is possible by stability). Then $M := M_\lambda$ is universal over N by Remark 2.3.8 (replace μ there by λ). □

While saturated models of the same size are isomorphic by a back-and-forth argument, a similar version can be said for limit models (over the same model) of the same length.

Proposition 2.3.11. *Let $\delta < \lambda^+$. Let M^1, M^2 be (λ, δ)-limit models over N, witnessed by $\langle M_i^l : i \leq \delta \rangle$, $l = 1, 2$. Then there are $\langle f_i : M_{2i}^1 \underset{N}{\rightarrow} M_{2i+1}^2 : i < \delta \rangle$ and $\langle g_i : M_{2i+1}^2 \underset{N}{\rightarrow} M_{2i+2}^1 : i < \delta \rangle$ such that*

- *Both f_i, g_i are increasing and continuous.*

- *${f_i}^{-1} \subseteq g_i$ and ${g_i}^{-1} \subseteq f_{i+1}$ (where we restrict the codomains of f_i and g_i to be isomorphic to their domains).*

- *$f_\lambda : M^1 \cong_N M^2$ and $g_\lambda = {f_\lambda}^{-1}$*

Proof. The last point is guaranteed by the first two points. So it suffices to define the successor stages. We assume f_i, g_i are defined and continue to build f_{i+1}. The case for g_{i+1} will be symmetric.

Consider $g_i : M^2_{2i+1} \cong_N g_i[M^2_{2i+1}] \leq M^1_{2i+2}$, we can build an isomorphic copy of M^1_{2i+2} over M^2_{2i+1}. Namely, we extend ${g_i}^{-1}$ to $h : M^1_{2i+2} \cong_N M$ for some $M \geq M^2_{2i+1}$. As M^2_{2i+3} is universal over M^2_{2i+1}, there is $k : M \xrightarrow[M^2_{2i+1}]{} M^2_{2i+3}$. Define $f_{i+1} := k \circ h : M^1_{2i+2} \xrightarrow[N]{} M^2_{2i+3}$, which extends ${g_i}^{-1}$ because h does so and k does not change M^2_{2i+1}.

$$\begin{tikzcd}
 & & M^2_{2i+3} \\
M^1_{2i+2} \arrow[r, "h"] \arrow[urr, bend left, "f_{i+1}"] & M \arrow[ur, "k"] & \\
g_i[M^2_{2i+1}] \arrow[u] & & M^2_{2i+1} \arrow[ll, "g_i", "\cong"'] \arrow[ul] \arrow[uu, dash] \\
M^1_{2i+1} \arrow[u, dash] & &
\end{tikzcd}$$

□

In the above proof we require the subscripts of M^1 and M^2 to keep increasing under f_i and g_i, but it is just for convenience (we could have required $g_i : M^2_{2i+1} \xrightarrow[N]{} M^1_{2i+1}$ for example). Also, the construction only depends on the cofinality of δ, so we can state a slightly stronger version:

Proposition 2.3.12. *Let $\delta_1, \delta_2 < \lambda^+$ with $\mathrm{cf}(\delta_1) = \mathrm{cf}(\delta_2)$. Let M^l be (λ, δ_l)-limit models over N for $l = 1, 2$. Then $M^1 \cong_N M^2$.*

Proof. For $l = 1, 2$, obtain witnesses of limit models $\langle M^l_i : i \leq \delta_l \rangle$ and reduce the chain to $\langle M^l_i : i \leq \mathrm{cf}(\delta_l) \rangle$. Then apply the previous proof. □

2.4 MONSTER MODEL

In the definition of a Galois type, say $p = \mathrm{gtp}(a/M, N)$, we need to specify the bigger model N which contains both a and M. It can get quite tedious to keep track of the bigger models all the time when we discuss many types in a proof. The same can be said for the ambient models in nonforking/independence relations. When we have a *monster model*, it saves us a lot of book-keeping and shows the idea of the proof more clearly.

Definition 2.4.1. A monster model $\mathfrak{C} \in K$ is a sufficiently saturated and universal model such that every model $M \in K$ considered in a proof satisfies $\|M\| < \|\mathfrak{C}\|$.

If we need to prove a statement P_μ for a proper class of cardinals μ, we simply build a proper class of monster models of increasing saturation. Then to prove a specific P_μ, we work in a saturated and universal enough monster model.

The existence of monster models is guaranteed by $AP + JEP + NMM$, which we always assume in this paper. We state a relevant result:

Proposition 2.4.2. *Assume K satisfy AP. The following are equivalent:*

1. *For every $\kappa > \mathrm{LS}(\mathbf{K})$, there is a κ-saturated and κ-universal model.*

2. *K satisfies $JEP + NMM$.*

Proof. $\Leftarrow$: Pick any $M_0 \in K_{2^\kappa}$, we extend M_0 to a κ-saturated model: use Proposition 2.2.9 and AP recursively to define

- $\langle M_i \in K_{2^\kappa} : 2 \leq i \leq \kappa^+ \rangle$ increasing and continuous.

- For $2 \leq i \leq \kappa$, for $N \leq M_i$ with $|N| < \kappa$, M_{i+1} realizes all types in $S(N)$.

We show that M_{κ^+} is κ-saturated. Pick any $N \leq M_{\kappa^+}$ of size less than κ. As $\mathrm{cf}(\kappa^+) \geq \kappa$, we can find $i < \kappa^+$ such that $N \leq M_i$ (using coherence axiom). But types in $S(N)$ are already realized in $M_{i+1} \leq M_{\kappa^+}$.

By Proposition 2.3.7, M_{κ^+} is κ-model homogeneous. By Proposition 2.3.5, it is also κ-universal.

$\Rightarrow$: Let $M_1, M_2 \in K_{<\kappa}$ for some κ. Pick a κ-saturated and κ-universal model $\mathfrak{C}$. Then there are $g_i : M_i \to \mathfrak{C}$ for $i = 1, 2$. Fix $a \in \mathfrak{C} - g_1[M_1]$. By Löwenheim-Skolem axiom, we can find $N \in K_{<\kappa}$ such that $N \leq \mathfrak{C}$, $\{a\} \cup g_1[M_1] \cup g_2[M_2] \subseteq N$, which witnesses JEP. Since $a \in g_1[M_1]$, $g_1[M_1] < N$. Extend g_1 to an isomorphism g_1' of codomain N. Then ${g_1'}^{-1}[N]$ is a proper extension of M_1. $\square$

In the right-to-left direction, we could have replaced 2^κ and κ^+ by some $\lambda = \lambda^{<\kappa}$ (which guarantees $\mathrm{cf}(\lambda) \geq \kappa$ so we can still proceed with the cofinality argument).

If we do not assume AP, we need to be careful of the definition of saturation (whether an element realizes a type by $\sim$ or by $\equiv$ as in Definition 2.2.1). Also, the left-to-right direction of the proof of Proposition 2.3.7 breaks down if we do not assume AP. An alternative is to replace "saturated" by "model homogeneous" in Definition 2.4.1. We stick to "saturated" because we will assume AP and the notion is closer to the first-order version.

When we work in a monster model, a good practice is to verify whether every notion makes sense inside a monster model. For example, types work fine because AP already implies the equality of types over different ambient models, which come from a two-dimensional amalgam. If we want to consider multidimensional amalgamations/independence relations, AP (over single models) is not sufficient (see [SV18b] or Chapter 6). Either one works harder (perhaps with non-ZFC axioms) to derive a higher dimensional AP, or one simply assumes stronger forms of amalgamation in place of the usual AP.

2.5 DIRECTED AND COHERENT SYSTEMS

When building a resolution of a model M, say $\langle M_i : i < \|M\| \rangle$, we use Löwenheim-Skolem axiom to get models of increasing sizes. One can also break down a model into small models of the same size, via a directed system. Readers can also consult [She09a, II 1.23] and [Gro21, Chapter 2 Theorem 2.7].

Definition 2.5.1. Let $\langle I, \leq \rangle$ be a preorder. It is *directed* if for any $r, s \in I$, there is $t \in I$ with $t \geq r$ and $t \geq s$.

Proposition 2.5.2. *Let $M \leq N \in K$. There is a directed system $\langle I, \leq \rangle$ and $\langle M_i : i \in I \rangle$*

such that

1. $M_i \in K_{\mathrm{LS}(\mathbf{K})}$ *for all* $i \in I$.

2. *If* $i \leq j$ *in* I, *then* $M_i \leq M_j \leq N$.

3. $M = \bigcup_{i \in I} M_i$

Conversely, if $\langle I, \leq \rangle$, *and* $\langle M_i : i \in I \rangle$ *satisfy the above points, then* $M \in K$ *and* $M \leq N$.

Proof. For the forward direction, let $\langle A_i : i \in I \rangle$ list the finite subsets of M and order them by inclusion: $i \leq_I j$ iff $A_i \subseteq A_j$. We define M_i inductively on the cardinality of A_i. When $|A_i| = 1$, apply Löwenheim-Skolem axiom to obtain $M_i \in K_{\mathrm{LS}(\mathbf{K})}$, $M_i \leq M$, $M_i \supseteq A_i$. When $|A_i| > 1$, we do the same with the additional requirement that $M_i \supseteq \bigcup\{M_j : j <_I i\}$. M_i is still in $K_{\mathrm{LS}(\mathbf{K})}$ because there are finitely many $j <_I i$. Coherence guarantees $M_j \leq M_i$ for $j < i$. $M \subseteq \bigcup\{M_i : i \in I\}$ because each $a \in M$ is inside the singleton $\{a\}$, which is listed as some $A_i \subseteq M_i$.

For the reverse direction, we prove by induction. If $|I|$ is finite, then $M = M_{\max(I)} \leq N$ and is in $K_{\mathrm{LS}(\mathbf{K})}$. If $|I| = \aleph_0$, then we can write $I = \omega$ (perhaps with a different ordering than the usual one). Obtain a function $f : \omega \to I$ such that $f(0) = 0$ and $f(i) \geq_I f(i-1)$ and $f(i) \geq_I i$ for $i \geq 1$. Then $\langle M_{f(i)} : i < \omega \rangle$ is an increasing chain and the chain axioms guarantee $M \in K_{\mathrm{LS}(\mathbf{K})}$, $M \leq N$. Also $M_i \leq M_{f(i)} \leq M$.

Inductively assume the reverse direction is true for $|I| < \aleph_\alpha$ where $\alpha \geq 1$. Let $|I| = \aleph_\alpha$. Build $\langle I_k : k < \omega_\alpha \rangle$ such that

- $|I_k| < \aleph_\alpha$

- I_k is increasing and continuous in k

- I_k is directed suborder of I

- $I = \bigcup_{k < \omega_\alpha} I_k$

The construction is possible, for example, by applying first-order Löwenheim-Skolem Theorem to $\langle I, \leq \rangle$. By inductive hypothesis, $M_k := \bigcup_{i \in I_k} M_i \in K$ and $M_k \leq N$. When

$k < k' < \omega_\alpha$, the same argument (by taking $N = M_{k'}$) gives $M_k \leq M_{k'}$. Hence $\langle M_k : k < \omega_\alpha \rangle$ is increasing and continuous. By chain axioms, $M' := \bigcup_{k<\omega_\alpha} M_k \in K$ and $M' \leq N$. It remains to check $M = M'$: both are the union of M_i over all $i \in I$. $\square$

Remark 2.5.3.

1. Since for any $\kappa \geq \mathrm{LS}(\mathbf{K})$, $\langle K_{\geq\kappa}, \leq_{\mathbf{K}} \rangle$ is still an AEC with $LS(\langle K_{\geq\kappa}, \leq_{\mathbf{K}} \rangle) = \kappa$, we could have broken down M into a directed system of models of size κ instead.

2. The proposition also shows that $K_{\mathrm{LS}(\mathbf{K})}$ determines $K_{\geq \mathrm{LS}(\mathbf{K})}$. In particular, $M \leq N \in K_{\geq \mathrm{LS}(\mathbf{K})}$ iff there are directed systems $\langle M_i : i \in I \rangle$ and $\langle N_j : j \in J \rangle$ for M, N respectively such that for any M_i there is some $N_j \geq M_i$.

In first-order theories, we can take the union of a chain of types, which is still consistent by compactness. In AECs, as types are not syntactic, we need to build *coherent systems* to extend types beyond the limit stage (for successor stage, we can simply use AP). We follow the formulation in [Bon14a, Section 5]. See also [Bal09, Chapter 11] but the arrows there point in backward directions.

Definition 2.5.4. Let α be an ordinal, $\langle M_i : i < \alpha \rangle$ and $\mathbf{p} = \langle p_i \in \mathrm{gS}(M_i) : i < \alpha \rangle$ be increasing. We say $\mathbf{p}$ is *coherent* when

- For each $i < \alpha$, there are some a_i and N_i such that $p_i = \mathrm{gtp}(a_i/M_i, N_i)$.
- There are $\langle f_{i,j} : N_i \xrightarrow[M_i]{} N_j \mid i < j < \alpha \rangle$ with $f_{i,j}(a_i) = a_j$ for $i < j < \alpha$.
- For $i < j < k < \alpha$, $f_{j,k} \circ f_{i,j} = f_{i,k}$.

We say $\langle a_i, f_{i,j} : i < j < \alpha \rangle$ is a *coherent system* witnessing the coherence of $\mathbf{p}$.

We first show that a type extension generates a coherent system.

Proposition 2.5.5. *If there is $p_\alpha \in \mathrm{gS}(M_\alpha)$ extending all $p_i \in \mathrm{gS}(M_i)$ where $M_\alpha \geq \bigcup_{i<\alpha} M_i$, then it generates a coherent system $\langle f_{i,j} : i < j \leq \alpha \rangle$ for $\mathbf{p} \cup \{p_\alpha\}$.*

Proof. Let $p_i = \mathrm{gtp}(a_i/M_i, N_i)$ for $i < \alpha$. Recursively build $\langle g_i, N_i' : i < \alpha \rangle$ such that

1. N'_i is increasing in i (perhaps not continuous)

2. $a_\alpha \in N'_i$ for all $i < \alpha$.

3. $g_i : N_i \cong_{M_i} N'_i$, $g(a_i) = a_\alpha$

$$\begin{tikzcd}
 & N'_0 \arrow[r] & N'_1 \arrow[r] & N'_2 \arrow[r, dotted, "\bigcup"] & N''_\gamma \arrow[r] & N'_\gamma \arrow[r, dotted] & {} \\
a_\alpha \in N_\alpha \arrow[ur] & N_0 \ni a_0 \arrow[u, "g_0", "\cong"'] & N_1 \ni a_1 \arrow[u, "g_1", "\cong"'] & N_2 \ni a_2 \arrow[u, "g_2", "\cong"'] & & N_\gamma \ni a_\gamma \arrow[u, "g_\gamma", "\cong"'] & \\
M_\alpha \arrow[u] & M_0 \arrow[l] \arrow[r] \arrow[u] & M_1 \arrow[r] \arrow[u] & M_2 \arrow[rr, dotted] \arrow[u] & & M_\gamma \arrow[u] \arrow[r, dotted] & {}
\end{tikzcd}$$

For base stage and successor stage, we use type equality of a_α and a_i. Expand N_i so that g_i is an isomorphism. For limit stage, let γ be a limit ordinal. We define $N''_\gamma := \bigcup_{i<\gamma} N'_i$ and apply type equality to $a_\alpha \in N''_\gamma$ and $a_\gamma \in N_\gamma$. Expand N_γ so that g_γ is an isomorphism.

We can read $\langle f_{i,j} : i < j \leq \alpha \rangle$ from the diagram: for $i < j < \alpha$, let $f_{i,j} := {g_j}^{-1} \circ g_i$. $\square$

Remark 2.5.6. If we also want $f_{i,\alpha}$ or even $f_{i,j}$ for $j < i$, it suffices to replace N_α and all N'_i in the above proof by $\bigcup_{i<\alpha} N'_i$. Then expand each N_i to maintain g_i be isomorphisms.

It is always possible to extend an omega chain of types even without using coherent systems:

Proposition 2.5.7. *Let $\langle p_i \in \mathrm{gS}(M_i) : i < \omega \rangle$ be increasing. Then there is $p_\omega \in \mathrm{gS}(\bigcup_{i<\omega} M_i)$ extending all p_i. Moreover, we can define $\langle f_{i,j} : i < j \leq \omega \rangle$ to give a coherent system.*

Proof. Let $M_\omega := \bigcup_{i<\omega} M_i$. We build the following (part (b)'s are optional):

$$\begin{tikzcd}
 & & N'_2 \arrow[r] & N'_3 \arrow[r, dotted] & N'_\omega \\
 & N'_1 = N_{01} \arrow[ur] & N_{12} \arrow[u, "g_2"] & N_{23} \arrow[u, "g_3"] & N_\omega \ni a_\omega \arrow[u, "\cong"] \\
a_0 \in N'_0 = N_0 \arrow[ur, "g_0"] & a_1 \in N_1 \arrow[u, "g_1"] \arrow[ur, "h_{12}"] & a_2 \in N_2 \arrow[u] \arrow[ur, "h_{23}"] & a_3 \in N_3 \arrow[u] & \\
M_0 \arrow[r] \arrow[u] & M_1 \arrow[r] \arrow[u] & M_2 \arrow[r] \arrow[u] & M_3 \arrow[r, dotted] \arrow[u] & M_\omega \arrow[uu] \arrow[uuu, bend right, "\bigcup g_i \restriction M_i"']
\end{tikzcd}$$

1. $N_0' := N_0$

2. (a) By type equality obtain $g_1 : N_1 \underset{M_0}{\longrightarrow} N_{01}$ with $N_{01} \geq N_0'$ and $g_1(a_1) = a_0$. Set $N_1' := N_{01}$, $g_0 := \mathrm{id} : N_0' \to N_1'$.

   (b) Expand N_1 so that g_1 is an isomorphism.

   For $i \geq 1$,

3. (a) By type equality obtain $h_{i,i+1} : N_i \underset{M_i}{\longrightarrow} N_{i,i+1}$ with $N_{i,i+1} \geq N_{i+1}$ and $h_{i,i+1}(a_i) = a_{i+1}$.

   (b) Replace N_{i+1} by $N_{i,i+1}$.

4. (a) By AP, obtain $g_{i+1} : N_{i,i+1} \to N_{i+1}'$ with $N_{i+1}' \geq N_i'$ and $g_i \restriction N_i = g_{i+1} \circ h_{i,i+1} \restriction N_i$.

   (b) Expand $N_{i,i+1} (= N_{i+1})$ so that g_{i+1} is an isomorphism.

5. Let $N_\omega' := \bigcup_{i<\omega} N_i'$. Define $\bigcup_{i<\omega} g_i \restriction M_i : M_\omega \to N_\omega'$. Extend the embedding to an isomorphism $N_\omega \cong N_\omega'$. Let $a_\omega := (\bigcup_{i<\omega} g_i \restriction M_i)^{-1}(a_0) \in N_\omega$.

We verify that $\bigcup_{i<\omega} g_i \restriction M_i$: is well-defined and $p_\omega := \mathrm{gtp}(a_\omega/M_\omega, N_\omega)$ is the desired extension. Since the diagram commutes, $g_0 \restriction M_0 = g_1 \restriction M_0$. For $i \geq 1$, $g_{i+1} \circ h_{i,i+1} \restriction N_i = g_i \restriction N_i$ by (4) and $N_i \geq M_i$, so it suffices to check $h_{i,i+1}$ fixes M_i, which is true by (3). Chasing the diagram via the northwest and southeast routes from each M_i, we see that $p_\omega \supseteq p_i$.

By Proposition 2.5.5 and the remark that follows, we can derive $\langle f_{i,j} : i < j \leq \omega \rangle$. Alternatively, we could have included part (b)'s in the above steps, and set

- $f_{i,j} := h_{i,j}$ for $1 \leq i < j < \omega$.
- $f_{0,j} := {g_j}^{-1}$ for $0 < j < \omega$.
- $f_{i,\omega} := (\bigcup_{i<\omega} g_i \restriction M_i)^{-1} \circ g_i$ for $i < \omega$.

which gives a similar diagram as in the previous proposition. $\square$

Remark 2.5.8.

- The proof in [Bal09, Theorem 11.1] works in $\mathfrak{C}$, and define g_i to be i-many compositions of automorphisms of $\mathfrak{C}$. There one should be careful of restricting g_i to M_i before taking the union. Here we do not use any monster model to avoid sending embeddings to automorphisms of $\mathfrak{C}$ and then restricting them to smaller models.

- We could have combined the last statement of the proposition and the next result to find p_ω, but here the coherent system comes after p_ω is obtained.

Beyond omega stage, we need coherent systems to extend types.

Proposition 2.5.9. *Let* $\mathbf{p}$ *be given as in Definition 2.5.4. Then there is* $p_\alpha \in \mathrm{gS}(M_\alpha)$ *extending all* p_i *for* $i < \alpha$*. We can also extend the coherent system to the* α*-stage.*

We call $\langle a_\alpha, f_{i,\alpha} : i < \alpha \rangle$ the *direct limit* (or more accurately the *directed colimit*) of the system.

Proof. If α is a successor, then we can apply AP to get $a_\alpha \in N_\alpha \geq M_\alpha$ and $f_{\alpha-1,\alpha} : N_{\alpha-1} \xrightarrow[M_{\alpha-1}]{} N_\alpha$ which sends $a_{\alpha-1}$ to a_α. Then $p_\alpha := \mathrm{gtp}(a_\alpha/M_\alpha, N_\alpha)$ extends $p_{\alpha-1}$ which extends the rest of the types. For $i < \alpha - 1$, we can define $f_{i,\alpha} := f_{\alpha-1,\alpha} \circ f_{i,\alpha-1}$.

$$\begin{tikzcd}
{} \arrow[r, dotted] & a_{\alpha-1} \in N_{\alpha-1} \arrow[r, "f_{\alpha-1,\alpha}"] & N_\alpha \ni a_\alpha \\
{} \arrow[r, dotted] & M_{\alpha-1} \arrow[u] \arrow[r] & M_\alpha \arrow[u]
\end{tikzcd}$$

If α is a limit, there are two cases: $M_\alpha = \bigcup_{i<\alpha} M_i$ or $M_\alpha > \bigcup_{i<\alpha} M_i$. In the first case, we build

- $\langle N'_i : i < \alpha \rangle$ increasing, $N'_0 := N_0$, $N' := \bigcup_{i<\alpha} N'_i$.

- $\langle f_{i,0} : N_i \cong N'_i : i < \alpha \rangle$. Write $f_{0,i} := {f_{i,0}}^{-1}$ which extends $f_{j,i} \circ f_{0,j}$ coherently for any $0 < j < i$.

$$\begin{tikzcd}
N'_0 \arrow[r] & N'_1 \arrow[r] & N'_2 \arrow[r, dotted] & {} \\
N_0 \arrow[u, equal] \arrow[r, "f_{0,1}"] & N_1 \arrow[u, "f_{1,0}", "\cong"'] \arrow[r, "f_{1,2}"] & N_2 \arrow[u, "f_{2,0}", "\cong"'] \arrow[r, dotted] & {} \\
M_0 \arrow[u] \arrow[r] & M_1 \arrow[u] \arrow[r] & M_2 \arrow[u] \arrow[r, dotted] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
N'_i \arrow[r] \arrow[d, "f_{0,i}"', "\cong"] & N'_{i+1} \arrow[d, dashed, "f_{0,i+1}"', "\cong"] \\
N_i \arrow[r, "f_{i,i+1}"] & N_{i+1} \\
M_i \arrow[u] \arrow[r] & M_{i+1} \arrow[u]
\end{tikzcd}$$

Suppose stage i is finished, then we have $f_{i,0} : N_i \cong N'_i$. Since $N'_i \cong N_i \cong f_{i,i+1}[N_i] \leq N_{i+1}$, we can extend this isomorphism to $f_{0,i+1} : N'_{i+1} \cong N_{i+1}$. As $f_{j,i} \circ f_{0,j}$ is extended by $f_{0,i}$ for $j < i$ and $f_{i,i+1} \circ f_{0,i} = f_{0,i+1} \restriction N'_{i+1}$, we obtain $f_{j,i+1} \circ f_{0,j} = f_{i,i+1} \circ (f_{j,i} \circ f_{0,j})$ is extended by $f_{0,i+1}$. Coherence is also guaranteed by $f_{i,i+1} \circ f_{0,i} = f_{0,i+1} \restriction N'_{i+1}$.

Suppose stage j is finished for each $j < i$ where $i < \alpha$ is a limit ordinal. Consider $f_{j,i} \circ f_{0,j} : N'_j \cong f_{j,i}[N_j] \leq N_i$ for $j < i$, we check that it is increasing in j. Let $k < j < i$, we have

$$\begin{aligned} f_{k,i} \circ f_{0,k} \restriction N'_k &= (f_{j,i} \circ f_{k,j}) \circ f_{0,k} \restriction N'_k \quad \text{by coherence of the original system} \\ &= f_{j,i} \circ (f_{k,j} \circ f_{0,k}) \restriction N'_k \\ &= f_{j,i} \circ f_{0,j} \restriction N'_k \quad \text{by coherence in inductive hypothesis} \end{aligned}$$

Therefore the following is well-defined:

$$k := \bigcup_{j<i} (f_{j,i} \circ f_{0,j}) : \bigcup_{j<i} N'_j \cong \bigcup_{j<i} f_{j,i}[N_j] \leq N_i$$

Extend this isomorphism to $f_{0,i} : N'_i \cong N_i$, where $N'_i \geq \bigcup_{j<i} N'_j$. This completes the recursive construction.

$$\begin{tikzcd}
N'_k \arrow[r] \arrow[d, "f_{0,k}"'] & N'_j \arrow[r, dotted] \arrow[d, "f_{0,j}"'] & \bigcup_{j<i} N'_j \arrow[r] \arrow[dr, "k"] & N'_i \arrow[r, dotted, "\cup"] \arrow[d, "f_{0,i}"] & N' \arrow[dr, "f_{0,\alpha}"] & \\
N_k \arrow[r, "f_{k,j}"] \arrow[rrr, bend right, "f_{k,i}"] & N_j \arrow[rr, "f_{j,i}"] & & N_i & & N_\alpha \ni a_\alpha \\
M_k \arrow[r] \arrow[u] & M_j \arrow[rr, dotted] \arrow[u] & & M_i \arrow[r, dotted] \arrow[u] & M_\alpha \arrow[uu, "f"] \arrow[ur] &
\end{tikzcd}$$

Consider $f := \bigcup_{i<\alpha} f_{i,0} \restriction M_i : M_\alpha \to N'$. As in the limit step above, f is well-defined. Extend f to an isomorphism $f_{\alpha,0} : N_\alpha \to N'$. Write $f_{0,\alpha} := {f_{\alpha,0}}^{-1}$ and $a_\alpha := f_{0,\alpha}(a_0)$. The type $p_\alpha := \mathrm{gtp}(a_\alpha / M_\alpha, N_\alpha)$ extends all p_i because we can chase the diagram towards N'.

In the second case, reindex M_α to $M_{\alpha+1}$ and define new $M_\alpha := \bigcup_{i<\alpha} M_i$. Then apply the first case (to obtain N_α and a_α) and then the successor case. $\square$

Remark 2.5.10. • In [Bal09, Theorem 11.3(1)], since the arrows are backward there, one can skip to the last two paragraphs of the above proof. Here we have a *forward* coherent system and need to make sure that the backward direction constructed is also coherent (even when working in $\mathfrak{C}$).

- If we are given a chain of types $\langle p_i : i < \delta \rangle$ where $\delta > \omega$, it is not clear that the chain is witnessed by a coherent system. In other words, we do not know if p_ω is generated by the coherent system $\langle a_i, f_{i,j} : i < j < \omega \rangle$ constructed in Proposition 2.5.7. If we have a chain of non-forking types and the nonforking relation satisfies continuity and uniqueness, then we can guarantee p_γ is generated by the coherent system $\langle a_i, f_{i,j} : i < j < \gamma \rangle$ where γ is a limit ordinal [Bon14a, Proposition 5.2]. In Chapter 5 we make use of this idea to generalize superstability results by assuming continuity of nonsplitting (nonsplitting is used to construct a nonforking relation).

CHAPTER 3

HANF NUMBER OF THE FIRST STABILITY CARDINAL IN AECS

ABSTRACT

We show that $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ is the lower bound to the Hanf numbers for the length of the order property and for stability in stable abstract elementary classes (AECs). Our examples satisfy the joint embedding property, no maximal model, $(<\aleph_0)$-tameness but not necessarily the amalgamation property. We also define variations on the order and syntactic order properties by allowing the index set to be linearly ordered rather than well-ordered. Combining with Shelah's stability theorem, we deduce that our examples can have the order property up to any $\mu < \beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$. Boney conjectured that the joint embedding property is needed for two type-counting lemmas. We solved the conjecture by showing it is independent of ZFC. Using Galois Morleyization, we give syntactic proofs to known stability results assuming a monster model.

3.1 INTRODUCTION

Semantic order properties (Definition 3.3.2) in abstract elementary classes (AECs) are defined in terms of (semantic) Galois types instead of formulas. They are analogs to syntactic order properties in first-order and infinitary logics. In [She72], Shelah showed that in $L_{\lambda^+,\omega}$ the (syntactic) order property of length $\beth_{(2^\lambda)^+}$ implies the order property of arbitrary length. In [GS86], Grossberg and Shelah introduced the Hanf number of the order property of $L_{\lambda^+,\omega}$ and later [GS98, Theorem 2.8] gave a lower bound as $\beth_{\lambda^+}$. These bound the Hanf number of order property between $\beth_{\lambda^+}$ and $\beth_{(2^\lambda)^+}$. However, the example for the lower bound does not readily generalize to (semantic) order properties of AECs. Shelah [She99, Claim 4.6] hinted that the upper bound of the order property in AECs is $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ but it was not known whether it is tight. We present examples (Corollary 3.6.8) that $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ is exact. Our examples satisfy the joint embedding property, no maximal model and $(<\aleph_0)$-tameness but the amalgamation property fails. It is open whether the bound can be lowered when one assumes the amalgamation property.

Vasey [Vas16c] extended Shelah [She71], Grossberg and Lessmann's [GL02] results to

AECs and showed that assuming the amalgamation property and tameness, the first stability cardinal is bounded above by $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$. It is open whether this bound can be lowered under the amalgamation property. Our examples, which do not satisfy the amalgamation property, show that the lower bound in general is at least $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$. From instability, we can apply Vasey's techniques (which are based on [She09b, V.A.]) to derive the order property. This provides an alternative way other than finding the witnesses directly. It is open whether the amalgamation property can lower the bound for the first stability cardinal.

Vasey's result above relies on one direction of [Bon17, Theorem 3.1], which does not use the joint embedding property. The other direction involves lemmas that assume the joint embedding property, which Boney suspected to be necessary. As a side product of our construction, we show in Corollary 3.4.4 that the need for the joint embedding property is independent of ZFC; and we find an example and a counterexample under different set theoretic assumptions.

In Section 2, we state our notations and definitions. We also give a shorter proof to Boney's result to motivate Corollary 3.4.4. In Section 3, we review results concerning stability and the order property. We give more details for the proof of [She99, Claim 4.6]. In Section 4, we construct our main examples in Proposition 3.4.1 which set a lower bound to stability and a variation of the order property for stable AECs. The variation of the order property is slightly more general by allowing the index set to be linear ordered rather than well-ordered. We will also show Corollary 3.4.4 as a side product of our construction. In Section 5, we apply the same variation to the syntactic order property which can be combined with Galois Morleyization. We give analogs to Vasey's results with our variation on the order property. In Section 6, we write down the details of Vasey's observation [Vas16c, Fact 4.10] that Shelah's results in [She09b, V.A.] can be applied to AECs under Galois Morleyization. It allows us to deduce (Corollary 3.6.8) the order property up to $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ in our examples in Proposition 3.4.1, without finding explicit witnesses. We also apply such technique to bound the first stability cardinal under extra hypotheses. In Section 7, we use Galois Morleyization to recover common stability results where types can be over sets under $\mathrm{LS}(\mathbf{K})$. Vasey in [Vas16c, Section 5] has done similarly for coheir while we will work on splitting instead. In particular we prove Theorem 3.2.10 syntactically which is needed for Vasey's upper bound to the first stability cardinal, under the amalgamation property and tameness.

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3.2 PRELIMINARIES

We assume some familiarity with AECs, for example [Bal09, Chapter 4]. We will use $\kappa, \lambda, \mu, \chi$ to denote cardinals, α, β, γ to denote ordinals, n for natural numbers. We define κ^- to be the predecessor cardinal (if it exists) or κ itself. When we write $\alpha - n$, we assume $\alpha = \beta + n$ for some ordinal β.

Let $\mathbf{K} = \langle K, \leq_{\mathbf{K}} \rangle$ be an AEC. If the context is clear, we write $\leq$ in place of $\leq_{\mathbf{K}}$. We abbreviate by AP the *amalgamation property*, by JEP the *joint embedding property* and by NMM *no maximal model*. For $M \in K$, write $|M|$ the universe of M and $\|M\|$ the cardinality of M. For the set of Galois types (orbital types) of length $(< \alpha)$, we denote them by $\mathrm{gS}^{<\alpha}(\cdot)$ where the argument can be a set A in some model $M \in K$. In general $\mathrm{gS}^{<\alpha}(A) := \bigcup\{\mathrm{gS}^{<\alpha}(A; M) : M \in K, |M| \supseteq A\}$ (under AP, the choice of M does not matter). $\mathbf{K}$ is $(< \alpha)$*-stable in* λ if for any set A in some model $M \in K$, $|A| \leq \lambda$, then $|\mathrm{gS}^{<\alpha}(A; M)| \leq \lambda$. We omit "$(< \alpha)$" if $\alpha = 2$, while we omit "in λ" if there exists such a $\lambda \geq \mathrm{LS}(\mathbf{K})$. Similarly $\mathbf{K}$ is α*-stable in* λ if for any set A in some model $M \in K$, $|\mathrm{gS}^{\alpha}(A)| \leq \lambda$. Tameness will be defined in Definition 3.5.1. We allow stability and tameness under $\mathrm{LS}(\mathbf{K})$, especially in Section 3.7.

Given a $\forall\exists$ theory T and a set of $L(T)$-types Γ, $EC(T, \Gamma)$ is the class of models of T such that Γ is not realized by any elements. If we order $EC(T, \Gamma)$ by L-substructures, it forms an AEC with $\mathrm{LS}(\mathbf{K}) = |L(T)|$. $\delta(\lambda, \kappa)$ is the least ordinal δ such that: for any T, Γ with $|L(T)| \leq \lambda$, $|\Gamma| \leq \kappa$, $\{P, <\} \subseteq L(T)$ where P is a unary predicate, $<$ is a linear order on P, if there is a model $M \in EC(T, \Gamma)$ whose $(P^M, <^M)$ has order type $\geq \delta$, then there is a model $N \in EC(T, \Gamma)$ whose $(P, <)$ is not well-ordered.

Given a theory T, a set of $L(T)$-types Γ and a reduct L' of $L(T)$, $PC(T, \Gamma, L')$ is the class of L'-reducts of models of $EC(T, \Gamma)$.

Recall the classical theorem: notice in the proof that X can be a linear order while Y' can be its suborder.

Theorem 3.2.1 (Erdős-Rado Theorem)**.** *Let λ be an infinite cardinal. For $n < \omega$,*

$$\beth_n(\lambda)^+ \to (\lambda^+)^{n+1}_\lambda$$

In other words, for any $|X| \geq \beth_n(\lambda)^+$, any $f : [X]^{n+1} \to \lambda$, there is $X' \subseteq X$ such that $|X'| \geq \lambda^+$ and $f \restriction [X']^{n+1}$ is constant.

Proof. We adapt the proof in [Mar02, Theorem 5.1.4] because it does not require the set X to be well-ordered. We prove by induction: When $n = 0$, the statement is $\lambda^+ \to (\lambda^+)^1_\lambda$. Let X be of size $\geq \lambda^+$. We need to color its elements with λ-many colors. By pigeonhole principle, it is possible to find $X' \subseteq X$ of size $\geq \lambda^+$ such that $f \restriction X'$ is constant.

Assume the statement is true for $n - 1$. We need to show $\beth_n(\lambda)^+ \to (\lambda^+)^{n+1}_\lambda$. Let X be of size $\beth_n(\lambda)^+$, $f : [X]^{n+1} \to \lambda$. For $x \in X$, define $f_x : [X - \{x\}]^n \to \lambda$ by $f_x(Y) := f(Y \cup \{x\})$. We build $\langle X_\alpha : \alpha < \beth_{n-1}(\lambda)^+ \rangle$ increasing and continuous subsets of X such that for $\alpha < \beth_n(\lambda)^+$, $|X_\alpha| = \beth_n(\lambda)$. For the base step, take any $X_0 \subseteq X$ of size $\beth_n(\lambda)$. Suppose X_α is constructed, we build $X_{\alpha+1}$ satisfying:

1. $X_\alpha \subseteq X_{\alpha+1} \subseteq X$

2. $|X_{\alpha+1}| = \beth_n(\lambda)$

3. For any $Y \subseteq X_\alpha$ of size $\beth_{n-1}(\lambda)$, any $x \in X - Y$, there is $x' \in X_{\alpha+1} - Y$ such that $f_x \restriction [Y]^n = f_{x'} \restriction [Y]^n$.

The above is possible by a counting argument: the number of possible Y is

$$|X_\alpha|^{\beth_{n-1}(\lambda)} = \beth_n(\lambda)^{\beth_{n-1}(\lambda)} = (2^{\beth_{n-1}(\lambda)})^{\beth_{n-1}(\lambda)} = 2^{\beth_{n-1}(\lambda)} = \beth_n(\lambda).$$

Given Y, the number of possible $h : [Y]^n \to \lambda$ is bounded by

$$\lambda^{\beth_{n-1}(\lambda)} = 2^{\beth_{n-1}(\lambda)} = \beth_n(\lambda)$$

Therefore, it suffices to add $\beth_n(\lambda) \cdot \beth_n(\lambda) = \beth_n(\lambda)$-many witnesses from X to X_α. Define $X' = \bigcup\{X_\alpha : \alpha < \beth_{n-1}(\lambda)^+\}$. Notice that $|X'| = \beth_n(\lambda) < |X|$. For any $Y \subseteq X'$ of size $\beth_{n-1}(\lambda)$, by a cofinality argument $Y \subseteq X_\alpha$ for some $\alpha < \beth_{n-1}(\lambda)^+$. So for any $x \in X - Y$, there is $x' \in X_{\alpha+1} - Y \subseteq X' - Y$ such that $f_x \restriction [Y]^n = f_{x'} \restriction [Y]^n$.

Pick any $x \in X - X'$ and build $Y = \{y_\alpha : \alpha < \beth_{n-1}(\lambda)^+\} \subseteq X'$ such that $f_{y_\alpha} \restriction [\{y_\beta : \beta < \alpha\}]^n = f_x \restriction [\{y_\beta : \beta < \alpha\}]^n$ ($y_0 \in X'$ can be any element). By inductive hypothesis on Y and f_x, we can find $Y' \subseteq Y$ of size $\geq \lambda^+$ such that $f_x \restriction [Y']^n$ is constant. We check that Y' is as desired: let $A \in [Y']^{n+1}$ and write $A = \{y_{\alpha_1}, \ldots, y_{\alpha_{n+1}}\}$ where $\alpha_1 < \cdots < \alpha_{n+1} < \beth_{n-1}(\lambda)^+$.

$$f(A) = f_{y_{\alpha_{n+1}}}(A - \{y_{\alpha_{n+1}}\}) = f_x(A - \{y_{\alpha_{n+1}}\})$$

which is constant because f_x is constant on $[Y']^n \ni A - \{y_{\alpha_{n+1}}\}$. $\square$

The following Theorem 3.2.2 and Theorem 3.2.10 are only used in the proof of Corollary 3.6.6(1). We will streamline Boney's proof of Lemma 3.2.9 by omitting the ambient models (otherwise it would involve a lot of bookkeeping and direct limits). We will clarify the relationship between Lemma 3.2.9 and Theorem 3.2.2 and show that JEP in Theorem 3.2.2 is not needed. If we work in a monster model $\mathfrak{C}$, we can also allow stability over sets (of size $< \mathrm{LS}(\mathbf{K})$), but we keep the original formulation to state Remark 3.2.6 more clearly.

Theorem 3.2.2. *Let $\mathbf{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathbf{K})$. Suppose $\mathbf{K}$ has λ-AP and is stable in λ. For any ordinal $\alpha \geq 1$ with $\lambda^{|\alpha|} = \lambda$, $\mathbf{K}$ is α-stable in λ.*

The requirement $\lambda^{|\alpha|} = \lambda$ cannot be improved: let $\lambda^{|\alpha|} > \lambda$, take $\mathbf{K}$ be the well-orderings of type at most λ and $\leq_{\mathbf{K}}$ by initial segments. Then it is stable in λ because there are only λ-many elements in the unique maximal model (which witnesses AP). It is not α-stable because each element has different types, so the number of α-types is exactly $\lambda^{|\alpha|} > \lambda$.

We will prove the theorem through a series of lemmas. We may assume α to be a cardinal $\kappa = |\alpha|$. Denote $\mathrm{gS}^1_\lambda := \sup\{|\mathrm{gS}^1(M)| : M \in K, \|M\| = \lambda\}$ and similarly $\mathrm{gS}^\kappa_\lambda := \sup\{|\mathrm{gS}^\kappa(M)| : M \in K, \|M\| = \lambda\}$.

Lemma 3.2.3. *Suppose $\kappa \geq \lambda$, then $(\mathrm{gS}^1_\lambda)^\kappa = \mathrm{gS}^\kappa_\lambda = 2^\kappa$.*

Proof. $2^\kappa \leq (\mathrm{gS}^1_\lambda)^\kappa \leq (2^\lambda)^\kappa = 2^\kappa$. Pick any $\|M\| = \lambda$ and two distinct elements $\{a, b\}$ from $|M|$. Form binary sequences from $\{a, b\}$ of length κ, which shows $2^\kappa \leq \mathrm{gS}^\kappa_\lambda \leq 2^{\lambda+\kappa} = 2^\kappa$. $\square$

Lemma 3.2.4. [Bon17, Proposition 2.7] *Suppose $\kappa \leq \lambda$. If in addition $\mathbf{K}$ has λ-JEP and $\mathrm{cf}(\mathrm{gS}^\kappa_\lambda) \leq \lambda$, then there is $M \in K$, $\|M\| = \lambda$ such that $|\mathrm{gS}^\kappa(M)| = \mathrm{gS}^\kappa_\lambda$.*

Proof. Pick $\langle M_i : i < \mu\rangle$ (not necessarily increasing) witnessing $\mu := \mathrm{cf}(\mathrm{gS}^\kappa_\lambda) \leq \lambda$. By λ-AP and λ-JEP, obtain M of size λ such that $M \geq M_i$ for all M_i. $|\mathrm{gS}^\kappa(M)| \geq \sup_{i<\mu} |\mathrm{gS}^\kappa(M_i)| = \mathrm{gS}^\kappa_\lambda$. $\square$

Lemma 3.2.5. [Bon17, Theorem 3.2] *Suppose $\kappa \leq \lambda$. If in addition $\mathbf{K}$ has λ-JEP, then $(\mathrm{gS}^1_\lambda)^\kappa \leq \mathrm{gS}^\kappa_\lambda$.*

Proof. Given $M \in K$ of size λ. We show that $|\mathrm{gS}^\kappa(M)| \geq |\mathrm{gS}^1(M)|^\kappa$, which does not use λ-JEP. By λ-AP, pick $N \geq M$ (perhaps of size greater than λ) such that N realizes $\mathrm{gS}(M)$, say by $\langle a_i : i < |\mathrm{gS}(M)|\rangle$. Form sequences of length κ from the a_i, there are $|\mathrm{gS}(M)|^\kappa$-many sequences. They realize distinct types in $\mathrm{gS}^\kappa(M)$ by checking each coordinate.

Suppose $\mathrm{cf}(\mathrm{gS}^1_\lambda) \leq \kappa$, then $\mathrm{cf}(\mathrm{gS}^1_\lambda) \leq \lambda$. Substitute $\kappa = 1$ in Lemma 3.2.4 (which uses λ-JEP), there is $M^* \in K_\lambda$ such that $|\mathrm{gS}^1(M^*)| = \mathrm{gS}^1_\lambda$. Thus $(\mathrm{gS}^1_\lambda)^\kappa = |\mathrm{gS}^1(M^*)|^\kappa \leq |\mathrm{gS}^\kappa(M^*)| \leq \mathrm{gS}^\kappa_\lambda$.

Now suppose $\mathrm{cf}(\mathrm{gS}^1_\lambda) > \kappa$, then a cofinality argument shows the second equality below:

$$\begin{aligned}(\mathrm{gS}^1_\lambda)^\kappa :=& (\sup\{|\mathrm{gS}^1(M)| : M \in K, \|M\| = \lambda\})^\kappa \\ =& \sup\{|\mathrm{gS}^1(M)|^\kappa : M \in K, \|M\| = \lambda\} \\ \leq& \sup\{|\mathrm{gS}^\kappa(M)| : M \in K, \|M\| = \lambda\} =: \mathrm{gS}^\kappa_\lambda\end{aligned}$$

$\square$

Remark 3.2.6. After [Bon17, Proposition 2.7], Boney suggested that λ-JEP might be necessary. We will show in Corollary 3.4.4 that the need of λ-JEP is independent of ZFC for the above two lemmas.

Question 3.2.7. In [Bon17, Proposition 2.7], there is an alternative hypothesis to Lemma 3.2.4 where $\mathrm{cf}(\mathrm{gS}^{\kappa}_{\lambda}) \leq \lambda$ is replaced by a stronger assumption $I(K, \lambda) \leq \lambda$. Would λ-JEP be necessary in this case or is it again independent of ZFC? An answer would shed light on the relationship between stability and the number of nonisomorphic models.

Lemma 3.2.8. [Bon17, Theorem 3.5] *Suppose $\kappa \leq \lambda$, then $(\mathrm{gS}^1_{\lambda})^{\kappa} \geq \mathrm{gS}^{\kappa}_{\lambda}$.*

Proof. First we describe the proof strategy: for a fixed model M, we show that $\mathrm{gS}^{\kappa}(M)$ is bounded above by $(\mathrm{gS}^1_{\lambda})^{\kappa}$. To do so, we build a gS^1_{λ}-branching tree of models of height κ and list the possible 1-types of each model. For each κ-type in $\mathrm{gS}^{\kappa}(M)$, we map it injectively to a branch of the tree (which is a sequence in $(\mathrm{gS}^1_{\lambda})^{\kappa}$), according to the 1-types of the elements from that sequence.

Let $\mu := \mathrm{gS}^1_{\lambda}$. Fix an arbitrary $M \in K$ with $\|M\| = \lambda$. Write $\mathrm{gS}^{\kappa}(M) = \langle p_k : k < \chi \rangle$ where p_k are distinct. Fix $\bar{a}_k := \{a^{\alpha}_k : \alpha < \kappa\} \vDash p_k$. Construct a tree of models $\langle M_{\nu} \in K_{\lambda} : \nu \in \mu^{<\kappa} \rangle$ as follows: $M_{\langle\rangle} := M$, take union at limit stages. Suppose M_{ν} is built for some $\nu \in \mu^{<\kappa}$. Enumerate without repetition $\mathrm{gS}^1(M_{\nu}) = \langle q^{\nu}_i : i < \chi_{\nu} \rangle$ for some $\chi_{\nu} \leq \mu$. For $i < \chi_{\nu}$, define $M_{\nu^\frown i} \in K_{\lambda}$ with $M_{\nu^\frown i} \geq M_{\nu}$ and containing some $c^{\nu}_i \vDash q^{\nu}_i$. For $\chi_{\nu} \leq i < \mu$ (if there is any), give a default value to $M_{\nu^\frown i} := M_{\nu}$. Now we map each $p_k \in \mathrm{gS}^{\kappa}(M)$ to $\eta_k \in \mu^{\kappa}$ as follows: suppose $\nu := \eta_k \restriction \alpha$ has been defined for some $\alpha < \kappa$, we set $\eta_k[\alpha]$ to be the minimum $i < \chi_{\nu}$ (which is the same as requiring $i < \mu$) such that a^{α}_k realizes q^{ν}_i. In other words, we decide the α-th element of the branch based on the type of the α-th element of $\bar{a}_k$ over the current node (model).

It remains to check that the map is injective. Let $k < \chi$, we build $\langle f_{\alpha} : \alpha \leq \kappa \rangle$ increasing and continuous such that f_{α} maps a^{β}_k to $c^{\eta_k \restriction \beta}_{\eta_k[\beta]}$ for all $\beta \leq \alpha$ while fixing $|M|$. Take $f_{-1} := \mathrm{id}_M$ and we handle the successor case: suppose f_{α} has been constructed. There is $g : a^{\alpha}_k \mapsto c^{\eta_k \restriction \alpha}_{\eta_k[\alpha]}$ fixing $|M_{\eta_k \restriction \alpha}| \supseteq \left\{c^{\eta_k \restriction \beta}_{\eta_k[\beta]} : \beta < \alpha\right\}$ by type equality. Let $f_{\alpha+1} := g \circ f_{\alpha}$. Now f_{κ} witnesses that $\bar{a}_k$ and $\langle c^{\eta_k \restriction \alpha}_{\eta_k[\alpha]} : \alpha < \kappa \rangle$ realize the same type over M. Since the latter sequence only depends on the coordinates of η_k, our map $p_k \mapsto \eta_k$ is injective. Therefore, $|\mathrm{gS}^{\kappa}(M)| \leq \mu^{\kappa}$. Since M is arbitrary, $\mathrm{gS}^{\kappa}_{\lambda} \leq \mu^{\kappa} = (\mathrm{gS}^1_{\lambda})^{\kappa}$. $\square$

Lemma 3.2.9. [Bon17, Theorem 3.1] *Let $\mathbf{K}$ be an AEC with λ-AP and λ-JEP. Let $\kappa \geq 1$ be a cardinal. Then $(\mathrm{gS}^1_{\lambda})^{\kappa} = \mathrm{gS}^{\kappa}_{\lambda}$ where λ-JEP is used in the "$\leq$" direction.*

Proof. Combine Lemma 3.2.3, Lemma 3.2.5 and Lemma 3.2.8. $\square$

Proof of Theorem 3.2.2. Let $\kappa = |\alpha|$. By Lemma 3.2.9, $\lambda = \lambda^\kappa = (\mathrm{gS}^1_\lambda)^\kappa \geq \mathrm{gS}^\kappa_\lambda$ which shows that $\mathbf{K}$ is κ-stable in λ. By reordering the index κ, $\mathbf{K}$ is α-stable in λ. $\square$

Theorem 3.2.10. [GV06b, Corollary 6,4] *Let μ be an infinite cardinal and $\mathbf{K}$ be an AEC with AP. If $\mathbf{K}$ is μ-tame and stable in μ, then $\mathbf{K}$ is stable in all $\lambda = \lambda^\mu$.*

The original proof proceeds semantically and we will give a syntactic proof in Section 3.7, allowing stability over sets which can be of size $< \mathrm{LS}(\mathbf{K})$.

3.3 STABILITY AND NO ORDER PROPERTY

To find the upper bound of the first stability cardinal in stable complete first-order theories, one possible way is to establish:

Fact 3.3.1 (Shelah)**.** Let T be a complete first-order theory. The following are equivalent:

1. T is stable.
2. For all $\lambda = \lambda^{|T|}$, T is stable in λ.
3. T has no (syntactic) order property of length ω.

$2^{|T|}$ is an upper bound for the first stability cardinal. Notice that in showing (3), compactness is used to stretch the order property to arbitrary length. In AECs, we can use the Hanf number to bound the (Galois) order property length. The following definition is based on [She99, Definition 4.3] and [Vas16c, Definition 4.3]:

Definition 3.3.2. Let μ be an infinite cardinal, $\alpha \geq 2$ and $\beta \geq 1$ be ordinals.

1. K has the *β-order property of length μ* if there exists some $\langle a_i : i < \mu \rangle \subseteq M \in K$ such that $l(a_i) = \beta$, and for $i_0 < i_1 < \mu$, $j_0 < j_1 < \mu$, $\mathrm{gtp}(a_{i_0} a_{i_1}/\emptyset, M) \neq \mathrm{gtp}(a_{j_1} a_{j_0}/\emptyset, M)$.
2. K has the *$(< \alpha)$-order property of length μ* if there is a $\beta < \alpha$ witnessing (1).
3. K has the *$(< \alpha)$-order property* if for all μ, K has the $(< \alpha)$-order property of length μ. In other words, if we fix μ, we can find a suitable β_μ witnessing (1).

4. K has the *no* $(<\alpha)$*-order property* if (3) fails. In other words, for each $\beta<\alpha$, there is an upper bound to the length of the β-order property. We omit $(<\alpha)$ if $\alpha=\omega$.

The above definition works fine if one wants an abstract generalization of the order property from the first-order version, in which case the length can be fixed at ω. However, in AECs, it is hard to construct long well-ordered sets without breaking stability or raising $\mathrm{LS}(\mathbf{K})$. We propose the following definition instead:

Definition 3.3.3. In Definition 3.3.2, we replace all occurences of "order property" by "order property*" if we also allow sequences indexed by linear orders instead of well-orderings. For example in (1), we say K has the β*-order property* of length* μ if there exist some linear order I, some $\langle a_i : i\in I\rangle \subseteq M\in K$ such that $|I|=\mu$, $l(a_i)=\beta$, and for $i_0<i_1$ in I, $j_0<j_1$ in I, $\mathrm{gtp}(a_{i_0}a_{i_1}/\emptyset, M)\neq \mathrm{gtp}(a_{j_1}a_{j_0}/\emptyset, M)$. When μ is omitted, we mean for all μ, there is a linear order I of cardinality μ witnessing the β-order property* of length μ.

In the following proposition, item (1) applies Morley's method [Mor65b] (see also [Bal09, Theorem A.3(2)]). The statement we use is from [She99, Claim 4.6] which only hinted at the proof of the Hanf number for arbitrarily large models [She90, VII.5]. We add more details and explain how to adapt that proof. The proof of item (3) adapts the proof from [BGKV16, Fact 5.13].

Proposition 3.3.4. *Let* $\mathbf{K}$ *be an AEC,* $\beta\geq 2$ *be an ordinal.*

1. *If for all* $\mu<\beth_{(2^{<(\mathrm{LS}(\mathbf{K})^{+}+|\beta|)})^{+}}$*,* $\mathbf{K}$ *has the* $(<\beta)$*-order property of length* μ*, then* $\mathbf{K}$ *has the* $(<\beta)$*-order property (and the* $(<\beta)$*-order property*).*

2. *If for all* $\mu<\beth_{(2^{<(\mathrm{LS}(\mathbf{K})^{+}+|\beta|)})^{+}}$*,* $\mathbf{K}$*,* $\mathbf{K}$ *has the* $(<\beta)$*-order property* of length* μ*, then* $\mathbf{K}$ *has the* $(<\beta)$*-order property* (and* $(<\beta)$*-order property).*

3. *If* $\mathbf{K}$ *is* $(<\beta)$*-stable (in some* $\lambda\geq \mathrm{LS}(\mathbf{K})+|\beta|$*), then there is* $\mu<\beth_{(2^{<(\mathrm{LS}(\mathbf{K})^{+}+|\beta|)})^{+}}$ *such that* $\mathbf{K}$ *has no* $(<\beta)$*-order property* (and thus no* $(<\beta)$*-order property) of length* μ*.*

Proof sketch. 1. We adapt the usual Hanf number argument. Suppose $\mathbf{K}$ has the $(<\beta)$-order property and we fix $\gamma < \beta$ such that $\mathbf{K}$ has the γ-order property. By Shelah's Presentation Theorem, we may write $K = PC(T, \Gamma_1, \mathrm{L}(\mathbf{K}))$ for some first-order theory T in $L \supseteq \mathrm{L}(\mathbf{K})$ and some sets of L-types Γ_1. Now we refer to the construction of [She90, VII Theorem 5.3] or [Gro21, Chapter 2 Theorem 6.35]. For each $\alpha < (2^{\mathrm{LS}(\mathbf{K})+|\gamma|})^+$, instead of defining $F^{\mathfrak{B}}(\alpha)$ to be some $M \in K$ of size $\beth_\alpha$, we demand it to be the witness of the γ-order property of length $\beth_\alpha$ (we can also add another function $F_1^{\mathfrak{B}}(\alpha, \cdot)$ to enumerate the elements of $F^{\mathfrak{B}}(\alpha)$). At the end of the construction (which uses Erdős-Rado Theorem), we obtain an L-indiscernible sequence (of γ-tuples) $\langle a_i : i < \omega\rangle$ such that for $n < \omega$, $i_1 < i_2 < \cdots < i_n < \omega$, the first-order type of $a_{i_1} \dots a_{i_n}$ is realized by some $d_1 \dots d_n$ that witness the order property. This induces an L-isomorphism between $EM(a_{i_1} \dots a_{i_n}) \cong EM(d_1 \dots d_n)$. Its reduct to $\mathrm{L}(\mathbf{K})$ is also an isomorphism between $EM(a_{i_1} \dots a_{i_n}) \restriction \mathrm{L}(\mathbf{K}) \cong EM(d_1 \dots d_n) \restriction \mathrm{L}(\mathbf{K})$. Since the right-hand-side witnesses the order property in K, so is the left-hand-side. The same argument applies when the indiscernible sequence is stretched to arbitrary length (or any linear order).

2. The same proof of (1) goes through because Erdős-Rado Theorem applies to linear orders (actually any sets) besides well-orderings.

3. Otherwise by (1)(2), $\mathbf{K}$ has the $(<\beta)$-order property*. For any infinite cardinal $\lambda \geq \mathrm{LS}(\mathbf{K}) + |\beta|$, let I_λ be a linear order of size $> \lambda$ such that it has a dense suborder J_λ of size λ. We stretch the indiscernible $\langle a_i : i < \omega\rangle$ in (2) to be indexed by I_λ. Pick $i < i'$ in I_λ, we can find k in J_λ such that $i < j < i'$. Then $\mathrm{gtp}(a_i a_j/\emptyset) \neq \mathrm{gtp}(a_{i'} a_j/\emptyset)$ are distinct by the order property, which means $\mathrm{gtp}(a_i/a_j) \neq \mathrm{gtp}(a_{i'}/a_j)$. This shows that $\mathbf{K}$ is $(<\beta)$-unstable in λ. As λ is arbitrary, $\mathbf{K}$ is $(<\beta)$-unstable above $\mathrm{LS}(\mathbf{K}) + |\beta|$.

□

Remark 3.3.5. • If we have a specific K in mind, we may replace $(2^{\mathrm{LS}(\mathbf{K})})^+$ by $\delta(\mathrm{LS}(\mathbf{K}), \kappa)$ where $\kappa := |\Gamma_1| \leq 2^{\mathrm{LS}(\mathbf{K})}$ in Shelah's Presentation Theorem.

- We used Galois types over the empty set in proving (2). The same proof goes through if we require the domain of the types to be some fixed (nonempty) model.

3.4 LOWER BOUND FOR STABILITY AND NO ORDER PROPERTY*

From Proposition 3.3.4, we saw that a stable AEC cannot have the order property* of length μ for some $\mu < \beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$. Our goal is to show that this is also a lower bound for no order property* as well as stability.

Proposition 3.4.1. *Let λ be an infinite cardinal and α be an ordinal with $\lambda \leq \alpha < (2^\lambda)^+$. Then there is a stable AEC $\mathbf{K}$ such that $\mathrm{LS}(\mathbf{K}) = \lambda$, $\mathbf{K}$ has the order property* of length up to $\beth_\alpha(\lambda)$ and is unstable anywhere below $\beth_\alpha(\lambda)$. Moreover, $\mathbf{K}$ has JEP, NMM and $(< \aleph_0)$-tameness but not AP.*

The proof of Proposition 3.4.1 will come after Corollary 3.4.4 and use Lemma 3.4.2 below.

Lemma 3.4.2. [She90, VII Theorem 5.5(6)] *Let λ be an infinite cardinal and α be an ordinal with $\lambda \leq \alpha < (2^\lambda)^+$. There is a stable AEC $\mathbf{K_1}$ such that $\mathrm{LS}(\mathbf{K_1}) = \lambda$ and for all $M \in K_1$, M is well-ordered of order type at most α. Moreover, $\mathbf{K_1}$ has AP, JEP and is $(< \aleph_0)$-tame.*

Proof. Let $L_1 := \{<, P_i : i < \lambda\}$ where P_i are unary predicates. T_1 requires $<$ to be a linear order. For $\beta < \alpha$, pick distinct subsets $S_\beta \subseteq \lambda$. Given an element x, we define its *type* by the set of indices i such that $P_i(x)$. We require that each element is characterized by its type and the only possible types are among $\{S_\beta : \beta < \alpha\}$. If $\beta < \alpha$ and x, y have types S_β and S_α respectively, then we stipulate that $x < y$. These will be coded in the collection of types Γ_1.

More precisely, for $\beta < \gamma < \alpha$, $S \subseteq \lambda$,

$$\begin{aligned}
p_{\beta,\gamma}(x, y) :=& \{\neg(x < y)\} \cup \{P_i(x) : i \in S_\beta\} \cup \{\neg P_i(x) : i \notin S_\beta\} \\
& \cup \{P_i(y) : i \in S_\gamma\} \cup \{\neg P_i(y) : i \notin S_\gamma\} \\
p_S(x) :=& \{P_i(x) : i \in S\} \cup \{\neg P_i(x) : i \notin S\} \\
\Gamma_1 :=& \{p_{\beta,\gamma}(x, y) : \beta < \gamma < \alpha\} \cup \{p_{S'} : S' \subseteq \lambda \text{ such that for } \beta < \alpha,\, S' \neq S_\beta\} \\
& \cup \{x \neq y \wedge P_i(x) \leftrightarrow P_i(y) : i < \lambda\}
\end{aligned}$$

Let $\mathbf{K_1} := EC(T_1, \Gamma_1)$ ordered by substructures. Notice that $|L(T_1)| = \lambda$ and $\Gamma_1 = 2^\lambda$. By replacing $\mathbf{K_1}$ by $(\mathbf{K_1})_{\geq\lambda}$, we may assume $\mathrm{LS}(\mathbf{K_1}) = \lambda$. Then $M_1 = \alpha$ is the maximal model (every model can be extended to an isomorphic copy of it) where for $\beta \in M_1$, $i < \lambda$, $P_i^{M_1}(\beta)$ iff $i \in S_\beta$. Hence $\mathbf{K_1}$ satisfies AP, JEP (but not NMM). As $\mathbf{K_1}$ has the maximal model of size $|\alpha| \leq 2^\lambda$, $\mathbf{K_1}$ is trivially stable in $\geq (2^\lambda)^+$. It is $(< \aleph_0)$-tame because types are decided by the P_i's they belong to.

□

Remark 3.4.3. By [She90, VII Theorem 5.5(2)], for any λ and $\kappa \leq 2^\lambda$, $\delta(\lambda, \kappa) \leq (2^\lambda)^+$, so the threshold $(2^\lambda)^+$ cannot be improved. If we restrict $1 \leq \kappa < 2^\lambda$, then for $\alpha < \lambda^+$, we can still define $\mathbf{K_1}$ to be well-orderings of type at most α, where models are ordered by initial segments. Then we get a lower bound $\delta(\lambda, \kappa) \geq \lambda^+$. But the above proof does not go through because it requires $|\Gamma| = 2^\lambda$.

Using Lemma 3.4.2, we are able to answer Boney's conjecture in Remark 3.2.6. We will complete the proof of Proposition 3.4.1 after the proof of Corollary 3.4.4.

Corollary 3.4.4. *Under GCH, the λ-JEP assumption in Lemma 3.2.4 and Lemma 3.2.5 is not necessary. If $2^{\aleph_0} = \aleph_{\omega_1}$ and $\lambda = \aleph_0$, then λ-JEP is necessary in Lemma 3.2.4. If $2^{\aleph_0} = \aleph_1 =: \lambda$ and $2^{\aleph_1} = \aleph_{\omega_2}$, then λ-JEP is necessary in Lemma 3.2.5.*

Proof. By Lemma 3.2.3, we may assume $\kappa < \lambda$. Suppose $\mathbf{K}$ is κ-stable in λ, then Lemma 3.2.4 and Lemma 3.2.5 are always true: for *all* $\|M\| = \lambda$, $\mathrm{gS}^\kappa_\lambda = \lambda = |\,\mathrm{gS}^\kappa(M)|$. Also, by taking sequences of length κ from $|M|$ (which give the algebraic types), we have $\mathrm{gS}^\kappa_\lambda \geq |\,\mathrm{gS}^\kappa_\lambda(M)| \geq \lambda^\kappa \geq (\mathrm{gS}^1_\lambda)^\kappa$ where the last inequality is by stability.

Therefore, we may further assume κ-instability in λ, witnessed by M. Suppose GCH holds, then $|\,\mathrm{gS}^\kappa(M)| \geq \lambda^+ = 2^\lambda = 2^{\lambda+\kappa} \geq \mathrm{gS}^\kappa_\lambda \geq |\,\mathrm{gS}^\kappa(M)|$. Also, $\mathrm{gS}^\kappa_\lambda \geq |\,\mathrm{gS}^\kappa(M)| \geq \lambda^+ = 2^\lambda = (2^\lambda)^\kappa \geq (\mathrm{gS}^1_\lambda)^\kappa$. Hence it is consistent that the lemmas are always true (regardless of λ-JEP or $\mathrm{cf}(\mathrm{gS}^\kappa_\lambda)$).

We show that Lemma 3.2.4 can be false: let $\lambda = \aleph_0$, $\kappa = 1$ and suppose $2^{\aleph_0} = \aleph_{\omega_1}$ (which is consistent by Easton Theorem, see [Jec03, Theorem 15.18]). Define L_1, T_1 as in Lemma 3.4.2, pick $\aleph_\omega$ many distinct subsets of $\aleph_0$, say $\{S_\beta : \beta < \aleph_\omega\}$. We allow $\beta < \gamma < \aleph_\omega$

when constructing $p_{\beta,\gamma}$. Now for $n<\omega$, define Γ_1^n to be the same as Γ_1 except that α is replaced by $\aleph_n$. Define $\mathbf{K^n} := EC(T_1,\Gamma_1^n)$ and $\mathbf{K}$ be the disjoint union of all $\mathbf{K^n}$ (adding $\aleph_0$-many predicates and stipulate that no two elements belong to different predicates — this destroys $\aleph_0$-JEP). Notice that $\mathbf{K}$ is still a EC class where the language has size $\aleph_0$ and whose models omit $2^{\aleph_0}$-many types. For any $M\in K$, $M\in K^n$ for some $n<\omega$. Since the unique maximal model in $\mathbf{K^n}$ has size $\aleph_n$, $|\,\mathrm{gS}(M)|\leq\aleph_n<\aleph_\omega=\sup_{n<\omega}\aleph_n=\mathrm{gS}^1_{\aleph_0}$ where the last equality is due to the fact that distinct two elements in a model satisfy distinct subsets of $\{P_i: i<\aleph_0\}$. Hence $\mathrm{gS}^1_{\aleph_0}$ (and similarly all $\mathrm{gS}^1_{\aleph_n}$, $n<\omega$) is not attained by any model. $\mathrm{cf}(\mathrm{gS}^1_{\aleph_0})=\aleph_0$ which satisfies the hypothesis.

We show that Lemma 3.2.5 can also be false: let $\lambda=\aleph_1$, $\kappa=\aleph_0$ and suppose $2^{\aleph_0}=\aleph_1$ and $2^{\aleph_1}=\aleph_{\omega_2}$ (which is consistent by Easton Theorem). This time we pick $\aleph_\omega$ many distinct subsets from $\aleph_1$ rather than from $\aleph_0$. Form $\mathbf{K}$ as above which is an EC class whose language has size $\aleph_1$ and whose models omit $2^{\aleph_1}$-many types. Now $(\mathrm{gS}^1_\lambda)^\kappa=(\mathrm{gS}^1_{\aleph_1})^{\aleph_0}={\aleph_\omega}^{\aleph_0}>\aleph_\omega$. On the other hand, $\mathrm{gS}^\kappa_\lambda=\mathrm{gS}^{\aleph_0}_{\aleph_1}\leq\sup_{n<\omega}\sup\{|\,\mathrm{gS}^{\aleph_0}(M)|: M\in(K^n)_{\aleph_1}\}\leq\sup_{n<\omega}{\aleph_n}^{\aleph_0}=\sup_{n<\omega}(2^{\aleph_0}\cdot\aleph_n)=\sup_{n<\omega}\aleph_1\cdot\aleph_n=\aleph_\omega<(\mathrm{gS}^1_\lambda)^\kappa$ where the second equality is a speical case of the Hausdorff formula [Jec03, Equation 5.22]. $\square$

Proof of Proposition 3.4.1. Fix α as in Lemma 3.4.2, we use $\mathbf{K_1}$ to build $\mathbf{K}$ as follows: let $L:=L_1\cup\{E,Q,R,f,g,c\}$ where E,Q are binary predicates, R is a ternary predicate, f is a unary function, g is a symmetric binary function and c is a constant. Thus we have $|L|=\lambda$. A model in $\mathbf{K}=(\mathbf{K_0},\mathbf{K_1},\mathbf{K_2})$ has three sorts (M_0,M_1,M_2). M_1 is in K_1, M_2 will take care of NMM while M_0 is the iterated power sets of M_1. In details, we require:

1. $M_1\in EC(T_1,\Gamma_1)$ as in Lemma 3.4.2. We identify it as an ordinal $\leq\alpha$.

2. M_2 is an infinite model of the theory of pure equality.

3. If xEy, then x is in $M_0\cup M_1$ while y is in M_0. E also satisfies the extensionality axiom.

4. The first argument of Q is in M_1. We write $Q_i(\cdot):=Q(i,\cdot)$ and abbreviate by Q_i the elements x in M_0 with $Q(i,x)$. For limit ordinal σ in M_1, we require $Q_\sigma=\bigcup_{i<\sigma}Q_i$.

5. f is the rank function from M_0 to M_1 such that each x is sent to the smallest i with $Q_i(x)$. If $x \in y$, then $f(x) < f(y)$.

6. R, g and c code the total order of M_0: we define $R(\beta, x, y)$ and $g(x, y)$ as follows:

   (a) β in M_1, x, y in M_0

   (b) If $f(x) \neq f(y)$, then we say x is less than y when x has a smaller rank than y.

   (c) If $f(x) = f(y) = 0$, then $g(x, y)$ is the $<$-least element in the symmetric difference of x, y. $R(0, x, y)$ if $g(x, y)Ey$, in which case we say x is less than y.

   (d) If $f(x) = f(y) = \beta > 0$, then $g(x, y)$ is the least element in the symmetric difference of x, y. $R(\beta, x, y)$ if $g(x, y)Ey$, in which case we say x is less than y.

   (e) c is the default value for $g(x, y)$ when x, y are in M_1 or M_2, or when x, y in M_0 are equal or have different ranks.

(If we think of subsets as sequences, we are ordering M_0 by rank, and then by lexicographical order of each rank.)

We order $(M_0, M_1, M_2) \leq_{\mathbf{K}} (N_0, N_1, N_2)$ iff for $i \leq 2$, $M_i, N_i \in K_i$ and $M_i \subseteq N_i$. Notice that we can describe $\mathbf{K}$ as some $EC(T, \Gamma)$ where T is a $\forall\exists$ theory, $|L(T)| = \lambda$ and $|\Gamma| = 2^\lambda$. Also, $\mathrm{LS}(\mathbf{K}) = \lambda$ because $\mathrm{LS}(\mathbf{K_1}) = \lambda$ and we can close any set in M_0 to a model by adding witnesses for f, g in (5),(6) ω-many times. Therefore $\mathbf{K}$ is an AEC.

The maximal model in $(\mathbf{K_0}, \mathbf{K_1})$ is $M^* := (V_\alpha(\alpha), \alpha)$, where for $\beta < \alpha$, $Q_\beta^{M^*} := V_{1+\beta}(\alpha)$. M^* witnesses that $\mathbf{K}$ have JEP. $\mathbf{K}$ is $(< \aleph_0)$-tame because elements are determined either by the predicates P_i; or their ranks and their own elements. With $\mathbf{K_2}$, we know that $\mathbf{K}$ has NMM. Since M^* has size $\beth_\alpha(\lambda)$, $(\mathbf{K_0}, \mathbf{K_1})$ is trivially stable in $\geq \beth_\alpha(\lambda)$. As $\mathbf{K_2}$ is trivially stable everywhere, $\mathbf{K}$ is stable in $\geq \beth_\alpha(\lambda)$. We now show instability in $< \beth_\alpha(\lambda)$ and the order property of lengths up to $\beth_\alpha(\lambda)$.

For instability, $(\mathcal{P}(\lambda), \lambda, \omega)$ witnesses that $\mathbf{K}$ is unstable between $[\lambda, 2^\lambda)$. Consider the maximal model M^* above, for each $\beta < \alpha$, $|Q_\beta^{M^*}| = \beth_{1+\beta}(\lambda)$. By item (6) above, $Q_\beta^{M^*}$ can distinguish all elements in $Q_{\beta+1}^{M^*}$, so $(\mathbf{K_0}, \mathbf{K_1})$, and hence $\mathbf{K}$ is unstable in $\big[\beth_{1+\beta}(\lambda), \beth_{1+\beta+1}(\lambda)\big)$. Therefore, we have instability in $\big[\lambda, \beth_\alpha(\lambda)\big)$.

For the order property*, we apply the order in (6) to $V_\alpha(\alpha)$ of the maximal model. In details: let a less than b while c less than d. If a, b is mapped to d, c respectively, then $f(d) = f(a) \leq f(b) = f(c) \leq f(d)$. It cannot happen to rank 0 because their elements are well-ordered in M_1. Since a is less than b, $g(a,b) \in b$. Then by mapping $g(d,c)(= g(c,d)) \in c$ which shows that d is less than c, contradiction. As $|V_\alpha(\alpha)| = \beth_\alpha(\lambda)$, we have the order property* of length $\beth_\alpha(\lambda)$.

$(\mathbf{K_0}, \mathbf{K_1})$ does not have AP: Pick an element x from $(V_\alpha(\alpha), \alpha)$ which contains $\geq \lambda^+$ elements. Close x to a substructure N of size λ, then there is yEx in $(V_\alpha(\alpha), \alpha)$ but y is not in N. N can be included in $(V_\alpha(\alpha), \alpha)$ such that x is mapped to $(x - \{y\})$. Suppose the following amalgam exists:

$$\begin{tikzcd}
x \in (V_\alpha(\alpha), \alpha) \arrow[r, dotted] & x \in W \\
x \in N \arrow[u] \arrow[r] & x - \{y\} \in (V_\alpha(\alpha), \alpha) \arrow[u, dotted, "t"]
\end{tikzcd}$$

Without loss of generality, we may assume the top dotted arrow is identity (hence we can write the image of x to be x itself). Since $(V_\alpha(\alpha), \alpha)$ is maximal, $W = (V_\alpha(\alpha), \alpha)$. Therefore, $t \in \mathrm{Aut}((V_\alpha(\alpha), \alpha))$. By an induction argument, t must be the identity (which boils down to the fact that α, the maximal model in $\mathbf{K_1}$, is rigid). From the right dotted arrow, $x - \{y\}$ would be mapped to x, which is a contradiction. $\square$

Remark 3.4.5. 1. One way to save AP is to redefine $\leq_{\mathbf{K}}$ by the E-transitive closure, but it raises $\mathrm{LS}(\mathbf{K})$ to $V_{\alpha-1}(\lambda)$. In this case, instability and the order property* length are up to $2^{\mathrm{LS}(\mathbf{K})}$.

2. Our total order is ill-founded: $\alpha, \alpha - \{0\}, \alpha - \{0, 1\}, \ldots$ form an infinite descending sequence in $Q_0^{M^*}$. It is not clear how to extract a witness to the order property of length $> 2^\lambda$. We will see in Corollary 3.6.8 that we can refine our example to have the order property at least up to $\beth_{\alpha-3}(\lambda)$.

Corollary 3.4.6. *1. For stable AECs, the Hanf number for the order property* length is exactly* $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$.

2. *The Hanf number for stability is at least* $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$. *In other words, let* $\lambda \geq \aleph_0$ *and* $\mu < \beth_{(2^\lambda)^+}$, *there is a stable AEC* $\mathbf{K}$ *such that* $\mathrm{LS}(\mathbf{K}) = \lambda$ *and the first stability cardinal is greater than* μ.

Proof. Combine Proposition 3.4.1 and Proposition 3.3.4 and range α in $[\lambda, (2^\lambda)^+)$. □

In the next two sections, we develop the machinery to show that the lower bound in (2) is tight, based on the arguments in [Vas16c]. Then we conclude: our example witnesses that the bound for the order property is also tight.

3.5 GALOIS MORLEYIZATION AND SYNTACTIC ORDER PROPERTY

Galois Morleyization is a way to capture tameness syntactically by adding infinitary predicates. First recall the definition of tameness:

Definition 3.5.1. Let κ be an infinite cardinal.

1. Let $p = \mathrm{gtp}(a/A, N)$ where $a = \langle a_i : i < \alpha \rangle$ may be infinite, $I \subseteq \alpha$, $A_0 \subseteq A$. We write $l(p) := l(a)$, $p \upharpoonright A_0 := \mathrm{gtp}(a/A_0, N)$, $a^I = \langle a_i : i \in I \rangle$ and $p^I := \mathrm{gtp}(a^I/A, N)$.

2. K is $(< \kappa)$-*tame for* $(< \alpha)$-*types* if for any subset A in some model of K, any $p \neq q \in \mathrm{gS}^{<\alpha}(A)$, there is $A_0 \subseteq A$, $|A_0| < \kappa$ with $p \upharpoonright A_0 \neq q \upharpoonright A_0$. We omit $(< \alpha)$ if $\alpha = 2$.

3. K is $(< \kappa)$-*short* if for any $\alpha \geq 2$, any subset A in some model of K, $p \neq q \in \mathrm{gS}^{<\alpha}(A)$, there is $I \subseteq \alpha$, $|I| < \kappa$ with $p^I \neq q^I$.

4. κ-*tame* means $(< \kappa^+)$-tame. Similarly for shortness.

Now we construct Galois Morleyization:

Definition 3.5.2. [Vas16c, Definitions 3.3, 3.13] Let κ be an infinite cardinal and K be an AEC in a (finitary) language L. The $(< \kappa)$-*Galois Morleyization of* K is a class $\hat{K}$ of structures in a language $\hat{L}$ such that:

1. $\hat{L}$ is a $(< \kappa)$-ary language. For convenience we may require $L \subseteq \hat{L}$.

2. For each $p \in \mathrm{gS}^{<\kappa}(\emptyset)$, we add a predicate R_p of length $l(p)$ to $\hat{L}$.

3. For each $M \in K$, we define $\hat{M} \in \hat{K}$ with $|M| = |\hat{M}|$. For $p \in \mathrm{gS}^{<\kappa}(M)$, $a \in |\hat{M}|^{l(p)}$, $\hat{M} \vDash R_p[a]$ iff $a \vDash p$ in K. Extend the definition to quantifier-free formulas of $\hat{L}_{\kappa,\kappa}$.

4. The $(<\kappa)$-*syntactic type* of $a \in |\hat{M}|^{<\kappa}$ over $A \subseteq |\hat{M}|$ is defined by $\mathrm{tp}_{\text{qf-}\hat{L}_{\kappa,\kappa}}(a/A; \hat{M})$, namely the quantifier-free formulas of $\hat{L}_{\kappa,\kappa}$ over A that a satisfies. We will abbreviate it by $\mathrm{tp}_\kappa(a/A; \hat{M})$.

5. For $\hat{M}, \hat{N} \in \hat{K}$, we order $\hat{M} \leq_{\hat{\mathbf{K}}} \hat{N}$ iff $M \leq_{\mathbf{K}} N$. We will omit the subscripts.

Remark 3.5.3. 1. If we allow AECs to have infinitary languages, we can view $\hat{K}$ as an AEC.

2. The above is well-defined even for AECs without AP, but readers can assume the existence of a monster model $\mathfrak{C}$ for convenience.

3. $|\hat{L}| = |L| + |\mathrm{gS}^{<\kappa}(\emptyset)| \leq 2^{<(\kappa + \mathrm{LS}(\mathbf{K})^+)}$.

The following justifies the definition of Galois Morleyization in tame AECs:

Proposition 3.5.4. [Vas16c, Corollary 3.18(2)] *Let K be an AEC and $\hat{K}$ be its $(<\kappa)$-Galois Morleyization. For each $p = \mathrm{gtp}(b/A; M) \in \mathrm{gS}(A)$, define $p_\kappa := \mathrm{tp}_\kappa(b/A; \hat{M})$ to be its $(<\kappa)$-syntactic version. Then K is $(<\kappa)$-tame iff $p \mapsto p_\kappa$ is a 1-1 correspondence.*

Proof. $\Rightarrow$: The map is well-defined because Galois types are finer than syntactic types. It is a surjection by construction. Suppose $p \neq q \in \mathrm{gS}(A)$, by $(<\kappa)$-tameness we may assume the domain A has size $<\kappa$. Let $b \vDash p$ and $b' \vDash q$. Then bA and $b'A$ satisfy different types in $\mathrm{gS}^{<\kappa}(\emptyset)$, say r and s repsectively. Thus $bA \vDash R_r \wedge \neg R_s$ while $b'A \vDash R_s \wedge \neg R_r$.

$\Leftarrow$: Suppose $p = \mathrm{gtp}(b/A; M) \neq q = \mathrm{gtp}(b'/A; M')$. Then $p_\kappa \neq q_\kappa$ and we can find $r \in \mathrm{gS}^{<\kappa}(\emptyset)$ and (a suitable enumeration of) $A_0 \subseteq A$ such that $M \vDash R_r[b; A_0]$ but $M' \vDash \neg R_r[b'; A_0]$. This means $bA_0 \vDash r$ while $b'A_0 \nvDash r$. Hence $\mathrm{gtp}(b/A_0; M) \neq \mathrm{gtp}(b'/A_0; M')$ witnessing $(<\kappa)$-tameness. $\square$

Remark 3.5.5. There is a stronger version assuming $(<\kappa)$-shortness in [Vas16c, Corollary 3.18(1)] but we have no use of it here.

We now define an infinitary version of the syntactic order property:

Definition 3.5.6. Let $2 \leq \alpha \leq \kappa$ and $1 \leq \beta < \kappa$ be ordinals.

1. [Vas16c, Definition 4.2] In Definition 3.3.2, we replace all occurences of "order property" by "syntactic order property" while requiring the condition in (1) there be: there exist some $\langle a_i : i < \mu \rangle \subseteq \hat{M} \in \hat{K}$ and some quantifier-free $\hat{L}_{\kappa,\kappa}$ formula $\phi(x, y)$ such that $l(a_i) = \beta$, and for $i, j < \mu$, $i < j$ iff $\hat{M} \vDash \phi[a_i, a_j]$.
2. As in Definition 3.3.3, we define syntactic order property* if in (1) we allow the index set to be a linear order I with $|I| = \mu$, instead of being a well-ordering μ.

The following links the (Galois) order property in K with the syntactic order property in $\hat{K}$.

Proposition 3.5.7. *Let κ be an infinite cardinal and $\hat{K}$ be $(< \kappa)$-Galois Morleyization of K. Let $1 \leq \beta < \kappa$ be an ordinal and $M \in K$. Let λ, μ be infinite cardinals and $\chi := |\mathrm{gS}^{\beta+\beta}(\emptyset)|$.*

1. [Vas16c, Proposition 4.4]
   - (a) *If $\hat{M}$ has the syntactic the β-order property of length μ, then M has the β-order property of length μ.*
   - (b) *If M has the β-order property of length μ for some $\mu \geq (2^{\lambda+\chi})^+$, then $\hat{M}$ has the syntactic β-order property of length λ^+.*
2.
   - (a) *If $\hat{M}$ has the syntactic β-order property* of length μ, then M has the β-order property* of length μ.*
   - (b) *If M has the β-order property* of length μ for some $\mu \geq (2^{\lambda+\chi})^+$, then $\hat{M}$ has the syntactic β-order property* of length λ^+.*
3. *The following are equivalent:*
   - (a) *K has the β-order property.*
   - (b) *K has the β-order property*.*

(c) $\hat{K}$ *has the syntactic* β*-order property.*

(d) $\hat{K}$ *has the syntactic* β*-order property*.*

Proof. 1. (a) is true because Galois types are finer than syntactic types. For (b): suppose $\langle a_i : i < \mu \rangle$ witnesses the β-order property. By Erdős-Rado Theorem, we have $\mu \to (\lambda^+)^2_{2^\chi}$. Apply this to $(i < j) \mapsto \text{gtp}(a_i a_j / \emptyset; M)$ and then on $(j < i) \mapsto \text{gtp}(a_j a_i / \emptyset; M)$. We can find $\langle b_i : i < \lambda^+ \rangle$ subsequence of $\langle a_i : i < \mu \rangle$, $p \neq q \in \text{gS}^{\beta+\beta}(\emptyset)$ such that for $i < j < \lambda$, $\text{gtp}(b_i b_j / \emptyset; M) = p$ and $\text{gtp}(b_j b_i / \emptyset; M) = q$. We may choose R_p (or R_q) to witness the β-syntactic order property.

2. The same proof goes through, because Erdős-Rado Theorem applies to linear orders too.

3. (1) gives (a)⇔(c) while (2) gives (b)⇔(d). (a)⇔(b) is by Proposition 3.3.4(1),(2).

□

Definition 3.5.8. Let κ be an infinite cardinal, $2 \leq \alpha \leq \kappa$ and $1 \leq \beta < \kappa$ be ordinals, $\hat{K}$ be the $(< \kappa)$-Galois Morleyization of K. Then

1. For $\mu \geq \text{LS}(\mathbf{K}) + |\beta|$, $\hat{K}$ is β*-syntactically stable in* μ if

$$|\{p_\kappa : p \in \text{gS}^\beta(A; M), A \subseteq |M|, |A| \leq \mu, M \in K\}| \leq \mu.$$

2. For $\mu \geq \text{LS}(\mathbf{K}) + |\alpha|$, $\hat{K}$ is $(< \alpha)$*-syntactically stable in* μ if

$$|\{p_\kappa : p \in \text{gS}^{<\alpha}(A; M), A \subseteq |M|, |A| \leq \mu, M \in K\}| \leq \mu.$$

Corollary 3.5.9. *Let* $\beta \geq 1$ *be an ordinal and* $\mu \geq \text{LS}(\mathbf{K}) + |\beta|$ *be a cardinal.*

1. [Vas16c, Fact 4.9] *If* K *has the* β*-order property, then* K *is not* β*-stable in* μ*. If also* $\beta < \kappa$*, then* $\hat{K}$ *is not* β*-(syntactically) stable in* μ*.*

2. *The same conclusion holds when* K *has the* β*-order property*.*

Proof. By Proposition 3.5.7, either assumption gives the syntactic β-order property. This implies β-syntactic instability in μ, using the proof of Proposition 3.3.4 (in particular replace $F^{\mathfrak{B}}(\alpha)$ by a witness of the syntactic β-order property of length $\beth_\alpha$). □

3.6 SHELAH'S STABILITY THEOREM

We will connect syntactic stability with no syntactic order property. The original result due to Shelah was in a more general context but only proof sketches were given. Vasey [Vas16c, Fact 4.10] applied it to AECs without a complete proof so we write out all the details. We will also remove the requirement that the order property length be a successor (which was hinted in [She09b, Exercise 1.22]).

Theorem 3.6.1. [She09b, V.A. Theorem 1.19] *Let $\chi \geq 2^{<(\kappa+\mathrm{LS}(\mathbf{K})^+)}$, μ be an infinite cardinal such that $\mu = \mu^\chi + 2^{2^{<\chi}}$. Suppose $\hat{K}$ does not have the $(<\kappa)$-syntactic order property of length χ, then $\hat{K}$ is $(<\kappa)$-syntactically stable in μ.*

The proof will be given after Lemma 3.6.5. Before that we state some relevant definitions and lemmas.

Definition 3.6.2. Let κ be an infinite cardinal, Π be a set of quantifier-free formulas of $\hat{L}_{\kappa,\kappa}$ over A, and p_κ be a $(<\kappa)$-syntactic type over A. We say p_κ *splits over* Π if there are $\phi(x;b), \neg\phi(x;c) \in p_\kappa$ such that for any $\hat{M}$ containing b, c and the parameters from Π, any $\psi(y;d) \in \Pi$ with $l(y) = l(b) = l(c)$, we have $\hat{M} \vDash \psi[b;d] \Leftrightarrow \hat{M} \vDash \psi[c;d]$ (the choice of $\hat{M}$ does not matter because its interpretation of R_p is external).

If we require the witnesses $\phi(x;b), \neg\phi(x;c)$ to be from a fixed formula $\phi(x;y)$, then we say p_κ *ϕ-splits over* Π.

Lemma 3.6.3. [She09b, V.A. Fact 1.10(4)] *Using the above notation,*

$$|\{p_\kappa \restriction \phi : p_\kappa \text{ does not } \phi\text{-split over } \Pi\}| \leq 2^{2^{|\Pi|}}$$

$$|\{p_\kappa : p_\kappa \text{ does not split over } \Pi\}| \leq 2^{2^{|\Pi|}\cdot\chi}$$

where $\chi := |\hat{L}|^{<\kappa} = \left(|\,\mathrm{gS}^{<\kappa}(\emptyset)|\right)^{<\kappa} \leq 2^{<(\kappa+\mathrm{LS}(\mathbf{K})^+)}$ is the size of the set of quantifier-free formulas of $\hat{L}_{\kappa,\kappa}$.

Proof. We count the number of combinations to build a $p_\kappa \restriction \phi$ that does not ϕ-split over Π. Partition the parameters of ϕ by their Π-type. Namely, b, c are equivalent iff for any $\hat{M}$ containing b, c and the parameters from Π, any $\psi(y;d) \in \Pi$ with $l(y) = l(b) = l(c)$, we have $\hat{M} \vDash \psi[b;d] \Leftrightarrow \hat{M} \vDash \psi[c;d]$. Then there are $2^{|\Pi|}$-many classes. Within each class, say containing b, it remains to choose whether $\phi(x;b)$ or $\neg\phi(x;b)$ is in $p_\kappa \restriction \phi$. Hence we have $2^{2^{|\Pi|}}$-many choices.

The second part follows from the observation that a $(<\kappa)$-syntactic type p_κ is determined by its restrictions $p_\kappa \restriction \phi$ where ϕ is a quantifier-free formula of $\hat{L}_{\kappa,\kappa}$. $\square$

Definition 3.6.4. A set A is $(<\chi)$-*compact* if for any $\hat{M}$ containing A, any cardinal $\lambda < \chi$, any quantifier-free formulas $\{\phi_i(x) : i < \lambda\}$ in $\hat{L}_{\kappa,\kappa}$ with parameters from A, if $\hat{M} \vDash \bigwedge_{i<\lambda} \phi_i[b]$ for some $b \in \hat{M}$, then $\hat{M} \vDash \bigwedge_{i<\lambda} \phi_i[a]$ for some $a \in A$.

Lemma 3.6.5. [She09b, V.A. Theorem 1.12] *Let χ be an infinite cardinal, A be $(<\chi)$-compact with $A \subseteq |\hat{M}|$, $\phi(x;y)$ be a quantifier-free formula in $\hat{L}_{\kappa,\kappa}$. Either*

1. *For any $m \in |\hat{M}|$, there is a set $\Pi \subseteq \{\phi(x;a) : a \in [A]^{<\kappa}\}$ such that $|\Pi| < \chi$ and $\mathrm{tp}_\kappa(m/A;\hat{M}) \restriction \phi$ does not ϕ-split over Π; or*

2. *$A \subseteq |\hat{M}|$ witnesses the $(<\kappa)$-syntactic order property of length χ.*

Proof. Suppose (1) does not hold, then we can pick $m \in |\hat{M}|$ such that $\mathrm{tp}_\kappa(m/A;\hat{M}) \restriction \phi$ splits over any Π with $|\Pi| < \chi$. Thus we can recursively build

1. $\langle m_i, b_i, c_i : i < \chi\rangle$ inside A.

2. For $j < \chi$, $\hat{M} \vDash \phi[m;b_j] \leftrightarrow \neg\phi[m;c_j]$

3. For $i < j < \chi$, $\hat{M} \vDash \phi[m_i;b_j] \leftrightarrow \phi[m_i;c_j]$.

4. For $j < \chi$, $m_j \vDash \bigwedge_{i\leq j} \big(\phi(x;b_i) \leftrightarrow \neg\phi(x;c_i)\big)$.

The construction is possible by the definition of ϕ-splitting and by $(<\chi)$-compactness of A. The sequence $\langle m_i b_i c_i : i < \chi\rangle$ witnesses the $(<\kappa)$-syntactic order property of length χ via the formula $\phi(x;y) \leftrightarrow \phi(x;z)$. $\square$

Proof of Theorem 3.6.1. Let A be a set of size μ. As in Lemma 3.6.3, χ bounds the number of quantifier-free formulas of $\hat{L}_{\kappa,\kappa}$. Also $\mu = \mu^{<\chi}$, so we may assume A is $(<\chi)$-compact (see Definition 3.6.4). Since Lemma 3.6.5(2) fails, (1) must hold for each quantifier-free formula ϕ of $\hat{L}_{\kappa,\kappa}$.

Now we count the number of $(<\kappa)$-syntactic types. Each type p_κ is determined by its restrictions $\{p_\kappa \restriction \phi : \phi \text{ is a quantifier-free formula in } \hat{L}_{\kappa,\kappa}\}$. Since $\mu = \mu^\chi$, we may assume $p_\kappa = p_\kappa \restriction \phi$ for a fixed ϕ (this is where we need $\mu = \mu^\chi$ instead of $\mu = \mu^{<\chi}$). By Lemma 3.6.5(1), we can find some Π_{p_κ} of size $< \chi$ such that p_κ does not ϕ-split over Π_{p_κ}. There are $[A]^{<\chi} = \mu^{<\chi}$-many ways to choose Π_{p_κ}. For each fixed $\Pi = \Pi_{p_\kappa}$, Lemma 3.6.3 gives at most $2^{2^{|\Pi|}} = 2^{2^{<\chi}}$-many choices for p_κ. So in total there are $\mu^{<\chi} + 2^{2^{<\chi}} = \mu$-many choices. $\square$

Corollary 3.6.6. *Let K be a stable AEC.*

1. [Vas16c, Theorem 4.13] *If K is $(<\kappa)$-tame, has AP and is stable in some cardinal $\geq \kappa^-$, then the first stability cardinal is bounded above by $\beth_{(2^{<(\kappa+\mathrm{LS}(\mathbf{K})^+)})^+}$.*
2. *If K is $(<\kappa)$-tame and does not have $(<\kappa)$-order property of length $\chi := 2^{<(\kappa+\mathrm{LS}(\mathbf{K})^+)}$. then the first stability cardinal is bounded above by $2^{2^{<\chi}}$.*
3. *If K is $\mathrm{LS}(\mathbf{K})$-tame and does not have $\mathrm{LS}(\mathbf{K})$-order property of length $2^{\mathrm{LS}(\mathbf{K})}$, then the first stability cardinal is bounded above by $\beth_3(\mathrm{LS}(\mathbf{K}))$.*
4. *Let $|D(T)| := |\,\mathrm{gS}^{<\omega}(\emptyset)|$. If K is $(<\aleph_0)$-tame and does not have $(<\omega)$-order property of length $|D(T)|$, then the first stability cardinal is bounded above by $\beth_2(|D(T)|)$.*

Proof. We prove (1): Since $\mathbf{K}$ is $(<\kappa)$-tame, by Proposition 3.5.4 $(<\kappa)$-syntactic stability in $\hat{\mathbf{K}}$ is equivalent to $(<\kappa)$-stability in $\mathbf{K}$. Also, by the contrapositive of Proposition 3.5.7(1)(a), no $(<\kappa)$-order property of length χ in $\mathbf{K}$ implies no $(<\kappa)$-syntactic order property in $\hat{\mathbf{K}}$ of length χ.

Let $\mathbf{K}$ be stable in some $\lambda \geq \kappa^-$. Since it is $(<\kappa)$-tame, Theorem 3.2.10 gives stability in $\beth_{\lambda^+}(\lambda)$ and Theorem 3.2.2 gives $(<\kappa)$-stability in $\beth_{\lambda^+}(\lambda)$. By Proposition 3.3.4(3), there is $\chi < \beth_{(2^{<(\kappa+\mathrm{LS}(\mathbf{K})^+)})^+}$ such that $\mathbf{K}$ does not have $(<\kappa)$-order property of length χ. We may assume $\chi \geq 2^{<(\kappa+\mathrm{LS}(\mathbf{K})^+)}$. By the first paragraph and Theorem 3.6.1, K is $(<\kappa)$-stable in $2^{2^{<\chi}} < \beth_{(2^{<(\kappa+\mathrm{LS}(\mathbf{K})^+)})^+}$. $\square$

Remark 3.6.7. In particular (4) misses the actual lower bound by $\beth_2$ [She90, III Theorem 5.15].

We now show the promised result in Remark 3.4.5:

Corollary 3.6.8. *Let λ be an infinite cardinal and γ be an ordinal with $\lambda \leq \gamma < (2^\lambda)^+$. Then there is a stable AEC $\mathbf{K}$ such that $\mathrm{LS}(\mathbf{K}) = \lambda$, $\mathbf{K}$ has the order property of length up to $\beth_\gamma(\lambda)$. Moreover, $\mathbf{K}$ has JEP, NMM, $(< \aleph_0)$-tameness but not AP.*

Proof. Let $\chi := \beth_\gamma(\lambda)$. We use the example in Proposition 3.4.1 with $\alpha = \gamma + 3$. Suppose $\mathbf{K}$ has no $(< \omega)$-order property of length χ. Since $\mathbf{K}$ is $(< \aleph_0)$-tame, by Proposition 3.5.4 $(< \omega)$-syntactic stability in $\hat{\mathbf{K}}$ is equivalent to $(< \omega)$-stability in $\mathbf{K}$. Also, by the contrapositive of Proposition 3.5.7(1)(a), no $(< \omega)$-order property of length χ in $\mathbf{K}$ implies no $(< \omega)$-syntactic order property in $\hat{\mathbf{K}}$ of length χ. Since $\chi \geq 2^{<\aleph_0}$, by Theorem 3.6.1, $\mathbf{K}$ is $(< \omega)$-stable in all $\mu = \mu^\chi + 2^{2^{<\chi}}$. In particular, it is $(< \omega)$-stable in $\beth_2(\chi) = \beth_{\alpha-1}(\lambda) = \beth_{\alpha-1}(\lambda)$, contradicting the fact that $M^* \in K$ is unstable in any cardinal $< \beth_\alpha(\lambda)$. $\square$

Remark 3.6.9. In our example, where exactly is the witness to the $(< \omega)$-order property of $\beth_\gamma(\lambda)$? Tracing the proofs, the key is the recursive construction in Lemma 3.6.5, where a long splitting chain is utilized. Fix a cardinal $\chi \leq \beth_{\alpha-1}(\lambda)$. For our $\mathbf{K}$, we do not even need the b_i and can simply set $\langle c_i, m_i : i < \chi \rangle$ such that all elements are distinct and m_i contains exactly $\{c_j : j \leq i\}$. Then $\langle c_i m_i : i < \chi \rangle$ witnesses the 2-order property of length χ, via the formula $\phi(x_1 y_1; x_2 y_2) := x_1 E y_2$. Therefore, we have an explicit example of the order property up to length $\beth_{\alpha-1}(\lambda)$ (the subscript cannot go further because most elements on the top rank do not belong to any other elements).

We can conclude:

Corollary 3.6.10.

1. *For stable AECs, the Hanf number for the order property length is exactly $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$.*
2. *For stable AECs, the Hanf number for the order property* length is exactly $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$.*
3. *The Hanf number for stability is at least $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$. In other words, let $\lambda \geq \aleph_0$ and $\mu < \beth_{(2^\lambda)^+}$, there is a stable AEC $\mathbf{K}$ such that $\mathrm{LS}(\mathbf{K}) = \lambda$ and the first stability cardinal is greater than μ.*
4. *With $\mathrm{LS}(\mathbf{K})$-tameness and AP, the Hanf number for stability is at most $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$. In other words, if $\mathbf{K}$ is a stable AEC with AP and $\mathrm{LS}(\mathbf{K})$-tameness, then the first stability cardinal is at most $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$.*

Proof. 1. Lower bound is by Corollary 3.6.8 and upper bound is by Proposition 3.3.4(3).

2. Lower bound is by Proposition 3.4.1 and upper bound is by Proposition 3.3.4(3).

3. By Proposition 3.4.1.

4. By Corollary 3.6.6(1).

□

We finish this section with the following question: are the bounds in (3) and (4) optimal?

3.7 SYNTACTIC SPLITTING

We will give a syntactic proof to Theorem 3.2.10 using Galois Morleyization. The advantage is that types are syntactic and can be over sets of size less than $\mathrm{LS}(\mathbf{K})$; the disadvantage is that we have extra assumptions.

Assumption 3.7.1. We assume the existence of a monster model $\mathfrak{C}$ $(AP+JEP+NMM)$, where each set is inside some model in K and each (set) embedding/isomorphism is extended by a $\mathbf{K}$-embedding/isomorphism. We also assume AP over set bases: if $A \subseteq |M_1| \cap |M_2|$ and M_1, M_2 interpret A in the same way, then there are $M_3 \geq M_2$ and $f : M_1 \underset{A}{\rightarrow} M_3$.

Definition 3.7.2. Let μ be an infinite cardinal, A, B be sets in some models of K.

1. B is *universal* over A if $A \subset B$ and for any $|B'| \leq |B|$, $B' \supseteq A$, there is $f : B' \underset{A}{\rightarrow} B$. We write $A \subset_u B$.

2. B is *μ-universal* over A if B' in (1) must have size $\leq \mu$.

3. B is *μ-homogeneous* if it is $< \mu$-universal over any $C \subset B$ of size $< \mu$.

4. Let $A \subseteq B \in K$, $p \in \mathrm{gS}(B)$. We say p *μ-splits over* A if there exists $A \subseteq B_1, B_2 \subseteq B$, $\|B_1\| = \mu$, $f : B_1 \cong_A B_2$ with $f(p) \upharpoonright B_2 \neq p \upharpoonright B_2$.

[She09a, II 1.16(1)(a)] shows that universal *models* of size μ exist if $\mathbf{K}$ is stable in $\mu \geq \mathrm{LS}(\mathbf{K})$ (and has μ-AP, μ-NMM). We prove universal sets exist:

Proposition 3.7.3. *1. For any A, if $\mathbf{K}$ is stable in $|A|$, there is $|B| = |A|$, $B \supset_u A$.*

2. For any infinite λ, μ and any $|A| \leq \lambda$. If $\lambda^{<\mu} = \lambda$, then there is a μ-homogeneous $B \supset A$ of size λ.

Proof. 1. Let $\mu := |A|$. Build $\langle B_i : i < \mu \rangle$ increasing and continuous such that $B_0 := A$, $B_{i+1} \vDash \mathrm{gS}(B_i)$. For any $A' \supseteq A$, $|A'| \leq \mu$. We may assume $|A' - A| = \mu$ and write $A' = A \cup \{a_i : i < \mu\}$. Define $A_i := A \cup \{a_j : j \leq i\}$ and $\langle f_i : A_i \underset{A}{\rightarrow} B_i : i \leq \mu \rangle$ increasing and continuous partial embeddings such that $f_i(a_i) \in B_i$. Set $f_{-1} := \mathrm{id}_A$ and suppose f_i has been constructed, obtain $\tilde{A}'$, an isomorphic copy of A' over $\mathrm{ran}(f_i)$ and denote by $\tilde{a}_{i+1}$ the copy of a_{i+1}. Now $B_{i+1} \vDash \mathrm{gS}(B_i) \supseteq \mathrm{gS}(\mathrm{ran}(f_i))$ so it realizes the type of $\tilde{a}_{i+1}$ over $\mathrm{ran}(f_i)$, say by b_{i+1}. By AP there is $g : \tilde{a}_{i+1} \mapsto b_{i+1}$ fixing $\mathrm{ran}(f_i)$. Define $f_{i+1}(a_{i+1}) := b_{i+1}$.

$$\begin{tikzcd}
A' \arrow[r, "\cong"] & \tilde{A}' & & B \\
a_{i+1} \arrow[u] \arrow[d, dash] \arrow[rr, bend left] & \tilde{a}_{i+1} \arrow[u, "f_{i+1}"'] \arrow[r, "g"] \arrow[d, dash] & b_{i+1} \arrow[ur] & \\
A_i \arrow[r, "f_i"] & \mathrm{ran}(f_i) \arrow[ur, dash] & & \\
A \arrow[u] \arrow[ur] & & &
\end{tikzcd}$$

2. By an exhaustive argument, we can build a (set) saturated $B \supset A$. We check that it is μ-homogeneous. Let $C \subset B$, $C' \supseteq C$ both of size $< \mu$, the argument from the previous item applies because B is saturated and $|C'| < \mu$.

□

We notice a correspondence between μ-Galois splitting and $(< \mu^+)$-syntactic splitting (see Definition 3.6.2); a similar treatment for coheir has already been done in [Vas16c, Section 5]). We write *q μ-syn-splits over A* to mean q $(< \mu^+)$-syntactically splits over the quantifier-free formulas of $\hat{L}_{\mu^+,\mu^+}$ over A.

Proposition 3.7.4. *Let μ be an infinite cardinal, $\hat{\mathbf{K}}$ be the $(< \mu^+)$-Galois Morleyization of $\mathbf{K}$. For any $A \subseteq B \in K$, $p \in \mathrm{gS}(B)$, p μ-splits over A iff p_{μ^+} μ-syn-splits over A.*

Proof. Suppose $d \vDash p$ μ-splits over A, obtain witness $f : B_1 \cong_A B_2$ as above. Enumerate $A = a$ and B_i as b_i. Since $f(p) \restriction B_2 \neq p \restriction B_2$, $r_1 := \text{gtp}(f(d)b_2/\emptyset) \neq \text{gtp}(db_2/\emptyset)$, so $\hat{\mathfrak{C}} \vDash R_{r_1}[db_1] \wedge R_{r_1}[f(d)b_2] \wedge \neg R_{r_1}[db_2]$. b_1 and b_2 have the same syntactic type over a because of f. Therefore, $p_{\mu^+} = \text{tp}_{\mu^+}(d/B) \supseteq \text{tp}_{\mu^+}(d/B_1 \cup B_2)$ μ-syn-splits over A.

Conversely, suppose p_{μ^+} μ-syn-splits over A. There are $\phi(x; b_1), \neg\phi(x; b_2)$ in p_{μ^+} such that b_1, b_2 have the same syntactic type over a. Pick $d \vDash p_{\mu^+}$, then $\hat{\mathfrak{C}} \vDash \phi[d; b_1] \wedge \neg\phi[d; b_2]$, $\text{gtp}(db_1/\emptyset) \neq \text{gtp}(db_2/\emptyset)$ (actually ϕ might tell the exact Galois type of db_1). On the other hand, let $r = \text{gtp}(b_1 a/\emptyset)$ and consider $R_r(x; a)$. As b_1, b_2 have the same syntactic type over A, $\hat{\mathfrak{C}} \vDash R_r[b_1; a] \wedge R_r[b_2; a]$, which means $\text{gtp}(b_2 a/\emptyset) = r$. Thus there is $f : b_1 \cong_a b_2$. □

In the above proof, we did not use tameness simply because $(< \mu^+)$-Galois Morleyization is already large enough to code all types over domains of size μ.

Corollary 3.7.5. *Let μ be an infinite cardinal and* **K** *be μ-tame. Let $A \subseteq B$ with $|A| \leq \mu$. Then any $p \in S(B)$ $(\geq \mu)$-splits over A iff it μ-splits over A.*

Proof. We adopt the previous proof: suppose $d \vDash p$ $(\geq \mu)$-splits over A, obtain witness $f : B_1 \cong_A B_2$ as above. Enumerate $A = a$ and B_i as b_i. Since $f(p) \restriction B_2 \neq p \restriction B_2$, $\text{gtp}(f(d)b_2/\emptyset) \neq \text{gtp}(db_2/\emptyset)$. By Proposition 3.5.4, there is a quantifier-free formula ϕ in $\hat{L}_{\mu^+,\mu^+}$ so that $\hat{\mathfrak{C}} \vDash \phi[db_1] \wedge \phi[f(d)b_2] \wedge \neg\phi[db_2]$. b_1 and b_2 have the same syntactic type over a because of f. Therefore, $p_{\mu^+} = \text{tp}_{\mu^+}(d/B) \supseteq \text{tp}_{\mu^+}(d/B_1 \cup B_2)$ μ-syn-splits over A. This implies p μ-splits over A by the second paragraph of the previous proof. □

We now prove a series of results syntactically. The original proofs in [Van06, Theorems I 4.10,4.12], [GV06b, Section 6] are semantic.

Lemma 3.7.6. *Let μ be an infinite cardinal and* **K** *be μ-tame. For any $B \subseteq C$ both of size $\geq \mu$ and $p \in \text{gS}(C)$, we have $p_{\mu^+} \restriction B = (p \restriction B)_{\mu^+}$.*

Proof. Let $d \vDash p$,

$$\begin{aligned} p_{\mu^+} \restriction B &= \text{tp}_{\mu^+}(d/C) \restriction B \\ &= \text{tp}_{\mu^+}(d/B) \\ &= (p \restriction B)_{\mu^+} \text{ because } d \vDash p \restriction B \end{aligned}$$

□

Lemma 3.7.7. *Let μ be an infinite cardinal, $A \subset_u B \subseteq C$ all of size μ and $p, q \in \mathrm{gS}(C)$. Suppose p, q do not μ-split over A and $p \restriction B = q \restriction B$. Then $p = q$. We can allow $|B|, |C| \geq \mu$ if we assume μ-tameness.*

Proof. We comment the case $|C| > \mu$ in square brackets. Let $\hat{\mathbf{K}}$ be the $(< \mu^+)$-Galois Morleyization of $\mathbf{K}$. Since p, q do not μ-split over A, Proposition 3.7.4 shows that p_{μ^+}, q_{μ^+} do not μ-syn-split over A [use μ-tameness and Corollary 3.7.5]. Suppose $p_{\mu^+} \neq q_{\mu^+}$, [by μ-tameness] there is $d \subseteq C$ of size μ, $\phi(x; y)$ such that $\phi(x; d) \in p_{\mu^+} - q_{\mu^+}$. As $B \supset_u A$, we may pick $b \vDash \mathrm{tp}_{\mu^+}(d/A)$. By non-syn-splitting, $\phi(x; b) \in p_{\mu^+} - q_{\mu^+}$ contradicting $p_{\mu^+} \restriction B = q_{\mu^+} \restriction B$ [use Proposition 3.5.4 and Lemma 3.7.6]. □

Extension also holds but it is applicable to μ-sized models.

Lemma 3.7.8. *Let μ be an infinite cardinal, $A \subset_u B \subseteq C$ all of size μ. Let $p \in \mathrm{gS}(B)$ do not μ-split over A. Then there is $q \in \mathrm{gS}(C)$ extending p and does not μ-split over A. Also, if p is non-algebraic, we can have q non-algebraic.*

Proof. Let $\hat{\mathbf{K}}$ be the $(< \mu^+)$-Galois Morleyization of $\mathbf{K}$. First we decide whether $\pm\phi(x; c) \in q_{\mu^+}$ for each quantifier-free formula in $\hat{L}_{\mu^+,\mu^+}$ over C. Since $A \subset_u B$, there is a copy $C' \subset B$ of C. We want q does not μ-split over A, by Proposition 3.7.4 we must set $\phi(x; c) \in q_{\mu^+}$ iff $\phi(x; b_c) \in p_{\mu^+}$ where $b_c \in C'$ and $b_c \vDash \mathrm{tp}_{\mu^+}(c/A)$. Such q_{μ^+} is realized because p_{μ^+}, and thus $p_{\mu^+} \restriction C' \cong_A q_{\mu^+}$ is realized where $C' \subset B$ is the copy of C.

If p is non-algebraic and we can modify the argument above by extending C to a copy of B. Then p_{μ^+} is realized/algebraic iff q_{μ^+} is.

$$\begin{tikzcd}
p \arrow[d, dash, dotted] & q \arrow[d, dash, dotted] \\
B \arrow[r] & C \ni c \arrow[dl, dash, "\cong"] \\
b_c \in C' \arrow[u] & \\
A \arrow[u] &
\end{tikzcd}$$

$\square$

Lemma 3.7.9. *Let μ be an infinite cardinal,* $\mathbf{K}$ *be μ-tame and stable in μ. For any $|A| \leq \mu$, $A \subset_u C$,*

$$\chi := |\{p \in \mathrm{gS}(C) : p \textit{ does not} \geq \mu\textit{-split over } A\}| \leq \mu$$

Proof. Pick B of size μ with $A \subset_u B \subseteq C$. By Corollary 3.7.5, Lemma 3.7.7 and Lemma 3.7.8, $\chi = |\{p \in \mathrm{gS}(B) : p \text{ does not } \mu\text{-split over } A\}| \leq |\mathrm{gS}(B)| \leq \mu$. $\square$

The following originates from [She99, Claim 3.3] and is extended to longer types in [GV06b, Fact 4.6].

Lemma 3.7.10. *Let μ be an infinite cardinal and* $\mathbf{K}$ *be stable in μ. For any $p \in \mathrm{gS}(B)$, there is $A \subseteq B$ of size μ such that p does not μ-splits over A.*

Proof. Suppose the lemma is false, we can find $d \vDash p \in \mathrm{gS}(B)$, B of size $> \mu$ such that p μ-splits over all A of size μ. Let $\hat{\mathbf{K}}$ be the $(<\mu^+)$-Galois Morleyization of $\mathbf{K}$. By Proposition 3.7.4, p_{μ^+} μ-syn-splits over all A of size μ. Pick any $A_0 \subset B$ of size μ and choose minimum $\kappa \leq \mu$ with $2^\kappa > \mu$. By assumption we can build $\langle a_{\eta^\frown 0}, a_{\eta^\frown 1}, \phi_\eta, f_\eta : \eta \in 2^{<\kappa} - \{\langle\rangle\}\rangle$ and $\langle A_\alpha : \alpha < \kappa\rangle$ such that

1. $\langle A_\alpha : \alpha < \kappa\rangle$ is increasing and continuous. For $\alpha < \kappa$, $A_\alpha \subset B$ has size μ.

2. For $\eta \in 2^{<\kappa}$, $a_\eta \in A_{l(\eta)}$, $f_\eta : a_{\eta^\frown 0} \mapsto a_{\eta^\frown 1}$ and f_η fixes $A_{l(\eta)}$.

3. For $\eta \in 2^{<\kappa}$, $\phi_\eta(x; a_{\eta^\frown 0}), \neg\phi_\eta(x; a_{\eta^\frown 1}) \in p_{\mu^+}$

$$A_2 \qquad \phi_0(x;a_{00}) \xrightarrow[A_1]{f_0} \phi_0(x;a_{01}) \qquad \phi_1(x;a_{10}) \xrightarrow[A_1]{f_1} \phi_1(x;a_{11})$$

$$A_1 \qquad a_0 \xrightarrow[A_0]{f_{\langle\rangle}} a_1$$

For $\nu \in 2^\kappa$, define $q_\nu := \{\phi_\eta(x; a_{\eta^\frown i}) : \eta \in 2^{<\kappa}, \eta^\frown i \sqsubseteq \nu; i = 1, 2\} \cup \{\neg\phi_\eta(x; a_{\eta^\frown 1}) : \eta \in 2^{<\kappa}, \eta^\frown 0 \sqsubseteq \nu\}$. $\langle q_\nu : \nu \in 2^\kappa\rangle$ are obviously pairwise contradictory. It remains to show each of them is realized. For any $\nu \in 2^\kappa$, define g_ν to be the composition of

$\langle f^{\nu[\alpha]}_{\nu\restriction\alpha} : \alpha < \kappa\rangle$ where each $f^0_\eta := \mathrm{id}$ and $f^1_\eta := f_\eta$. g_ν is well-defined because for $\eta \in 2^{<\kappa}$, f_η fixes $A_{l(\eta)}$. Extend g_ν to an isomorphism containing d. Inductively we can show $g_\nu(d) \vDash q_\nu$: let $\alpha < \kappa$. If $\nu[\alpha] = 0$, then item (3) in the construction guarantees $g(p^+_\mu) \supseteq g_\nu(\{\phi_{\nu\restriction\alpha}(x; a_{\nu\restriction\alpha\frown 0}), \neg\phi_{\nu\restriction\alpha}(x; a_{\nu\restriction\alpha\frown 1})\}) = \{\phi_{\nu\restriction\alpha}(x; a_{\nu\restriction\alpha\frown 0}), \neg\phi_{\nu\restriction\alpha}(x; a_{\nu\restriction\alpha\frown 1})\}$. If $\nu[\alpha] = 1$, then $g(p^+_\mu) \supseteq g_\nu(\{\phi_{\nu\restriction\alpha}(x; a_{\nu\restriction\alpha\frown 0})\}) = \{\phi_{\nu\restriction\alpha}(x; a_{\nu\restriction\alpha\frown 1})\}$. $\square$

Proof of Theorem 3.2.10. Let C be of size λ. By an exhaustive argument, we may extend C to be μ^+-saturated. Let $A \subset C$ of size μ. By Proposition 3.7.3(1) and (2), we can build B of size μ such that $A \subset_u B \subset C$. Thus Lemma 3.7.9 applies. Also by Lemma 3.7.10, each $p \in \mathrm{gS}(C)$ does not μ-split over some $A_p \subset C$ of size μ. There are at most λ^μ of such A_p. Thus

$$|\mathrm{gS}(C)| = \Big|\bigcup\{p \in \mathrm{gS}(C) : p \text{ does not } \mu\text{-split over } A_p\}\Big| \leq \lambda^\mu \cdot \mu = \lambda$$

$\square$

Corollary 3.7.11. *If* $\mathbf{K}$ *is stable in some* $\lambda < \mathrm{LS}(\mathbf{K})$*. Then the first stability cardinal* $\geq \mathrm{LS}(\mathbf{K})$ *is bounded above by* $2^{\mathrm{LS}(\mathbf{K})}$.

Proof. Apply Theorem 3.2.10 to $(2^{\mathrm{LS}(\mathbf{K})})^\lambda = 2^{\mathrm{LS}(\mathbf{K})}$. $\square$

Our final application is the upward transfer of stability. The original proof of (1) below uses weak tameness (tameness over saturated models). [Vas16b, Lemma 5.5] proves a stronger version of (2) with chain local character instead of set local character, but we do not assume the former here.

Proposition 3.7.12. *Let* $\mu < \lambda$ *be infinite cardinals. Assume* $\mathbf{K}$ *is* μ*-tame and stable in* μ.

1. [BKV06, Theorem 4.5] $\mathbf{K}$ *is also stable in* μ^+.
2. *If in addition* $\mathrm{cf}(\lambda) > \mu$ *and* $\mathbf{K}$ *is stable in unbounded many cardinals below* λ*, then it is stable in* λ.

Proof. 1. Suppose $|A| = \mu^+$ has μ^{++} many types $\langle p_\alpha : \alpha < \mu^{++} \rangle$. Write $A = \bigcup_{i<\mu^+} A_i$ increasing and continuous, with $|A_i| = \mu$. We may assume $A_{i+1} \vDash \mathrm{gS}(A_i)$ by the following: define another chain $\langle A'_i : i \leq \mu^+ \rangle$ increasing and continuous such that $|A'_{i+1}| = |A_{i+1}| = \mu$ and $A'_{i+1} \supset_u A_i \cup A'_i$ (using Proposition 3.7.3(1)). Replace A by A'_{μ^+}.

   By Lemma 3.7.10 and $\mathrm{cf}\, \mu^{++} > \mu^+$, we may assume all p_α does not μ-split over A_0. By stability in μ and pigeonhole principle, we may assume all p_α has the same type over A_1. Together with Lemma 3.7.7, all p_α are equal, contradiction.

2. We consider limit cardinal λ. Pick a cofinal sequence $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ to λ such that $\mathbf{K}$ is stable in all λ. Repeat the same argument as (1) but with $|A| = \lambda$ and $A = \bigcup_{i<\mathrm{cf}(\lambda)} A_i$.

$\square$

CHAPTER 4
AXIOMATIZING AECS AND APPLICATIONS

ABSTRACT

For any abstract elementary class (AEC) $\mathbf{K}$ with $\lambda = \mathrm{LS}(\mathbf{K})$, the following holds:

1. K has an axiomatization in $L_{(2^\lambda)^+,\lambda^+}$, allowing game quantification. If $\mathbf{K}$ has arbitrarily large models, the λ-amalgamation property and is categorical both in λ and λ^+, then it has an axiomatization in L_{λ^+,λ^+} with game quantification. These extend Kueker's [Kue08] result which assumes finite character and $\lambda = \aleph_0$.
2. If K is universal and categorical in λ, then it is axiomatizable in L_{λ^+,λ^+}.
3. Shelah's celebrated presentation theorem asserts that for any AEC $\mathbf{K}$ there is a first-order theory in an expansion of $\mathrm{L}(\mathbf{K})$, and a set Γ of 2^λ many T-types such that $K = PC(T, \Gamma, \mathrm{L}(\mathbf{K}))$. We provide a better bound on $|\Gamma|$ in terms of $I_2(\lambda, \mathbf{K})$.
4. We present additional applications which extend, simplify and generalize results of Shelah [She87, She01] and Shelah-Vasey [SV18a]. Some of our main results generalize to μ-AECs.

4.1 INTRODUCTION

In the proof of Shelah's presentation theorem [She09a, I Lemma 1.9], functions are added to capture isomorphism axioms and Löwenheim-Skolem axiom. [SV21, Theorem 2.1] claimed that any abstract elementary class (AEC) $\mathbf{K}$ can be axiomatized by an $L_{\beth_2(\lambda)^{+3},\lambda^+}$ sentence where λ is the Löwenheim-Skolem number, and such an axiomatization is in $\mathrm{L}(\mathbf{K})$.

Let $\chi := \lambda + I_2(\lambda, \mathbf{K})$, where $I_2(\lambda, \mathbf{K})$ is the number of pairs (M, N) that are nonisomorphic where $M, N \in K_\lambda$ and $M \leq_{\mathbf{K}} N$. In Main Theorem 4.3.7, we will axiomatize an AEC $\mathbf{K}$ by a sentence $\sigma_{\mathbf{K}}$ in L_{χ^+,λ^+}, allowing game quantification. As $\chi \leq 2^\lambda$, we have that $\sigma_{\mathbf{K}}$ is in $L_{(2^\lambda)^+,\lambda^+}$, improving Shelah and Villaveces' result. Modulo the use of game quantification, our result is optimal for uncountable λ as it is known that there is an AEC that cannot be axiomatized by an $L_{\infty,\lambda}$ sentence [Hen19]. Under extra assumptions, we can axiomatize an AEC by a sentence in L_{λ^+,λ^+} (Proposition 4.3.10 and Theorem 4.3.12).

By slightly modifying $\sigma_{\mathbf{K}}$, we can encode the $\mathbf{K}$-substructure relation (Proposition 4.3.16 and Proposition 4.3.18) by a formula $\sigma_{\leq}$ in L_{χ^+,λ^+}. As above, we can improve the results under extra assumptions (Corollary 4.3.20).

As an application of our axiomatization of AECs, we derive a variation on the presentation theorem, where any AEC is a PC_χ class (Theorem 4.4.1) and game quantification is not used. Our presentation theorem is stronger than Shelah's as the bound of $|\Gamma|$ in some cases is smaller than $2^{\mathrm{LS}(\mathbf{K})}$. It also lowers the threshold of the existence theorem from successive categoricity (Theorem 4.4.8). The axiomatization strategy is also applicable to the μ-AEC analogs, giving a stronger presentation theorem (Theorem 4.5.6) than [BGL$^+$16, Theorem 3.2].

In the following, we provide two tables. The first table summarizes the known results in literature. The definition of $L(\omega)$ can be found in [Kue08, Definition 1.12] and $L_{\chi^+,\lambda^+}(\omega\cdot\omega)$ is defined in Definition 4.2.6. The second table summarizes the main results in this paper. We write AL = arbitrarily large models, AP = amalgamation property, JEP = joint embedding property, NMM = no maximal models. Monster model means $AP + JEP + NMM$. The entries of the second table link to the related theorems (see Observation 4.3.15 for cases where we do not assume λ-categoricity).

<u>Known results</u>

Assumptions on $\mathbf{K}$	K is	References
None	axiomatizable in $L_{(2^{2^{\lambda^+}})^{+++},\lambda^+}$	[SV21, Theorem 2.1]
None	$PC_{\lambda,2^\lambda}$	[She87, Lemma 1.8]
None	reducts of a theory in L'_{χ^+,λ^+} where $L' \supseteq L$	[BB16, Theorem 3.2.3]
$\lambda = \aleph_0$, $\aleph_0$-AP, stable in $\aleph_0$, $I(\aleph_0, \mathbf{K}) \leq \aleph_0$	$PC_{\aleph_0}$	[SV18a, Theorem 4.2]
$\lambda = \aleph_0$	closed under $\equiv_{\infty,\omega_1}$	[Kue08, Theorem 2.5]
$\lambda = \aleph_0$, $\exists\kappa = \kappa^{\aleph_0}(I(\kappa, \mathbf{K}) \leq \kappa)$	axiomatizable in L_{∞,ω_1}	[Kue08, Theorem 2.11]
$\lambda = \aleph_0$, $\exists\kappa = \kappa^{\aleph_0}(I(< \kappa, \mathbf{K}) \leq \kappa)$	axiomatizable in L_{κ^+,ω_1}	[Kue08, Theorem 2.11]
Finitary, $\lambda = \aleph_0$	closed under $\equiv_{\infty,\omega}$	[Kue08, Theorem 3.4]
Finitary	closed under $\equiv_{\infty,\omega}$	[Joh10, Theorem 3.7]
Finitary, $\lambda = \aleph_0$	axiomatizable in $L(\omega)$	[Kue08, Theorem 3.7]
Finitary, $\lambda = \aleph_0$, $\exists\kappa(I(\kappa, \mathbf{K}) \leq \kappa)$	axiomatizable in $L_{\infty,\omega}$	[Kue08, Theorem 3.10]
Finitary, $\exists\kappa = \kappa^{<\lambda}(I(\kappa, \mathbf{K}) \leq \kappa)$	axiomatizable in $L_{\infty,\lambda}$	[Joh10, Theorem 3.10]
Finitary, $\lambda = \aleph_0$, $\exists\kappa(I(< \kappa, \mathbf{K}) \leq \kappa)$	axiomatizable in $L_{\kappa^+,\omega}$	[Kue08, Theorem 3.10]
Finitary, $\exists\kappa = \kappa^{<\lambda}(I(< \kappa, \mathbf{K}) \leq \kappa)$	axiomatizable in $L_{\kappa^+,\lambda}$	[Joh10, Theorem 3.10]
$\lambda = \aleph_0$, $\exists\kappa(I(\kappa, \mathbf{K}) = 1)$	$K_{\geq\kappa}$ is closed under $\equiv_{\infty,\omega}$	[Kue08, Theorem 5.1]
Finitary, $\lambda = \aleph_0$, $\exists\kappa(I(\kappa, \mathbf{K}) = 1)$	$K_{\geq\kappa}$ axiomatizable in $L_{\omega_1,\omega}$	[Kue08, Theorem 5.2]
$\lambda = \aleph_0$, $\exists\kappa = \kappa^{\aleph_0}(I(\kappa, \mathbf{K}) = 1)$	Models in $K_{\geq\kappa}$ are L_{∞,ω_1}-equivalent	[Kue08, Theorem 5.3a]
Previous row + monster model	$K_{\geq\kappa}$ axiomatizable in $L_{(2^\omega)^+,\omega_1}$	[Kue08, Theorem 5.3c]
$\lambda > \aleph_0$	K is closed under $\equiv_{\infty,\lambda^+}$	[Kue08, Theorem 7.2]
Monster model, $\lambda > \aleph_0$, $\exists\kappa(I(\kappa, \mathbf{K}) = 1 \wedge \mathrm{cf}(\kappa) > \lambda)$	$K_{\geq\kappa}$ axiomatizable in $L_{2^{\kappa^+},\kappa^+}$	[Kue08, Theorem 7.4]
Finitary, monster model, $\exists\kappa(I(\kappa, \mathbf{K}) = 1 \wedge \mathrm{cf}(\kappa) > \lambda)$	$K_{\geq\kappa}$ axiomatizable in $L_{\infty,\lambda}$	[Joh10, Theorem 3.11]

<u>New results</u>

Assumptions on $\mathbf{K}$	Axiomatization in	K is
None	$L_{\chi^+,\lambda^+}(\omega\cdot\omega)$	PC_χ
Universal class, $I(\lambda,\mathbf{K})\leq\lambda$	L_{λ^+,λ^+}	PC_λ
AL, λ-AP, $I(\lambda,\mathbf{K})\leq\lambda$, $I(\lambda^+,\mathbf{K})=1$	$L_{\lambda^+,\lambda^+}(\omega\cdot\omega)$	PC_λ
$2^\lambda<2^{\lambda^+}$, AL, $I(\lambda,\mathbf{K})=1$, $I(\lambda^+,\mathbf{K})=1$	$L_{\lambda^+,\lambda^+}(\omega\cdot\omega)$	PC_λ
λ-AP, $I(\lambda,\mathbf{K})\leq\lambda$, $I(\lambda^+,\mathbf{K})=1$ stable in λ	$L_{\lambda^+,\lambda^+}(\omega\cdot\omega)$	PC_λ
$2^\lambda<2^{\lambda^+}$, $I(\lambda,\mathbf{K})=1$, $I(\lambda^+,\mathbf{K})=1$, stable in λ	$L_{\lambda^+,\lambda^+}(\omega\cdot\omega)$	PC_λ

Note that the last row is a significant improvement of [SV18a, Theorem 4.2], using much simpler and general methods while covering the case when λ is uncountable. We highlight the differences between our result and [BB16, Theorem 3.2.3]:

1. They expand the base vocabulary to τ^* by adding new predicates of arity λ, and their theory T^* in the expanded language is more semantic and longer; our axiomatization keeps the original language $\mathrm{L}(\mathbf{K})$ and is purely syntactic.

2. Their relational presentation theorem characterizes K as reducts of models of T^*, $\mathbf{K}$-substructure as reducts of τ^*-substructure; our axiomatization is simply in $L_{\chi^+,\lambda^+}(\omega\cdot\omega)$ and we pin down the formula that determines $\mathbf{K}$-substructure.

3. Their expanded language τ^* has size χ; our axiomatization uses the original language so has size $\leq\lambda$ (but both approaches require taking χ-conjunctions). We expand the language to size χ only when we derive a variation on Shelah's presentation theorem.

4. In [BB16, Theorem 3.2.3], they do not require types to be omitted because their theory T^* is in the infinitary logic. We omit types in our variation on Shelah's presentation theorem so as to represent K as a first-order PC class.

Our approach in this paper was inspired by Villaveces' question of the complexity of the example in Proposition 3.4.1, which has high instability but low complexity $\gamma=1$.

Also, Grossberg suggested in May 2021 that [SV21] could have a significant improvement. This motivated us to look for a simpler axiomatization of an AEC, without using trees or other combinatorial machinery in [SV21, Theorem 2.4]. At the cost of game quantification, we lower the complexity of junctions in their paper from $\beth_2(\lambda)^{+3}$ to χ^+.

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4.2 PRELIMINARIES

Let L be a finitary language, $\lambda_1 \geq \lambda_2$ be infinite cardinals. We write L_{λ_1,λ_2} the set of formulas generated by L, allowing $< \lambda_2$ free variables and $< \lambda_2$ quantifiers, in addition to conjunctions and disjunctions of $< \lambda_1$ subformulas. Given an L-structure M, we write $|M|$ the universe of M and $\|M\|$ the cardinality of M.

Definition 4.2.1. Let $L \subseteq L'$ be two languages (they can be infinitary), T be an L'-theory and Γ be a set of L'-types. Let μ be a regular cardinal. If L, L' are $(< \mu)$-ary, we define

$$EC^\mu(T, \Gamma) := \{M : M \text{ is an } L'\text{-structure}, M \vDash T\ , M \text{ omits } \Gamma\}$$

$$PC^\mu(T, \Gamma, L) := \{M \restriction L : M \text{ is an } L'\text{-structure}, M \vDash T\ , M \text{ omits } \Gamma\}$$

When $\mu = \aleph_0$, we omit the superscript $\aleph_0$.

Let λ, χ be infinite cardinals, and assume $|T| \leq \lambda$ and $|\Gamma| \leq \chi$. If $K = EC^\mu(T, \Gamma)$, we call K an $EC^\mu_{\lambda,\chi}$ class. If $K = PC^\mu(T, \Gamma, L)$, we call K a $PC^\mu_{\lambda,\chi}$ class. We omit the superscript $\aleph_0$ when $\mu = \aleph_0$. We omit λ, χ if the sizes of T and Γ are not specified. PC_λ stands for $PC_{\lambda,\lambda}$.

Definition 4.2.2. Let L be a finitary language. An abstract elementary class $\mathbf{K} = \langle K, \leq_{\mathbf{K}} \rangle$ in L satisfies the following axioms:

1. K is a class of L-structures and $\leq_{\mathbf{K}}$ is a partial order on $K \times K$.

2. For $M_1, M_2 \in K$, $M_1 \leq_{\mathbf{K}} M_2$ implies $M_1 \subseteq M_2$ (as L-substructures).

3. Isomorphism axioms:

(a) If $M \in K$, N is an L-structure, $M \cong N$, then $N \in K$.

(b) Let $M_1, M_2, N_1, N_2 \in K$. If $f : M_1 \cong M_2$, $g : N_1 \cong N_2$, $g \supseteq f$ and $M_1 \leq_{\mathbf{K}} N_1$, then $M_2 \leq_{\mathbf{K}} N_2$.

4. Coherence: Let $M_1, M_2, M_3 \in K$. If $M_1 \leq_{\mathbf{K}} M_3$, $M_2 \leq_{\mathbf{K}} M_3$ and $M_1 \subseteq M_2$, then $M_1 \leq_{\mathbf{K}} M_2$.

5. Löwenheim-Skolem axiom: There exists an infinite cardinal $\lambda \geq |\mathrm{L}(\mathbf{K})|$ such that: for any $M \in K$, $A \subseteq |M|$, there is some $N \in K$ with $A \subseteq |N|$, $N \leq_{\mathbf{K}} M$ and $\|N\| \leq \lambda + |A|$. We call the minimum such λ the Löwenheim-Skolem number $\mathrm{LS}(\mathbf{K})$.

6. Chain axioms: Let α be an ordinal and $\langle M_i : i < \alpha \rangle \subseteq K$ such that for $i < j < \alpha$, $M_i \leq_{\mathbf{K}} M_j$.

(a) Then $M := \bigcup_{i<\alpha} M_i$ is in K and for all $i < \alpha$, $M_i \leq_{\mathbf{K}} M$.

(b) Let $N \in K$. If in addition for all $i < \alpha$, $M_i \leq_{\mathbf{K}} N$, then $M \leq_{\mathbf{K}} N$.

Let $\lambda \geq \mathrm{LS}(\mathbf{K})$ be a cardinal. We define $K_\lambda := \{M \in K : \|M\| = \lambda\}$ and $\mathbf{K}_\lambda := \langle K_\lambda, \leq_{\mathbf{K}} \restriction K_\lambda \times K_\lambda \rangle$. When the context is clear, we omit the subscript of $\leq_{\mathbf{K}}$ and write $\leq$. We will only consider $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$ in place of $\mathbf{K}$, which is still an AEC.

Definition 4.2.3. 1. Let I be an index set. A directed system $\langle M_i : i \in I \rangle \subseteq K$ indexed by I satisfies the following: for any $i, j \in I$, there is $k \in I$ such that $M_i \leq M_k$ and $M_j \leq M_k$.

2. Let μ be an infinite cardinal. A μ-directed system $\langle M_i : i \in I \rangle \subseteq K$ indexed by I satisfies the following: for any $J \subseteq I$ of size $< \mu$, there is $k \in I$ such that for all $j \in J$, $M_j \leq M_k$, (thus a system is directed iff it is $\aleph_0$-directed.)

Fact 4.2.4. [She09a, I Observation 1.6] Let $\langle M_i : i \in I \rangle \subseteq K$ be a directed system. Then

1. $M := \bigcup_{i \in I} M_i \in K$

2. For all $i \in I$, $M_i \leq M$.

3. Let $N \in K$. If in addition for all $i \in I$, $M_i \leq N$, then $M \leq N$.

Fact 4.2.5. [She09a, II Claim 1.8(2)] If $M \leq N$ in K, then there are index sets I_1 and I_2, directed systems $\langle M_i : i \in I_1\rangle$ and $\langle N_i : i \in I_2\rangle$ of union M, N respectively, $I_1 \subseteq I_2$ and $M_i = N_i$ for all $i \in I_1$.

Definition 4.2.6. Let L be a language, λ, χ be infinite cardinals and δ be an ordinal. $L_{\lambda,\chi}(\delta)$ extends $L_{\lambda,\chi}$ by allowing δ-game quantification: if ϕ is a formula in $L_{\lambda,\chi}(\delta)$ with free variables $(x_\alpha, y_\alpha)_{\alpha<\delta}$ and $l(x_\alpha), l(y_\alpha) < \chi$, then $(\forall x_\alpha \exists y_\alpha)_{\alpha<\delta}\, \phi$ is a formula in $L_{\lambda,\chi}(\delta)$. An L-structure M satisfies $(\forall x_\alpha \exists y_\alpha)_{\alpha<\delta}\, \phi$ if Player II has a winning strategy in the following game of δ rounds: in the α-th round, Player I chooses some tuple $a_\alpha \subseteq M$ of length $l(x_\alpha)$ and Player II responds by choosing some tuple $b_\alpha \subseteq M$ of length $l(y_\alpha)$. Player II wins if $M \vDash \phi[a_\alpha, b_\alpha]_{\alpha<\delta}$ where for $\alpha < \delta$, x_α is substituted by a_α and y_α is substituted by b_α.

If δ above is finite, then $L_{\lambda,\chi}(\delta) = L_{\lambda,\chi}$. The use of game quantifiers in AECs can be found in [Kue08, Theorems 2.9, 3.7] which handle the case $\mathrm{LS}(\mathbf{K}) = \aleph_0$. Our version is consistent with $L(\omega)$ there and is called a *closed game quantifier* in [Kol85, Chapter X.2].

4.3 ENCODING AN AEC

In this section, we fix an AEC $\mathbf{K}$ in a language L.

Definition 4.3.1.

1. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. $I(\lambda, \mathbf{K}) := |\{M/_{\cong} : M \in K_\lambda\}|$.
2. Let $M_1 \leq N_1$, $M_2 \leq N_2$. We write $(M_1, N_1) \cong (M_2, N_2)$ if there exists $f : N_1 \cong N_2$ such that $f \restriction M_1 : M_1 \cong M_2$.
3. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. $I_2(\lambda, \mathbf{K}) := |\{(M, N)/_{\cong} : M \leq N \text{ in } K_\lambda\}|$.

Example 4.3.2. Depending on λ and $\mathbf{K}$, $I(\lambda, \mathbf{K})$ and $I_2(\lambda, \mathbf{K})$ may not be the same:

1. If $\mathbf{K}$ is the class of the $L_{\omega_1,\omega}$ theory $\forall x \bigvee_{i<\omega}(x = c_i)$ where c_i are constants, then $I(\aleph_0, \mathbf{K}) = I_2(\aleph_0, \mathbf{K}) = 1$.
2. If $\mathbf{K}$ is the class of the first-order theory of pure equality, then $I(\lambda, \mathbf{K}) = 1$ but $I_2(\lambda, \mathbf{K}) = \lambda$ for any infinite λ.

3. (The following argument is due to the referee.) If $\mathbf{K}$ is the class of the first-order theory of dense linear orders without endpoints, then $I(\aleph_0, \mathbf{K}) = 1$ but $I_2(\aleph_0, \mathbf{K}) = 2^{\aleph_0}$. To see the latter, fix $M = \mathbb{Q}$ and countably many cuts of $\mathbb{Q}$. Those cuts form a countable order with respect to the usual ordering. We require a countable elementary extension N to add either a single realization or infinitely many realizations to each cut. Hence there are $2^{\aleph_0}$ such N. Since an order automorphism of $\mathbb{Q}$ extends uniquely to an automorphism of the reals, pairs of the form $(\mathbb{Q}, N)$ are not isomorphic.

Proposition 4.3.3. $I(\lambda, \mathbf{K}) \leq I_2(\lambda, \mathbf{K}) \leq 2^\lambda$.

Proof. For any $M \leq N_1$ and $M \leq N_2$ in K, if $N_1 \ncong N_2$, then $(M, N_1) \ncong (M, N_2)$ by definition, hence the first inequality. Using the fact that $I(\lambda, \mathbf{K}) \leq 2^\lambda$, we can bound $I_2(\lambda, \mathbf{K}) \leq \lambda^\lambda \cdot I(\lambda, \mathbf{K}) \leq 2^\lambda$. □

Question 4.3.4. Assuming stability or categoricity, is it possible to obtain a better bound than Proposition 4.3.3?

Until the end of this section, we write $\lambda := \mathrm{LS}(\mathbf{K})$.

Observation 4.3.5. List the representatives of $\{M/_{\cong} : M \in K_\lambda\}$ by $\langle M_i : i < I(\lambda, \mathbf{K})\rangle$ and those of $\{(M, N)/_{\cong} : M \leq N \text{ in } K_\lambda\}$ by $\langle (M_j, N_j) : j < I_2(\lambda, \mathbf{K})\rangle$. For $i < I(\lambda, \mathbf{K})$, let $\phi_i(x)$ be an L_{λ^+,λ^+} formula that encodes the isomorphism type of M_i with a fixed enumeration of the universe $|M_i| = \{m^i_k : k < \lambda\}$. For variables $x = \langle x_k : k < \lambda\rangle$,

$$\phi_i(x) := \bigwedge\{\theta(x_{\alpha_0}, \ldots, x_{\alpha_{s-1}}) : M_i \vDash \theta[m^i_{\alpha_0}, \ldots, m^i_{\alpha_{s-1}}], s < \omega, \alpha_0, \ldots, \alpha_{s-1} < \lambda,$$
$$\theta \text{ is an atomic } L\text{-formula or its negation with } s \text{ free variables}\}$$

Namely for any L-structure N and any $a \in |N|$ of length λ, if $N \vDash \phi_i[a]$ then $a \cong M_i$ (with the fixed enumeration).

Similarly, for $j < I_2(\lambda, \mathbf{K})$, let $\psi_j(x, y)$ be an L_{λ^+,λ^+} formula that encodes the isomorphism type of (M_j, N_j) with fixed enumerations, where $|M_j| = \langle m^j_k : k < \lambda\rangle$,

$|N_j| = \{n^j_k : k < \lambda\}$. For variables $x = \langle x_k : k < \lambda\rangle$ and $y = \langle y_k : k < \lambda\rangle$,

$$\begin{aligned}\psi_j(x; y) := \bigwedge\{\theta(&x_{\alpha_0}, \dots, x_{\alpha_{s-1}}; y_{\beta_0}, \dots, y_{\beta_{t-1}}) : N_j \vDash \theta[m^j_{\alpha_0}, \dots, m^j_{\alpha_{s-1}}; n^j_{\beta_0}, \dots, n^j_{\beta_{t-1}}],\\ &s < \omega,\ t < \omega,\ \alpha_0, \dots, \alpha_{s-1}, \beta_0, \dots, \beta_{t-1} < \lambda,\\ &\theta \text{ is an atomic } L\text{-formula or its negation with } s+t \text{ free variables}\}\end{aligned}$$

Namely for any L-structure N and any $a, b \in |N|$ both of length λ, if $N \vDash \psi_j[a, b]$ then $a \cong M_j$, $b \cong N_j$ (with the fixed enumerations) and $\mathrm{ran}(a) \subseteq \mathrm{ran}(b)$.

It is also possible to encode the re-enumerations of the isomorphism types in ϕ_i and ψ_j, but we will do that directly in the sentence $\sigma_{\mathbf{K}}$ in Main Theorem 4.3.7 and $\sigma_{\leq}$ in Proposition 4.3.18(1), so as to be more consistent with the format of Theorem 4.4.1.

Definition 4.3.6. Let $\alpha, \beta < \lambda^+$, $a = \langle a_i : i < \alpha\rangle$ and $b = \langle b_i : i < \beta\rangle$. $a \subseteq b$ stands for $\mathrm{ran}(a) \subseteq \mathrm{ran}(b)$, which can be expressed by the L_{λ^+,λ^+} formula

$$\bigwedge_{i<\alpha}\bigvee_{j<\beta} a_i = b_j$$

$a \approx b$ stands for $\mathrm{ran}(a) = \mathrm{ran}(b)$, which can be expressed by the L_{λ^+,λ^+} formula

$$\Big(\bigwedge_{i<\alpha}\bigvee_{j<\beta} a_i = b_j\Big) \wedge \Big(\bigwedge_{j<\beta}\bigvee_{i<\alpha} b_j = a_i\Big)$$

Main Theorem 4.3.7. *K is axiomatizable by an $L_{(\lambda+I_2(\lambda,\mathbf{K}))^+,\lambda^+}(\omega\cdot\omega)$ sentence $\sigma_{\mathbf{K}}$. In other words, for any L-structure M, $M \in K$ iff $M \vDash \sigma_{\mathbf{K}}$.*

Proof. The following variables $(x_\alpha, y_\alpha)_{\alpha<\omega\cdot\omega}$ are all of length λ.

$$\begin{aligned}\sigma_{\mathbf{K}} := (\forall x_\alpha \exists y_\alpha)_{\alpha<\omega\cdot\omega} \bigwedge_{\alpha<\omega\cdot\omega} \Big[&(x_\alpha \subseteq y_\alpha) \wedge \exists z_\alpha\Big((y_\alpha \approx z_\alpha) \wedge \bigvee_{i<I(\lambda,\mathbf{K})} \phi_i(z_\alpha)\Big) \wedge\\ &\bigwedge_{\beta<\alpha<\omega\cdot\omega} \exists u_{\beta,\alpha}\exists v_{\beta,\alpha}\Big((y_\beta \approx u_{\beta,\alpha}) \wedge (y_\alpha \approx v_{\beta,\alpha}) \wedge \bigvee_{j<I_2(\lambda,\mathbf{K})} \psi_j(u_{\beta,\alpha}, v_{\beta,\alpha})\Big)\Big]\end{aligned}$$

In words, $\sigma_{\mathbf{K}}$ stipulates the iterated use of Löwenheim-Skolem and coherence axioms $(\omega\cdot\omega)$ many times.

Suppose $M \in K$, we show that Player II can win the associated game in $\sigma_{\mathbf{K}}$. In the α-th round, Player I provides some x_α of length λ. By Löwenheim-Skolem axiom, pick any

$y_\alpha \leq M$ of size λ such that $\mathrm{ran}(x_\alpha) \cup \bigcup_{\beta<\alpha} \mathrm{ran}(y_\beta) \subseteq \mathrm{ran}(y_\alpha)$. By inductive hypothesis, for $\beta < \alpha$, we have $y_\beta \leq M$. By coherence axiom, $y_\beta \leq y_\alpha$ as desired.

Suppose $M \vDash \sigma_{\mathbf{K}}$. By Fact 4.2.4(1), it suffices to build a directed system $\langle M_a \in K_\lambda : a \in I\rangle$ of union M. We choose I to be the set of finite tuples a in M. Let a, b be finite tuples in M, we pre-order $a \leq_I b$ iff $\mathrm{ran}(a) \subseteq \mathrm{ran}(b)$. We will inductively build all M_a in ω stages. At stage n we handle finite tuples of length $n+1$.

- Stage 0: apply $\sigma_{\mathbf{K}}$ to each singleton s in M and substitute $x_0 = s$. We obtain $y_0 = M_s$ which is a $\mathbf{K}$-structure. Only the 0-th round of the game is used for each singleton.

- Inductive hypothesis: for some $n < \omega$, M_a has been constructed for each $l(a) \leq n+1$ with the following requirements:

  1. For some $k < \omega$ and some singleton s, M_a is the union $\bigcup_{\alpha<1+\omega\cdot k} y_\alpha$ where y_α comes from the game of s (i.e. $x_0 = s$; the "1+" is to handle the $k = 0$ case).

  2. Let a, b both of length $\leq n+1$. If $\mathrm{ran}(b) \subseteq \mathrm{ran}(a)$, then $M_b \leq M_a$.

Before we move onto the inductive step, we show that given M_a and M_b constructed in previous stages, we can find M^* such that $M^* \geq M_a$ and $M^* \geq M_b$. By inductive hypothesis, there is a singleton s in M, $m_s < \omega$ such that M_a is the union $\bigcup_{\alpha<1+\omega\cdot m_s} y_\alpha$ from the game of s. Similarly, we can find some singleton t in M and $m_t < \omega$ such that M_b is the union $\bigcup_{\alpha<1+\omega\cdot m_t} y_\alpha$ from the game of t. Using ω more rounds in the games of s and of t, we recursively build $\langle N_k : k < \omega\rangle$ $\subseteq$-increasing such that

1. $\langle N_{2l} : l < \omega\rangle$ and $\langle N_{2l+1} : l < \omega\rangle$ are both $\leq_{\mathbf{K}}$-increasing.

2. $N_{-1} := M_a$ and $N_0 := M_b$.

3. If $k = 2l+1$, then use the $(1 + \omega \cdot m_s + l)$-th round in the game of s to obtain $y_{1+\omega\cdot m_s+l} = N_k$ from $x_{1+\omega\cdot m_s+l} = N_{k-1}$. Notice that if $m_s > 0$, $N_{-1} \leq N_1$ by the second chain axiom.

4. If $k = 2l+2$, then use the $(1 + \omega \cdot m_t + l)$-th round in the game of t to obtain $y_{1+\omega\cdot m_t+l} = N_k$ from $x_{1+\omega\cdot m_t+l} = N_{k-1}$. Notice that if $m_t > 0$, $N_0 \leq N_2$ by the second

chain axiom.

$$\begin{tikzcd}[column sep=small]
\bigcup_{\alpha<1+\omega\cdot m_s} y_\alpha & x_{1+\omega\cdot m_s} & y_{1+\omega\cdot m_s} & x_{1+\omega\cdot m_s+1} & y_{1+\omega\cdot m_s+1} & & \\
M_a = N_{-1} \arrow[rr] & & N_1 \arrow[rr] \arrow[dr, dashed] & & N_3 \arrow[r] \arrow[dr, dashed] & \cdots & \\
& M_b = N_0 \arrow[rr] & & N_2 \arrow[rr] \arrow[ur, dashed] & & N_4 \arrow[r] & \cdots \\
& \bigcup_{\alpha<1+\omega\cdot m_t} y_\alpha & x_{1+\omega\cdot m_t} & y_{1+\omega\cdot m_t} & x_{1+\omega\cdot m_t+1} & y_{1+\omega\cdot m_t+1} &
\end{tikzcd}$$

In the above diagram, a solid arrow stands for **K**-substructure while a dashed arrow stands for L-substructure. The first row represents the game of s while the last row represents the game of t. Each vertical column contains identical **K**-structures.

Define $M^* := \bigcup_{k<\omega} N_k$. By requirements (1), (2) and chain axioms, $M^* \geq M_a$ and $M^* \geq M_b$. Also, s has used the first $\omega \cdot (m_s+1)$ rounds while t has used the first $\omega \cdot (m_t+1)$ rounds. This finishes the construction of M^*.

- Stage $n+1$: Now for tuples c of length $n+2$, we build M_c. Break down c as the union of two tuples of length $\leq n+1$ (there might be more than one way), say a and b. As above assume M_a is generated by some singleton s and M_b by some singleton t. Then we can find M^* with $M^* \geq M_a$, $M^* \geq M_b$ and M^* is the union of bounded many y_α's of the game of some singleton s. We cannot immediately define $M_c := M^*$ because M^* depends on the choice of the decomposition a, b. Since there are finitely many possible decompositions of a finite tuple c, we can continue the game of s and extend M^* to M_c which includes all M^* from other decompositions of c (M_c might not be unique but it is a **K**-superstructure to all those M_a with $\mathrm{ran}(a) \subseteq \mathrm{ran}(c)$; M_c is also generated by other games of singletons but we just need one representative s for M_c).

After the construction is completed, $\langle M_a \in K_\lambda : a \in I\rangle$ is directed by our inductive step. Their union is M because for each element u in M, $u \in M_{\{u\}}$. $\square$

Remark 4.3.8. Our theorem generalizes [Kue08, Theorems 2.9, 3.7] which use the ω-game quantification. Modulo the $(\omega \cdot \omega)$-game quantification, our result also generalizes [Kue08, Theorems 5.3, 7.4] which assumes a monster model and categoricity in a higher cardinal.

We will encode the $\mathbf{K}$-substructure relation in Proposition 4.3.18. Before that we investigate possible improvements of Main Theorem 4.3.7. The following questions were suggested by Grossberg:

Question 4.3.9. Is it possible to axiomatize an AEC $\mathbf{K}$ in L_{λ^+,λ^+} instead of $L_{(\lambda+I_2(\lambda,\mathbf{K}))^+,\lambda^+}$ (with or without game quantification), assuming

1. stability?
2. categoricity in λ and λ^+?

For (2), Grossberg also suggested that [MAV18] should allow improvements of Main Theorem 4.3.7. Indeed it is possible when $\mathbf{K}$ is a universal class. A partial converse can be found in [MAV18, Theorem 3.5].

Proposition 4.3.10. *If $\mathbf{K}$ is a universal class, then it is axiomatizable by an $L_{(\lambda+I(\lambda,\mathbf{K}))^+,\lambda^+}$ sentence $\sigma_{\mathbf{K}}$. In particular if $\mathbf{K}$ is categorical in λ, then it is axiomatizable by an L_{λ^+,λ^+} sentence.*

Proof. Since models are ordered by L-substructures, we can avoid game quantification and replace the ψ_j's by subset relations when defining $\sigma_{\mathbf{K}}$ in Main Theorem 4.3.7. Namely,

$$\begin{aligned}\sigma_{\mathbf{K}} := \forall a\, \exists m_a \Big(a \subseteq m_a \wedge \bigvee_{i<I(\lambda,\mathbf{K})} \phi_i(m_a)\Big) \wedge \\ \forall m_b \forall m_c \Big[\Big(\bigvee_{i<I(\lambda,\mathbf{K})} \phi_i(m_b) \wedge \bigvee_{j<I(\lambda,\mathbf{K})} \phi_j(m_c)\Big) \\ \to \exists m_d \Big(\bigvee_{k<I(\lambda,\mathbf{K})} \phi_k(m_d) \wedge (m_b \subseteq m_d) \wedge (m_c \subseteq m_d)\Big)\Big]\end{aligned}$$

□

We also have some approximations to Question 4.3.9(2), using game quantification. In the following we abbreviate *amalgamation property* as *AP* and *arbitrarily large models* as *AL*.

Fact 4.3.11. 1. If $\mathbf{K}$ has *AL*, λ-*AP*, is categorical in λ^+, then it is stable in λ. Hence for any $M \in K_\lambda$, there is $N \in K_\lambda$ which is a (λ, ω)-limit model over M.

2. Let $\mathbf{K}$ have λ-AP and $M_1, M_2, M_3 \in K_\lambda$. If $M_1 \leq M_2$ and M_3 is a (λ, ω)-limit model over M_2, then M_3 is also a (λ, ω)-limit model over M_1.

3. Let $M, N, N' \in K_\lambda$. If N, N' are both (λ, ω)-limit models over M, then $N \cong_M N'$.

Theorem 4.3.12. *If $\mathbf{K}$ has AL, λ-AP and is categorical in λ and λ^+, then it is axiomatizable by an $L_{\lambda^+,\lambda^+}(\omega \cdot \omega)$ sentence σ. AL can be replaced by stability in λ.*

Proof. By Fact 4.3.11(1), fix $M, N \in K_\lambda$ such that N is (λ, ω)-limit over M. Let $\phi(x)$ code the isomorphism type of M and $\psi(x, y)$ code the isomorphism type of (M, N). From the proof of Main Theorem 4.3.7, it suffices to replace $\bigvee_{i<I(\lambda,\mathbf{K})} \phi_i$ and $\bigvee_{j<I_2(\lambda,\mathbf{K})} \psi_j$ there by some λ-junctions. Since $\mathbf{K}$ is λ-categorical, we can replace the first disjunction by ϕ. We also replace the second disjunction by a coherence formula involving ψ. Finally we add a disjunction to $\sigma_{\mathbf{K}}$ specifying models of size λ:

$$\begin{aligned}
\sigma := \exists w\big(\phi(w) \wedge \forall x(x \subseteq w)\big) \vee \Big\{ \\
(\forall x_\alpha \exists y_\alpha)_{\alpha<\omega\cdot\omega} \bigwedge_{\alpha<\omega\cdot\omega} \Big[(x_\alpha \subseteq y_\alpha) \wedge \exists z_\alpha\Big((y_\alpha \approx z_\alpha) \wedge \phi(z_\alpha)\Big) \wedge \bigwedge_{\beta<\alpha<\omega\cdot\omega} \exists u_{\beta,\alpha} \exists v_{\beta,\alpha} \exists w \exists z \\
\Big((u_{\beta,\alpha} \subseteq v_{\beta,\alpha}) \wedge (y_\beta \approx u_{\beta,\alpha}) \wedge (y_\alpha \approx v_{\beta,\alpha}) \wedge (w \approx z) \wedge \psi(u_{\beta,\alpha}, w) \wedge \psi(v_{\beta,\alpha}, z))\Big)\Big]\Big\}
\end{aligned}$$

Suppose M^* is an L-structure and $M^* \vDash \sigma$. By coherence, the last line of σ implies $y_\beta \leq y_\alpha$. Then either $M^* \in K_\lambda$ or it can build a directed system $\langle M_\alpha \in K_\lambda : \alpha \in I \rangle$ of union M^* as in Main Theorem 4.3.7. By Fact 4.2.4(1) $M^* \in K$.

Suppose $M^* \in K_\lambda$, then it satisfies the first disjunct of σ. Otherwise $M^* \in K_{>\lambda}$. We need to verify that for any $M_1, M_2 \in K_\lambda$, if $M_1 \leq M_2 \leq M^*$ then there is $M_3 \leq M^*$ such that $(M_1, M_3) \cong (M_2, M_3) \cong (M, N)$. By AL and categoricity in λ^+, M^* is λ^+-saturated, so we can build $M_3 \in K_\lambda$ which is (λ, ω)-limit over M_2.

$$\begin{tikzcd}
M_3 \arrow[rr, "\cong"] & & N' \arrow[r, "\cong"] & N \\
M_2 \arrow[u, "{(\lambda,\omega)}"] & & & \\
& M_1 \arrow[ul] \arrow[uul, dotted, bend right, "{(\lambda,\omega)}"'] \arrow[rr, "\cong"] & & M \arrow[uul] \arrow[uu, "{(\lambda,\omega)}"']
\end{tikzcd}$$

By Fact 4.3.11(2), M_3 is (λ, ω)-limit to both M_1. Since $\mathbf{K}$ is categorical in λ, $M_1 \cong M$ and we can extend this isomorphism to $M_3 \cong N'$ for some $N' \geq M$. Then N' is a (λ, ω)-limit over M. By Fact 4.3.11(3), $N' \cong_M N$ so $(M_1, M_3) \cong (M, N)$. Similarly $(M_2, M_3) \cong (M, N)$. Therefore, $M^* \models \sigma$ as desired. □

Using a well-known result of Shelah, we can replace the assumption by AP by a set-theoretic one.

Fact 4.3.13. [She87, Theorem 3.5] Assume $2^\lambda < 2^{\lambda^+}$. If $I(\lambda, \mathbf{K}) = 1$ and $1 \leq I(\lambda^+, \mathbf{K}) < 2^{\lambda^+}$, then $\mathbf{K}$ has λ-AP.

Corollary 4.3.14. *Assume $2^\lambda < 2^{\lambda^+}$. If $\mathbf{K}$ has AL and is categorical in λ and λ^+, then it is axiomatizable by an $L_{\lambda^+,\lambda^+}(\omega \cdot \omega)$ sentence σ'. AL can be replaced by stability in λ.*

Proof. Combine Fact 4.3.13 and Theorem 4.3.12. □

In other words, under WGCH Question 4.3.9 has a positive answer when we assume both (1) and (2) of the hypotheses there and use game quantification.

Observation 4.3.15.

1. As in Proposition 4.3.10, we can replace categoricity in λ by $I(\lambda, \mathbf{K}) \leq \lambda$. We keep the original format of the theorem statements to better answer Question 4.3.9.

2. Let $\kappa \geq \lambda^+$. In Theorem 4.3.12, if we replace categoricity in λ^+ by κ, and further assume $(< \kappa)$-AP, then models in K_κ are saturated [Vas17d, Corollary 4.11(3)]. The same argument allows us to axiomatize $K_{\geq\kappa}$ by an $L_{\lambda^+,\lambda^+}(\omega \cdot \omega)$ sentence. If κ is regular, then we can replace AL by stability in $[\lambda, \kappa)$.

3. John Baldwin pointed out that [She99] can reduce the successive categoricity assumption to a single categoricity. Indeed, assuming AP and categoricity in a successor above H_2 (the second Hanf number), we have categoricity in H_2 and AL. Thus we can axiomatize $K_{\geq H_2}$ by a sentence in $L_{H_2^+,H_2^+}(\omega \cdot \omega)$.

4. Marcos Mazari-Armida observed that [She01] provides another variation on Corollary 4.3.14: if we assume categoricity in λ, λ^+, λ^{++} as well as $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$, then

we have stability in λ and λ-AP. Hence we can axiomatize K by a sentence in $L_{\lambda^+,\lambda^+}(\omega \cdot \omega)$.

We now encode the $\mathbf{K}$-substructure relation. First we handle the case when the smaller model has size λ. In [Vas18a, Section 6], the language is expanded by adding a new predicate for the substructure relation. In [BB16, Theorem 3.2.3], $I_2(\lambda, \mathbf{K})$ many new predicates are added. Here we explicitly define the predicates in $L_{(\lambda+I_2(\lambda,\mathbf{K}))^+,\lambda^+}(\omega \cdot \omega)$ without expanding the language.

Proposition 4.3.16. *There is an $L_{(\lambda+I_2(\lambda,\mathbf{K}))^+,\lambda^+}(\omega \cdot \omega)$ formula $\sigma_\leq(x)$ that encodes the $\mathbf{K}$-substructure relation: for any $M \in K$, $a \subseteq |M|$ of size λ, $M \vDash \sigma_\leq[a]$ iff $a \leq M$ (the enumeration of a does not matter).*

Proof. Our definition $\sigma_\leq(x)$ will be similar to that of $\sigma_\mathbf{K}$:

$$\begin{aligned}\sigma_\leq(x) := (\forall x_\alpha \exists y_\alpha)_{\alpha<\omega\cdot\omega} &\Big(\exists u \exists v (x \approx u) \wedge (y_0 \approx v) \wedge \bigvee_{j<I_2(\lambda,\mathbf{K})} \psi_j(u,v)\Big) \wedge \\ &\bigwedge_{\alpha<\omega\cdot\omega} \Big[(x_\alpha \subseteq y_\alpha) \wedge \exists z_\alpha \Big((y_\alpha \approx z_\alpha) \wedge \bigvee_{i<I(\lambda,\mathbf{K})} \phi_i(z_\alpha)\Big) \wedge \\ &\bigwedge_{\beta<\alpha<\omega\cdot\omega} \exists u_{\beta,\alpha} \exists v_{\beta,\alpha} \Big((y_\beta \approx u_{\beta,\alpha}) \wedge (y_\alpha \approx v_{\beta,\alpha}) \wedge \bigvee_{j<I_2(\lambda,\mathbf{K})} \psi_j(u_{\beta,\alpha}, v_{\beta,\alpha})\Big)\Big]\end{aligned}$$

The enumeration of a does not matter by our definition of $\approx$. If $a \in K_\lambda$ and $a \leq M$, then $M \vDash \sigma_\leq[a]$ by Löwenheim-Skolem and coherence axioms. In particular, given x_0 in $\sigma_\leq[a]$, we choose $y_0 \leq M$ that contains both x_0 and a. Then coherence guarantees that $a \leq y_0$. Conversely suppose $a \subseteq |M|$ of size λ and $M \vDash \sigma_\leq[a]$. As in Main Theorem 4.3.7, we can build a directed system $\langle M_\alpha \in K_\lambda : \alpha \in I \rangle$ of union M with the additional requirement that for any $\alpha \in I$, $M_\alpha \geq a$. By Fact 4.2.4(1)(2), $M \in K$ and $M_\alpha \leq M$ for all $\alpha \in I$. By transitivity of $\leq$, $a \leq M$ as desired. $\square$

Remark 4.3.17. Using the terminology in [SV21, Theorem 2.1], we only used the first two levels of the canonical tree because a K-substructure relation only concerns two levels. The price to pay is game quantification.

Since our infinitary language only allows λ many free variables, it cannot directly encode substructures of size greater than λ. We propose two solutions: the first solution is a substructure relation whose underlying language is the singleton $\{\sigma_{\leq}\}$. The second solution involves relativizing $\sigma_{\leq}$ to a new predicate.

Proposition 4.3.18. *Let $M, N \in K$ and $\sigma_{\leq}$ be defined as in Proposition 4.3.16.*

1. *$M \leq N$ iff $M \subseteq_{\{\sigma_{\leq}\}} N$ (if $a \subseteq |M|$ is of size λ, then $M \vDash \sigma_{\leq}[a]$ iff $N \vDash \sigma_{\leq}[a]$).*

2. *Let R be a new predicate where $N^R = |M|$ closed under permutations. $M \leq N$ iff $(N, R) \vDash \forall b \left(\sigma^R_{\leq}(b) \to \sigma_{\leq}(b)\right)$ where $\sigma^R_{\leq}$ is the relativized version of $\sigma_{\leq}$ inside R. Namely,*

$$
\begin{aligned}
\sigma^R_{\leq}(x) := (\forall x_\alpha \exists y_\alpha)_{\alpha<\omega\cdot\omega} \Big(\bigwedge_{\alpha<\omega\cdot\omega} R(x_\alpha)\Big) \to \Big\{ \Big(\bigwedge_{\alpha<\omega\cdot\omega} R(y_\alpha)\Big) \wedge \\
\Big(\exists u \exists v (x \approx u) \wedge (y_0 \approx v) \wedge \bigvee_{j<I_2(\lambda,\mathbf{K})} \psi_j(u,v)\Big) \wedge \\
\bigwedge_{\alpha<\omega\cdot\omega} \Big[(x_\alpha \subseteq y_\alpha) \wedge \exists z_\alpha \Big((y_\alpha \approx z_\alpha) \wedge \bigvee_{i<I(\lambda,\mathbf{K})} \phi_i(z_\alpha)\Big) \wedge \\
\bigwedge_{\beta<\alpha<\omega\cdot\omega} \exists u_{\beta,\alpha} \exists v_{\beta,\alpha} \Big((y_\beta \approx u_{\beta,\alpha}) \wedge (y_\alpha \approx v_{\beta,\alpha}) \wedge \bigvee_{j<I_2(\lambda,\mathbf{K})} \psi_j(u_{\beta,\alpha}, v_{\beta,\alpha})\Big) \Big] \Big\}
\end{aligned}
$$

Proof. 1. If $M \leq N$ and let $a \subseteq M$. Using Proposition 4.3.16, if $M \vDash \sigma_{\leq}[a]$, then $a \leq M \leq N$ showing $N \vDash \sigma_{\leq}[a]$. If $N \vDash \sigma_{\leq}[a]$, then $a \leq N$. By coherence, $a \leq M$ and so $M \vDash \sigma_{\leq}[a]$. Conversely, build a directed system $\langle M_\alpha \in K_\lambda : \alpha \in I\rangle$ inside M such that for all $\alpha \in I$, $M_\alpha \leq M$. Then $M \vDash \sigma_{\leq}[M_\alpha]$. Since $M \subseteq_{\{\sigma_{\leq}\}} N$, we have $N \vDash \sigma_{\leq}[M_\alpha]$ and $M_\alpha \leq N$. The result follows from Fact 4.2.4(3).

2. If $M \leq N$ and $N \vDash \sigma^R_{\leq}[b]$ for some $b \subseteq |N|$, we need to show that $N \vDash \sigma_{\leq}[b]$. By assumption we can build a directed system of union M and have $b \leq M$. By transitivity of $\leq$, $b \leq N$ and the conclusion follows. Conversely, by Fact 4.2.4(3), it suffices to build a directed system $\langle M_\alpha \in K_\lambda : \alpha \in I\rangle$ of union M such that for all $\alpha \in I$, $M_\alpha \leq N$. Since $(N, R) \vDash \forall b \left(\sigma^R_{\leq}(b) \to \sigma_{\leq}(b)\right)$, we can require $M_\alpha \leq M$ instead of $M_\alpha \leq N$. Such construction is possible by Löwenheim-Skolem and coherence axioms.

□

Using game quantification, we derive a simple proof to [Kue08, Theorem 7.2], which uses back-and-forth arguments. [Vas18a, Theorem 6.21] proved similarly by transfering the AEC to its substructure expansion and translating results between $\equiv_{\infty,\lambda^+}$ and λ^+ back-and-forth systems.

Corollary 4.3.19. *Let M, N be L-structures. If either M or N is in K and $M \preceq_{L_{\infty,\lambda^+}(\omega\cdot\omega)} N$, then $M \leq N$ (and both are in K).*

Proof. Since $M \preceq_{L_{\infty,\lambda^+}(\omega\cdot\omega)} N$, $M \subseteq_{L_{(\lambda+I_2(\lambda,\mathbf{K}))^+,\lambda^+}(\omega\cdot\omega)} N$. In particular $M \vDash \sigma_{\mathbf{K}}$ iff $N \vDash \sigma_{\mathbf{K}}$. By Main Theorem 4.3.7, either M, N is in K implies both are in K. On the other hand, the assumption implies $M \subseteq_{\{\sigma_\leq\}} N$. By Proposition 4.3.18(1), $M \leq N$. □

One can ask the same Question 4.3.9 for $\mathbf{K}$-substructure relation instead of models of K. The following are variations on Proposition 4.3.16:

Corollary 4.3.20. *There is a formula $\sigma_\leq(x)$ in $L_{\lambda^+,\lambda^+}(\omega\cdot\omega)$ that encodes the $\mathbf{K}$-substructure relation (for any $M \in K$, $a \subseteq |M|$ of size λ, $M \vDash \sigma_\leq[a]$ iff $a \leq M$), assuming one of the following:*

1. *$\mathbf{K}$ is a universal class, in which case $\sigma_\leq(x)$ is in L_{λ^+,λ^+} (categoricity is not needed).*
2. *$\mathbf{K}$ has AL, λ-AP and is categorical in λ, λ^+ and we restrict the use of $\sigma_\leq$ to models of size $\geq \lambda^+$. AL can be replaced by stability in λ.*
3. *$2^\lambda < 2^{\lambda^+}$, $\mathbf{K}$ has AL and is categorical in λ, λ^+ and we restrict the use of $\sigma_\leq$ to models of size $\geq \lambda^+$. AL can be replaced by stability in λ.*

Proof.

1. $\sigma_\leq(x)$ requires that x is closed under functions.
2. Combine the proofs of Theorem 4.3.12 and Proposition 4.3.16: from Proposition 4.3.16, it suffices to encode $a \leq b$ where both have length λ (then we can replace $\bigvee_{j<I_2(\lambda,\mathbf{K})} \psi_j(a,b)$). Fix $M, N \in K_\lambda$ such that N is (λ,ω)-limit over M. Let $\phi(x)$ code the isomorphism type of M and $\psi(x,y)$ code the isomorphism type of (M,N). From the proof of Theorem 4.3.12, ψ is the only isomorphism type of pairs inside a λ^+-saturated model. Thus $a \leq b$ can be encoded as

$$a \subseteq b \wedge \exists z_0\ \exists z_1\ \exists a'\ \exists b'\big((a \approx a') \wedge (b \approx b') \wedge (z_0 \approx z_1) \wedge \psi(a', z_0) \wedge \psi(b', z_1)\big).$$

3. Combine (2) and Fact 4.3.13.

$\square$

With the exact same proof as in Proposition 4.3.18, we can show:

Corollary 4.3.21. *Let $M, N \in K$ and $\sigma_{\leq}$ be defined as in Corollary 4.3.20.*

1. *$M \leq N$ iff $M \subseteq_{\{\sigma_{\leq}\}} N$ (if $a \subseteq |M|$ is of size λ, then $M \vDash \sigma_{\leq}[a]$ iff $N \vDash \sigma_{\leq}[a]$).*

2. *Let R be a new predicate where $N^R = |M|$ closed under permutations. $M \leq N$ iff $(N, R) \vDash \forall b \left(\sigma^R_{\leq}(b) \to \sigma_{\leq}(b)\right)$ where $\sigma^R_{\leq}$ is the relativized version of $\sigma_{\leq}$ inside R.*

4.4 A VARIATION ON SHELAH'S PRESENTATION THEOREM

We will give a variation to Shelah's presentation theorem via our encoding of AECs. The presentation theorem statement is adapted from [Bal09, Theorem 4.15] (see also [Gro02, Theorem 3.4]).

Theorem 4.4.1. *Let $\mathbf{K}$ be an AEC in L and with Löwenheim-Skolem number* $\mathrm{LS}(\mathbf{K})$*. Define $\chi := \mathrm{LS}(\mathbf{K}) + I_2(\mathrm{LS}(\mathbf{K}), \mathbf{K})$. There exists an expansion $L' \supseteq L$ of size χ, an L'-theory T and a set of L'-types Γ of size χ such that*

1. *$K = PC(T, \Gamma, L)$.*

2. *If $M', N' \in EC(T, \Gamma)$ and $M' \subseteq_{L'} N'$, then $M' \restriction L \leq_{\mathbf{K}} N' \restriction L$.*

3. *If $M \leq_{\mathbf{K}} N$, there are L'-expansions of M, N to M', N' such that $M' \subseteq_{L'} N'$.*

Remark 4.4.2. When $I_2(\mathrm{LS}(\mathbf{K}), \mathbf{K}) < 2^{\mathrm{LS}(\mathbf{K})}$, our result is stronger than Shelah's presentation theorem as his encoding totally ignores the model theoretic complexity of $\mathbf{K}$ and is using $2^{\mathrm{LS}(\mathbf{K})}$ many types in Γ.

Proof. We will adapt $\sigma_{\mathbf{K}}$ in Main Theorem 4.3.7 to the first-order language. For a tuple of variables a and $n < \omega$, we write a^n to emphasize a has length n. For $k < n$, we write $a^n(k)$ the k-th coordinate of a^n.

As in the original presentation theorem, expand L to L' which includes $\{f^n_k : k < \mathrm{LS}(\mathbf{K}), n < \omega\}$ where for each $n < \omega$, $k < \mathrm{LS}(\mathbf{K})$, f^n_k is an n-ary function. For $n < \omega$, we will require that $\{f^n_k : k < \mathrm{LS}(\mathbf{K})\}$ maps an n-tuple to a K-structure of size $\mathrm{LS}(\mathbf{K})$ containing that tuple. This will be achieved by

$$\sigma^n := \forall a^n \bigwedge_{l<n} \bigvee_{k<\mathrm{LS}(\mathbf{K})} \big(f^n_k(a^n) = a^n(l)\big) \wedge \bigvee_{i<I(\lambda,\mathbf{K})} \phi_i\big(\{f^n_k(a^n) : k < \mathrm{LS}(\mathbf{K})\}\big)$$

In the above definition, although ϕ_i's have $\mathrm{LS}(\mathbf{K})$ many free variables, it is just an $\mathrm{LS}(\mathbf{K})$-conjunction of (negation of) atomic formulas with n free variables (from a^n). So each ϕ_i is inside $L'_{\mathrm{LS}(\mathbf{K})^+,\omega}$.

Also, we want to require that the K-structures generated are directed with respect to the tuple input. However, $\{f^n_k : k < \mathrm{LS}(\mathbf{K}), n < \omega\}$ might not be compatible with the enumerations of *pairs* of models, say $M_j \leq N_j$. Hence we expand L' further to include $\{g^{m,l}_k, h^{m,l}_k : k < \mathrm{LS}(\mathbf{K}), m + l < \omega\}$ where for $m + l < \omega$, $k < \mathrm{LS}(\mathbf{K})$, $g^{m,l}_k$ and $h^{m,l}_k$ are $(m + l)$-ary functions and correctly enumerate a pair of models. The following will take care of the re-enumerations of $\{f^n_k : k < \mathrm{LS}(\mathbf{K})\}$ for each $n < \omega$.

$$\begin{aligned}\sigma^{m,n,l} := \forall b^m\ \forall c^n\ \forall d^l \Big[&\mathrm{ran}(b^m) \cup \mathrm{ran}(c^n) \subseteq \mathrm{ran}(d^l) \to \\ &\Big(\{f^m_k(b^m) : k < \mathrm{LS}(\mathbf{K})\} \approx \{g^{m,l}_k(b^m; d^l) : k < \mathrm{LS}(\mathbf{K})\} \wedge \\ &\{f^l_k(d^l) : k < \mathrm{LS}(\mathbf{K})\} \approx \{h^{m,l}_k(b^m; d^l) : k < \mathrm{LS}(\mathbf{K})\} \wedge \\ &\{f^n_k(c^n) : k < \mathrm{LS}(\mathbf{K})\} \approx \{g^{n,l}_k(c^n; d^l) : k < \mathrm{LS}(\mathbf{K})\} \wedge \\ &\{f^l_k(d^l) : k < \mathrm{LS}(\mathbf{K})\} \approx \{h^{n,l}_k(c^n; d^l) : k < \mathrm{LS}(\mathbf{K})\} \wedge \\ &\bigvee_{i<I_2(\mathrm{LS}(\mathbf{K}),\mathbf{K})} \psi_i\big(\{g^{m,l}_k(b^m; d^l) : k < \mathrm{LS}(\mathbf{K})\}, \{h^{m,l}_k(b^m; d^l) : k < \mathrm{LS}(\mathbf{K})\}\big) \wedge \\ &\bigvee_{j<I_2(\mathrm{LS}(\mathbf{K}),\mathbf{K})} \psi_j\big(\{g^{n,l}_k(c^n; d^l) : k < \mathrm{LS}(\mathbf{K})\}, \{h^{n,l}_k(c^n; d^l) : k < \mathrm{LS}(\mathbf{K})\}\big)\Big)\Big]\end{aligned}$$

Similar to the case of σ^n, the formulas ϕ_i, ψ_j and the connective $\approx$ are simply $\mathrm{LS}(\mathbf{K})$-junctions of (negation of) atomic formulas, which are inside $L'_{\mathrm{LS}(\mathbf{K})^+,\omega}$.

To convert $\{\sigma^n : n < \omega\} \cup \{\sigma^{m,n,l} : m, n, l < \omega\}$ into first-order sentences, we use Chang's presentation theorem (see [Gro21, Chapter 1 Theorem 8.16]) which adds χ-many new predicates to L' to represent the χ-conjunctions and disjunctions, and χ-many L'-types to omit. This gives our final T, Γ and L'.

It remains to check the three items in the theorem statement.

1. $K \subseteq PC(T, \Gamma, L)$ by Löwenheim-Skolem and coherence axioms. Let $M \in PC(T, \Gamma, L)$ and $M' \in EC(T, \Gamma)$ such that $M' \restriction L = M$. Then $M' \restriction L$ is the union of a directed system of K-structures of size $\mathrm{LS}(\mathbf{K})$. By Fact 4.2.4(1) $M \in K$. Hence $PC(T, \Gamma, L) \subseteq K$.

2. By Fact 4.2.4(3).

3. By Fact 4.2.5.

$\square$

Another question raised by Grossberg is the following:

Question 4.4.3. Is it possible to lower the bound of $|\Gamma|$ below $\mathrm{LS}(\mathbf{K}) + I_2(\mathrm{LS}(\mathbf{K}), \mathbf{K})$ in general? What if we also assume tameness or stability?

Remark 4.4.4. 1. In the above proof, we did not use $\sigma_{\leq}$ because $\{f_k^n : k < \mathrm{LS}(\mathbf{K}), n < \omega\}$ already plays its role. We could have done the same in Proposition 4.3.18 but the approach via $\sigma_{\leq}$ is cleaner and does not add new function symbols.

2. One might want to encode $\bigvee_{i<I_2(\lambda,\mathbf{K})} \psi_i$ etc by omitting types without raising $|T|$ above $\mathrm{LS}(\mathbf{K})$. However, it amounts to list all pairs of isomorphism types that are not any of the ψ_i's. This will raise $|\Gamma|$ to $2^{\mathrm{LS}(\mathbf{K})}$ which is equivalent to the original presentation theorem.

3. In [BB16, Theorem 3.2.3], new predicates are essentially added for our ϕ_i and ψ_j. Also, the requirement in [BB16, Definition 3.2.1(4)] encodes our $\{\sigma^{m,n,l} : m, n, l < \omega\}$ to build a directed system, but our version is purely syntactic and more direct. We used Chang's representation theorem to bring down the infinitary logic to first-order with omitting types. In [BB16], their theory T^* is still in the infinitary logic and thus does not need to omit types.

We derive the known results in the following corollary. Item (1) appeared for the first time in [Gro21] while (2) and (3) were undoubtedly known to Chang [Cha68].

Corollary 4.4.5. *Let* $\mathbf{K}$ *be an AEC and* $\chi := \mathrm{LS}(\mathbf{K}) + I_2(\mathrm{LS}(\mathbf{K}), \mathbf{K})$.

1. *K is a $PC_\chi (= PC_{\chi,1})$ class.*

2. *The Hanf number of $\mathbf{K}$ is bounded above by $\beth_{\delta(\chi,1)}$.*

3. *If $\chi = \aleph_0$ or χ is a strong limit with $\mathrm{cf}(\chi) = \aleph_0$, then the Hanf number of $\mathbf{K}$ is bounded above by $\beth_{\chi^+}$.*

Proof. 1. Combine Theorem 4.4.1 and Shelah's 1-type coding [She90, VII Lemma 5.1(4)].

2. Combine (1) and [She90, VII Theorem 5.3].

3. Combine (2) and the fact that $\delta(\chi, 1) = \chi^+$ [She90, VII Theorem 5.5(5)].

$\square$

We finish this section with one more application.

Definition 4.4.6. Let $\mathbf{K}$ be an AEC. $K^< := \{\langle |M|, |N| \rangle : N < M\}$ is a class of structures whose language consists of a single unary predicate.

In 1994, motivated by [She87, Theorem 3.8], Grossberg suggested the following conjecture (see Problem (5) in [She01, Introduction]):

Conjecture 4.4.7. Let $\mathbf{K}$ be an AEC, $\lambda \geq \mathrm{LS}(\mathbf{K})$. If $I(\lambda, \mathbf{K}) = I(\lambda^+, \mathbf{K}) = 1$, then $K_{\lambda^{++}} \neq \emptyset$.

[She87, Theorem 3.8] has two additional hypotheses:

1. Both K and $K^<$ are PC_λ; and

2. $\delta(\lambda, 1) = \lambda^+$.

Much of [She01] is dedicated to the special cases of Grossberg's conjecture under various strong assumptions (including non-ZFC axioms).

Here we delete hypothesis (1) above and work in ZFC. In addition to hypothesis (2), we assume that $\lambda \geq \mathrm{LS}(\mathbf{K}) + I_2(\mathrm{LS}(\mathbf{K}), \mathbf{K})$.

Theorem 4.4.8. *Let* $\mathbf{K}$ *be an AEC,* $\chi := \mathrm{LS}(\mathbf{K}) + I_2(\mathrm{LS}(\mathbf{K}), \mathbf{K})$*,* $\lambda \geq \chi$ *with* $\delta(\lambda, 1) = \lambda^+$*. If* $\mathbf{K}$ *is categorical in* λ *and* λ^+*, then* $\mathbf{K}$ *has a model of cardinality* λ^{++}*.*

Remark 4.4.9. Our theorem applies to the case $\mathrm{LS}(\mathbf{K}) = \aleph_1$, $\lambda = \beth_\omega$ while [She87, Theorem 3.8] cannot handle uncountable $\mathrm{L}(\mathbf{K})$ or $\mathrm{LS}(\mathbf{K})$.

Proof. We check that hypothesis (1) above is satisfied. Since $\lambda \geq \chi$, it suffices to show that both K and $K^<$ are PC_χ classes. K is a PC_χ class by Corollary 4.4.5. To show that $K^<$ is also a PC_χ class, we will use the proof of Theorem 4.4.1, add a new predicate R in L' and encode Proposition 4.3.18(2) by the new functions $\{f^n_k : k < \mathrm{LS}(\mathbf{K}), n < \omega\}$ (to lighten the notation, we omit the encoding of re-enumerations, but it is the same strategy as in Theorem 4.4.1). At the end, we will only leave R in the reduct of the language.

The details are as follows: we expand the language L to L' which includes a new predicate R and the functions $\{f^n_k : k < \mathrm{LS}(\mathbf{K}), n < \omega\}$ as in Theorem 4.4.1. For $n < \omega$, we abbreviate $\{f^n_k : k < \mathrm{LS}(\mathbf{K})\}$ as $\bar{f}^n$ and require that it maps an n-tuple to a model of size $\mathrm{LS}(\mathbf{K})$ containing the tuple. This can be achieved by

$$\sigma^n := \forall a^n \bigwedge_{l<n} \bigvee_{k<\mathrm{LS}(\mathbf{K})} \big(f^n_k(a^n) = a^n(l)\big) \wedge \bigvee_{i<I(\lambda,\mathbf{K})} \phi_i\big(\bar{f}^n(a^n)\big)$$

We also require that given an n-tuple inside R, the model generated is within R:

$$\sigma^n_R := \forall a^n \subseteq R\big(\bar{f}^n(a^n) \subseteq R\big)$$

Next, we want to require that the models generated are directed with respect to the tuple input. For $m, n, l < \omega$,

$$\sigma^{m,n,l} := \forall b^m\ \forall c^n\ \forall d^l \Big[\mathrm{ran}(b^m) \cup \mathrm{ran}(c^n) \subseteq \mathrm{ran}(d^l) \to \Big(\bigvee_{i<I_2(\lambda,\mathbf{K})} \psi_i\big(\bar{f}^m(b^m), \bar{f}^l(d^l)\big) \wedge \bigvee_{j<I_2(\lambda,\mathbf{K})} \psi_j\big(\bar{f}^n(c^n), \bar{f}^l(d^l)\big)\Big)\Big]$$

The final requirement is that R is a proper subset of the model:

$$\sigma_p := \exists x(\neg R(x))$$

Notice that for $m, n, l < \omega$, the sentences $\sigma^n, \sigma^n_R, \sigma^{m,n,l}$ are χ-junctions of (negation of) atomic formulas, so we can use Chang's presentation theorem to convert them to first-order formulas, by adding χ-many new predicates to L' to represent the χ-conjunctions and disjunctions, and χ-many L'-types to omit. This gives our T, Γ and L'.

We check that $K^{<} = PC(T, \Gamma, \{R\})$. If $\langle |M|, |N| \rangle$ is in $K^{<}$, then $N < M$ are in K. Expand the language to L' and define $R^M := N$. Inside N, build a directed system of $\mathbf{K}$-substructures of size $\mathrm{LS}(\mathbf{K})$, indexed by the finite tuples of N. This determines $\bar{f}^n \restriction R^n$ for $n < \omega$. Now inside M, we extend the directed system to be indexed by the finite tuples of M. This determines $\bar{f}^n$ completely for $n < \omega$. Also, M satisfies σ^n, σ^n_R and $\sigma^{m,n,l}$ for $m, n, l < \omega$. Hence M under the expanded language is in $EC(T, \Gamma)$ and its reduct to $\{R\}$ is in $PC(T, \Gamma, \{R\})$. Conversely, if $M \in PC(T, \Gamma, \{R\})$, expand M to M' such that $M' \in EC(T, \Gamma)$ and define $N' := R^{M'}$. By $\{\sigma^n, \sigma^n_R, \sigma^{m,n,l} : m, n, l < \omega\}$, $N' \restriction L$ is the union of the directed system of $\mathbf{K}$-structures of size $\mathrm{LS}(\mathbf{K})$. By Fact 4.2.4(1), $N' \restriction L \in K$. By $\{\sigma^n, \sigma^{m,n,l} : m, n, l < \omega\}$, the directed system can be extended to union $M' \restriction L$. By Fact 4.2.4(1) again, $M' \restriction L \in K$. By Fact 4.2.4(2), each $\mathbf{K}$-structure of the directed system is a $\mathbf{K}$-substructure $M' \restriction L$. But then the models of the original system that generates $N' \restriction L$ are all $\mathbf{K}$-substructures of $M' \restriction L$. By Fact 4.2.4(3), $N' \restriction L \leq M' \restriction L$. By σ_p, $N' \restriction L < M' \restriction L$. In other words, $\langle |M'|, |N'| \rangle = \langle |M|, R^M \rangle = M \in K^{<}$. $\square$

As in Section 3, we can add extra assumptions to improve our results:

Corollary 4.4.10. *1. If* $\mathbf{K}$ *is a universal class, then* $\chi := \mathrm{LS}(\mathbf{K}) + I_2(\mathrm{LS}(\mathbf{K}), \mathbf{K})$ *in Theorem 4.4.1, Corollary 4.4.5 and Theorem 4.4.8 can be replaced by* $\chi := \mathrm{LS}(\mathbf{K})$.

2. If $\mathbf{K}$ *is categorical in* $\mathrm{LS}(\mathbf{K})$ *and* $\mathrm{LS}(\mathbf{K})^+$*, has AL and either*

(a) $\mathbf{K}$ *has* $\mathrm{LS}(\mathbf{K})$*-AP; or*

(b) $2^{\mathrm{LS}(\mathbf{K})} < 2^{\mathrm{LS}(\mathbf{K})^+}$

then K *and* $K^{<}$ *are both* $PC_{\mathrm{LS}(\mathbf{K})}$ *classes when restricted to models of size* $\geq \mathrm{LS}(\mathbf{K})^+$*. In either case, AL can be replaced by stability in* $\mathrm{LS}(\mathbf{K})$*.*

3. In (2), K *can be made to a* $PC_{\mathrm{LS}(\mathbf{K})}$ *class.*

Proof sketch. 1. Combine the proof of Theorem 4.4.1, Corollary 4.4.5 and Theorem 4.4.8 with Proposition 4.3.10. The point is that we do not need to encode $\mathbf{K}$-substructure relation.

2. In Theorem 4.3.12 and Corollary 4.3.14, we used coherence to encode $a \leq b$ in the infinitary language: $a \subseteq b$ and there is c which is $(\mathrm{LS}(\mathbf{K}), \omega)$-limit over a and b. Thus we can add two more sets of functions $\{d_k^{m,l}, e_k^{m,l} : k < \mathrm{LS}(\mathbf{K}), m + l < \omega\}$ (which represent c) in addition to the original $\{g_k^{m,l}, h_k^{m,l} : k < \mathrm{LS}(\mathbf{K}), m + l < \omega\}$ (which represent a and b) in Theorem 4.4.1.

3. It remains to handle the case when the models are of size $\mathrm{LS}(\mathbf{K})$: add a disjunct to the theory T in Theorem 4.4.1 which stipulates that the model is generated by an element a^1:

$$\exists a^1 \Big(\forall x^1 \bigvee_{k < \mathrm{LS}(\mathbf{K})} (f_k^1(a^1) = x^1) \wedge \phi\big(\{f_k^1(a^1) : k < \mathrm{LS}(\mathbf{K})\}\big)\Big).$$

$\square$

Remark 4.4.11. 1. Observation 4.3.15 also applies to the above corollary. If $\mathrm{LS}(\mathbf{K}) = \aleph_0$ in (3), then we obtain: if $\mathbf{K}$ is stable in $\aleph_0$, has $\aleph_0$-AP, $I(\aleph_0, \mathbf{K}) \leq \aleph_0$ and $I(\aleph_1, \mathbf{K}) = 1$, then K is $PC_{\aleph_0}$. This special case (with the extra assumption of categoricity in $\aleph_1$) provides an alternative proof to [SV18a, Theorem 4.2] which uses results from descriptive set theory.

2. We do not know if (3) also applies to $K^{<}$, for a similar reason in Corollary 4.3.20(2),(3).

4.5 GENERALIZATION TO μ-AECS

Our strategy of encoding AECs is also applicable to μ-AECs.

Definition 4.5.1. [BGL+16, Definitions 2.1,3.1] Let L be a $(< \mu)$-ary language. A μ-AEC $\mathbf{K} = \langle K, \leq_{\mathbf{K}} \rangle$ in L satisfies the axioms (1)(2)(3)(4) in Definition 4.2.2 in addition to

a. Directed system axioms: if $\langle M_i \in K : i \in I \rangle$ is a μ-directed system, then $M := \bigcup_{i \in I} M_i \in K$ and for all $i \in I$, $M_i \leq_{\mathbf{K}} M$. If in addition $N \in K$ with $M_i \leq_{\mathbf{K}} N$ for all $i \in I$, then $M \leq_{\mathbf{K}} N$.

b. Löwenheim-Skolem axiom: there exists a cardinal $\lambda = \lambda^{<\mu} \geq |\operatorname{L}(\mathbf{K})| + \mu$ such that for any $M \in K$, $A \subseteq |M|$, there is $N \leq_{\mathbf{K}} M$ such that $A \subseteq |N|$ with $\|N\| \leq |A|^{<\mu} + \lambda$. We call the minimum such cardinal the Löwenheim-Skolem number $\operatorname{LS}(\mathbf{K})$.

The analogs of Main Theorem 4.3.7, Proposition 4.3.16 and Proposition 4.3.18 hold in μ-AECs.

Proposition 4.5.2. *Let $\mathbf{K}$ be a μ-AEC in L and $\lambda := \operatorname{LS}(\mathbf{K})$. K is axiomatizable by an $L_{(\lambda+I_2(\lambda,\mathbf{K}))^+,\lambda^+}(\mu\cdot\mu)$ sentence $\sigma_{\mathbf{K}}$. In other words, for any L-structure M, $M \in K$ iff $M \vDash \sigma_{\mathbf{K}}$.*

Proof. Similar to the proof in Main Theorem 4.3.7. The difference is that we allow the iterated use of Löwenheim-Skolem and coherence axioms $(\mu\cdot\mu)$-many times instead of $(\omega\cdot\omega)$-many. We give the details below:

As usual, list the isomorphism types $\{M/_{\cong} : M \in K_\lambda\}$ by $\langle M_i : i < I(\lambda,\mathbf{K})\rangle$ and those of $\{(M,N)/_{\cong} : M \leq N \text{ in } K_\lambda\}$ by $\langle (M_j,N_j) : j < I_2(\lambda,\mathbf{K})\rangle$. For $i < I(\lambda,\mathbf{K})$, let $\phi_i(x)$ be an L_{λ^+,λ^+} formula that encodes the isomorphism type of M_i with a fixed enumeration of the universe $|M| = \langle m^i_k : k < \lambda\rangle$. For variables $x = \langle x_k : k < \lambda\rangle$,

$$\begin{aligned}\phi_i(x) := \bigwedge\{\theta(x_{\alpha_0},\dots,x_{\alpha_\xi}) : M_i \vDash \theta[m^i_{\alpha_0},\dots,m^i_{\alpha_\xi}];\ \xi<\mu,\ \alpha_0,\dots,\alpha_\xi<\lambda,\\ \theta \text{ is an atomic } L\text{-formula or its negation with } s \text{ free variables}\}\end{aligned}$$

Notice that ϕ_i is a conjunction of $\lambda^{<\mu} = \lambda$ many formulas so it is inside L_{λ^+,λ^+}. Similarly for $j < I_2(\lambda,\mathbf{K})$, let $\psi_j(x,y)$ be an L_{λ^+,λ^+} formula that encodes the isomorphism type of (M_j,N_j) with fixed enumerations, where $|M_j| = \{m^j_k : k<\lambda\}$, $|N_j| = \{n^j_k : k < \lambda\}$. For variables $x = \langle x_k : k<\lambda\rangle$ and $y = \langle y_k : k < \lambda\rangle$,

$$\begin{aligned}\psi_j(x;y) := \bigwedge\{\theta(x_{\alpha_0},\dots,x_{\alpha_\xi};y_{\beta_0},\dots,y_{\beta_{\xi'}}) : N_j \vDash \theta[m^j_{\alpha_0},\dots,m^j_{\alpha_\xi};n^j_{\beta_0},\dots,n^j_{\beta_{\xi'}}],\\ \xi,\xi'<\mu;\ \alpha_0,\dots,\alpha_\xi,\beta_0,\dots,\beta_{\xi'}<\lambda,\\ \theta \text{ is an atomic } L\text{-formula or its negation with } \xi+\xi' \text{ free variables}\}\end{aligned}$$

ψ_j is also a conjunction of $\lambda^{<\mu} = \lambda$ many formulas so it is inside L_{λ^+,λ^+}. The axiomatization

$\sigma_{\mathbf{K}}$ consists of two components (the variables all have length λ):

$$\sigma_{\mathbf{K}} := (\forall x_\alpha \exists y_\alpha)_{\alpha<\mu\cdot\mu} \bigwedge_{\alpha<\mu\cdot\mu} \Big[(x_\alpha \subseteq y_\alpha) \wedge \exists z_\alpha \Big((y_\alpha \approx z_\alpha) \wedge \bigvee_{i<I(\lambda,\mathbf{K})} \phi_i(z_\alpha)\Big) \wedge$$
$$\bigwedge_{\beta<\alpha<\mu\cdot\mu} \exists u_{\beta,\alpha} \exists v_{\beta,\alpha} \Big((y_\beta \approx u_{\beta,\alpha}) \wedge (y_\alpha \approx v_{\beta,\alpha}) \wedge \bigvee_{j<I_2(\lambda,\mathbf{K})} \psi_j(u_{\beta,\alpha}, v_{\beta,\alpha})\Big)\Big]$$

Suppose $M \in K$, we show that Player II can win the associated game in $\sigma_{\mathbf{K}}$. In the α-th round, Player I provides some x_α of length λ. By Löwenheim-Skolem axiom, pick any $y_\alpha \leq M$ of size λ such that $\operatorname{ran}(x_\alpha) \cup \bigcup_{\beta<\alpha} \operatorname{ran}(y_\beta) \subseteq \operatorname{ran}(y_\alpha)$. By inductive hypothesis, for $\beta < \alpha$, we have $y_\beta \leq M$. By coherence axiom, $y_\beta \leq y_\alpha$ as desired.

Suppose $M \vDash \sigma_{\mathbf{K}}$. We will build a μ-directed system $\langle M_a \in K_\lambda : a \in I\rangle$ of union M, with I being the set of tuples of length $(< \mu)$ in M, ordered by inclusion. By directed system axioms, $M \in K$. In the proof of Main Theorem 4.3.7, we showed that given M_a and M_b generated by the games of the singletons s and t, it is possible to find $M^* \geq M_a, M_b$ by extending those games by ω-many rounds. In the μ-AEC case, without the usual chain axioms we do not know if $M^* \in K$, so we extend those games by μ-many rounds instead to obtain an increasing (but not necessarily continuous) chain $\langle N_k : k < \mu\rangle$ and define $M^* = \bigcup_{k<\mu} N_k \in K$. A similar argument shows: let $\delta < \mu$ and if for $\alpha < \delta$, M_α is generated by the game of some tuple a_α of length $< \mu$, then we can extend the games by μ-many rounds to obtain M^* that extends all M_α. This allows us to get past the limit stages which were absent in the original proof, and continue to build M_a for $l(a) < \mu$. Given a tuple c of length $< \mu$, there are less than μ-many ways to decompose c into a union of a singleton and a tuple of length $< \mu$. Thus we can still combine all copies of M^* from the decompositions of M_c as in the original proof. $\square$

Proposition 4.5.3. *Let* $\mathbf{K}$ *be a* μ*-AEC in* L *and* $\lambda := \operatorname{LS}(\mathbf{K})$. *There is a formula in* $L_{(\lambda+I_2(\lambda,\mathbf{K}))^+,\lambda^+}(\mu\cdot\mu)$ *that encodes the* $\mathbf{K}$*-substructure relation: for any* $M \in K$, $a \subseteq |M|$ *of size* λ, $M \vDash \sigma_{\leq}[a]$ *iff* $a \leq M$ *(the enumeration of* a *does not matter).*

Proof. Define $\sigma_{\leq}(x)$ as in Proposition 4.3.16 but replace $\omega\cdot\omega$ by $\mu\cdot\mu$. The enumeration of a does not matter by our definition of $\approx$. If $a \in K_\lambda$ and $a \leq M$, then $M \vDash \sigma_{\leq}[a]$ by Löwenheim-Skolem and coherence axioms. Conversely suppose $a \subseteq |M|$ of size λ and

$M \vDash \sigma_{\leq}[a]$. As in Proposition 4.5.2, we can build a μ-directed system $\langle M_\alpha \in K_\lambda : \alpha \in I\rangle$ of union M such that for any $\alpha \in I$, $M_\alpha \geq a$. By directed system axioms, $M \in K$ and $M_\alpha \leq M$ for all $\alpha \in I$. By transitivity of $\leq$, $a \leq M$ as desired. $\square$

Proposition 4.5.4. *Let* $\mathbf{K}$ *be a* μ*-AEC in* L *and* $\lambda := \mathrm{LS}(\mathbf{K})$. *Let* $M, N \in K$.

1. $M \leq N$ *iff* $M \subseteq_{\{\sigma_{\leq}\}} N$ *(if* $a \subseteq |M|$ *is of size* λ*, then* $M \vDash \sigma_{\leq}[a]$ *iff* $N \vDash \sigma_{\leq}[a]$*).*
2. *Let* R *be a new predicate where* $N^R = |M|$ *closed under permutations.* $M \leq N$ *iff* $(N, R) \vDash \forall b \left(\sigma^R_{\leq}(b) \to \sigma_{\leq}(b)\right)$ *where* $\sigma^R_{\leq}$ *is the relativized version of* $\sigma_{\leq}$ *inside* R *(replace* $(\omega \cdot \omega)$ *by* $(\mu \cdot \mu)$ *in the definition of* $\sigma^R_{\leq}$ *in Proposition 4.3.18).*

Proof. Similar to the proof in Proposition 4.3.18. The difference is that instead of building $\aleph_0$-directed systems, we build μ-directed systems. We give details below:

1. If $M \leq N$ and let $a \subseteq M$. If $M \vDash \sigma_{\leq}[a]$, then $a \leq M \leq N$ showing $N \vDash \sigma_{\leq}[a]$. If $N \vDash \sigma_{\leq}[a]$, then $a \leq N$. By coherence, $a \leq M$ and so $M \vDash \sigma_{\leq}[a]$. Conversely, build a μ-directed system $\langle M_\alpha \in K_\lambda : \alpha \in I\rangle$ inside M such that for all $\alpha \in I$, $M_\alpha \leq M$. Then $M \vDash \sigma_{\leq}[M_\alpha]$. Since $M \subseteq_{\{\sigma_{\leq}\}} N$, we have $N \vDash \sigma_{\leq}[M_\alpha]$ and $M_\alpha \leq N$. The result follows from directed system axioms.
2. If $M \leq N$ and $N \vDash \sigma^R_{\leq}[b]$ for some $b \subseteq |N|$, we need to show that $N \vDash \sigma_{\leq}[b]$. By assumption we can build a μ-directed system of union M and have $b \leq M$. By transitivity of $\leq$, $b \leq N$ and the conclusion follows. Conversely, by directed system axioms, it suffices to build a μ-directed system $\langle M_\alpha \in K_\lambda : \alpha \in I\rangle$ of union M such that for all $\alpha \in I$, $M_\alpha \leq N$. Since $(N, R) \vDash \forall b \left(\sigma^R_{\leq}(b) \to \sigma_{\leq}(b)\right)$, we can require $M_\alpha \leq M$ instead of $M_\alpha \leq N$. Such construction is possible by Löwenheim-Skolem and coherence axioms.

$\square$

As an application of Proposition 4.5.2 and Proposition 4.5.4, we generalize Corollary 4.3.19:

Corollary 4.5.5. *Let $\mathbf{K}$ be a μ-AEC in L, $\lambda := \mathrm{LS}(\mathbf{K})$ and M, N be L-structures. If either M or N is in K and $M \preceq_{L_{\infty,\lambda^+}(\mu\cdot\mu)} N$, then $M \leq N$ (and both are in K).*

Proof. Same proof as in Corollary 4.3.19: Since $M \preceq_{L_{\infty,\lambda^+}} N$, $M \subseteq_{L_{(\lambda+I_2(\lambda,\mathbf{K}))^+,\lambda^+}(\mu\cdot\mu)} N$. In particular $M \vDash \sigma_{\mathbf{K}}$ iff $N \vDash \sigma_{\mathbf{K}}$. By Proposition 4.5.2, either M, N is in K implies both are in K. On the other hand, the assumption implies $M \subseteq_{\{\sigma_\leq\}} N$. By Proposition 4.5.4(1), $M \leq N$. $\square$

We now state the μ-AEC version of Theorem 4.4.1, which is a variation to [BGL+16, Theorem 3.2].

Theorem 4.5.6. *Let $\mathbf{K}$ be a μ-AEC in L and with Löwenheim-Skolem number $\mathrm{LS}(\mathbf{K})$. Define $\chi := \mathrm{LS}(\mathbf{K}) + I_2(\mathrm{LS}(\mathbf{K}), \mathbf{K})$. There exists a $(< \mu)$-ary expansion $L' \supseteq L$ of size χ, an L'-theory T and a set of L'-types Γ of size χ such that*

1. *$K = PC^\mu(T, \Gamma, L)$.*
2. *If $M', N' \in EC(T, \Gamma)$ and $M' \subseteq_{L'} N'$, then $M' \restriction L \leq_{\mathbf{K}} N' \restriction L$.*
3. *If $M \leq_{\mathbf{K}} N$, there are L'-expansions of M, N to M', N' such that $M' \subseteq_{L'} N'$.*

Proof sketch. Repeat the same argument in Theorem 4.4.1 by replacing ω by μ, in particular:

1. Superscripts of f, g, h will be $\alpha, \beta, \gamma < \mu$ instead of $m, n, l < \omega$.
2. We require that the K-substructures generated by $\{f_k^\alpha : k < \mathrm{LS}(\mathbf{K}), \alpha < \omega\}$ are μ-directed instead of $\aleph_0$-directed.
3. The sentences $\{\sigma^\alpha : \alpha < \mu\} \cup \{\sigma^{\alpha,\beta,\gamma} : \alpha, \beta, \gamma < \mu\}$ are in $L'_{\chi^+,\mu}$.
4. For $i < I(\mathrm{LS}(\mathbf{K}), \mathbf{K})$ and $j < I_2(\mathrm{LS}(\mathbf{K}), \mathbf{K})$, the formulas ϕ_i, ψ_j are still $\mathrm{LS}(\mathbf{K})$-conjunctions because $\mathrm{LS}(\mathbf{K})^{<\mu} = \mathrm{LS}(\mathbf{K})$.
5. Chang's presentation theorem generalizes to μ-AECs and converts a $L'_{\chi^+,\mu}$ theory of size χ into a PC^μ_χ.

6. When checking the items of the theorem statement, notice that by definition of a μ-AEC, directed system axioms (instead of chain axioms) are built-in. Meanwhile, Fact 4.2.5 generalizes to μ-directed systems.

$\square$

Unlike Corollary 4.4.5, the above result does not lead to the Hanf number computation because the languages are not finitary while well-ordering is definable. In particular there is no reasonable bound to the Hanf number of $L_{\aleph_1,\aleph_1}$ [Dic75, Chapter 5.1B]. As asked in [BGL$^+$16, Remark 3.3]:

Question 4.5.7. Let $\mu \geq \aleph_1$. Does the Hanf number exist for μ-AECs?

CHAPTER 5

STABILITY RESULTS ASSUMING TAMENESS, MONSTER MODEL AND CONTINUITY OF NONSPLITTING

ABSTRACT

Assuming the existence of a monster model, tameness and continuity of nonsplitting in an abstract elementary class (AEC), we extend known superstability results: let $\mu > \mathrm{LS}(\mathbf{K})$ be a regular stability cardinal and let χ be the local character of μ-nonsplitting. The following holds:

1. When μ-nonforking is restricted to $(\mu, \geq \chi)$-limit models ordered by universal extensions, it enjoys invariance, monotonicity, uniqueness, existence, extension and continuity. It also has local character χ. This generalizes Vasey's result [Vas18a, Corollary 13.16] which assumed μ-superstability to obtain same properties but with local character $\aleph_0$.
2. There is $\lambda \in [\mu, h(\mu))$ such that if $\mathbf{K}$ is stable in every cardinal between μ and λ, then $\mathbf{K}$ has μ-symmetry while μ-nonforking in (1) has symmetry. In this case
   (a) $\mathbf{K}$ has the uniqueness of $(\mu, \geq \chi)$-limit models: if M_1, M_2 are both $(\mu, \geq \chi)$-limit over some $M_0 \in K_\mu$, then $M_1 \cong_{M_0} M_2$;
   (b) any increasing chain of μ^+-saturated models of length $\geq \chi$ has a μ^+-saturated union. These generalize [VV17] and remove the symmetry assumption in [BV15a, Vas18c] .

Under $(< \mu)$-tameness, the conclusions of (1), (2)(a)(b) are equivalent to $\mathbf{K}$ having the χ-local character of μ-nonsplitting.

Grossberg and Vasey [GV17, Vas18c] gave eventual superstability criteria for tame AECs with a monster model. We remove the high cardinal threshold and reduce the cardinal jump between equivalent superstability criteria. We also add two new superstability criteria to the list: a weaker version of solvability and the boundedness of the U-rank.

5.1 INTRODUCTION

Good frames in abstract elementary classes (AECs) were constructed in [She09a, IV Theorem 4.10], assuming categoricity and non-ZFC axioms. Later Boney and Grossberg [BG17] built a good frame from coheir with the assumption of tameness and extension property of coheir in ZFC. Vasey [Vas16c, Section 5] further developed on coheir and [Vas16a] managed to construct a good frame at a high categoricity cardinal (categoricity can be replaced by superstability and type locality, but the initial cardinal of the good frame is still high).

Another approach to building a good frame is via nonsplitting. It is in general not clear whether uniqueness or transitivity hold for nonsplitting (where models are ordered by universal extensions). To resolve this problem, Vasey [Vas16b] constructed nonforking from nonsplitting, which has nicer properties: assuming superstability in K_μ, tameness and a monster model, nonforking gives rise to a good frame over the limit models in K_{μ^+} [VV17, Corollary 6.14]. Later it was found that uniqueness of nonforking also holds for limit models in K_μ [Vas17c].

We will generalize the nonforking results by replacing the superstability assumption by continuity of nonsplitting. A key observation is that the extension property of nonforking still holds if we have continuity of nonsplitting and stability. This allows us to replicate extension, uniqueness and transitivity properties. Since the assumption of continuity of nonsplitting applies to universal extensions only, we only get continuity and local character for universal extensions. Hence we can build an approximation of a good frame which is over the skeleton (see Definition 5.2.4) of long enough limit models ordered by universal extensions. We state the known result and our result for comparison:

Theorem 5.1.1. *Let $\mu \geq \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ have a monster model, be μ-tame and stable in μ. Let χ be the local character of μ-nonsplitting.*

1. *[Vas18a, Corollary 13.16] If $\mathbf{K}$ is μ-superstable, then there exists a good frame over the skeleton of limit models in K_μ ordered by $\leq_u$, except for symmetry;*

2. *(Corollary 5.4.13) If μ is regular and $\mathbf{K}$ has continuity of μ-nonsplitting, then there exists a good μ-frame over the skeleton of $(\mu, \geq \chi)$-limit models ordered by $\leq_u$, except*

for symmetry. The local character is χ in place of $\aleph_0$.

We assumed that μ is regular to guarantee that $\chi \leq \mu$. In the superstable case, $\chi = \aleph_0 \leq \mu$ by the definition of μ-superstability.

To obtain symmetry for our frame, we look at the argument in [VV17]. In [Van16a, Van16b], VanDieren defined a stronger version of symmetry called μ-symmetry and proved its equivalence with the continuity of reduced towers. [VV17, Lemma 4.6] noticed that a weaker version of symmetry is sufficient in one direction and deduced the weaker version of symmetry via superstability. To generalize these arguments, in Section 5.5 we replace superstability by continuity of nonsplitting and stability in a range of cardinals (the range depends on the no-order-property of $\mathbf{K}$, see Proposition 5.5.9). Then we can obtain a local version of μ-symmetry, which implies symmetry of our frame for long enough limit models. Notice that in the superstable case, $\chi = \aleph_0$ while (μ, χ)-symmetry is the same as μ-symmetry.

Theorem 5.1.2. *Let $\mu \geq \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ be μ-tame and stable in μ. Let χ be the local character of μ-nonsplitting.*

1. *[VV17, Corollary 6.9] If $\mathbf{K}$ is μ-superstable, then it has μ-symmetry;*
2. *(Corollary 5.5.13) If μ is regular and $\mathbf{K}$ has continuity of μ-nonsplitting. There is $\lambda < h(\mu)$ such that if $\mathbf{K}$ is stable in every cardinal between μ and λ, then $\mathbf{K}$ has (μ, χ)-symmetry.*

Continuity of nonsplitting and the localization of symmetry were already exploited in [BV15a, Theorem 20] to obtain the uniqueness of long enough limit models (see Fact 5.6.1). They simply assumed the local symmetry while we used the argument in [VV17] to deduce it from extra stability and continuity of nonsplitting (Corollary 5.6.2). On the other hand, [Vas18c, Section 11] used continuity of nonsplitting to deduce that a long enough chain of saturated models of the same cardinality is saturated. There he assumed saturation of limit models and managed to satisfy this assumption using his earlier result with Boney [BV17a], which has a high cardinal threshold. Since we already have local symmetry under continuity of nonsplitting and extra stability, we immediately have uniqueness of long limit models, and hence Vasey's argument can be applied to obtain the above result of saturated models (see Proposition 5.6.6; a comparison table of the approaches can be found in Remark 5.6.8(2)).

Vasey [Vas18c, Lemma 11.6] observed that a localization of VanDieren's result [Van16a] can give: if the union of a long enough chain of μ^+-saturated models is μ^+-saturated, then local symmetry is satisfied. Assuming more tameness, we use this observation to obtain converses of our results (see Main Theorem 5.8.1(4)⇒(3)). In particular local symmetry will lead to uniqueness of long limit models, which implies local character of nonsplitting (Main Theorem 5.8.1(3)⇒(1)). Despite the important observation by Vasey, he did not derive these corollaries.

Theorem 5.1.3. *Let $\mu > \mathrm{LS}(\mathbf{K})$, $\delta \leq \mu$ be regular, $\mathbf{K}$ have a monster model, be $(< \mu)$-tame, stable in μ and has continuity of μ-nonsplitting. If any increasing chain of μ^+-saturated models of cofinality $\geq \delta$ has a μ^+-saturated union, then $\mathbf{K}$ has δ-local character of μ-nonsplitting.*

The equivalent properties of a stable AEC with continuity of nonsplitting can be specialized to a superstable AEC, because superstability implies stability and continuity of nonsplitting. In [GV17], equivalent superstability properties were listed using the machinery of averages, leading to a high cardinal threshold for the equivalences to take place, and a high cardinal jump when moving from one property to another. In comparison, the equivalent properties we obtained in Main Theorem 5.8.1 and Main Theorem 5.8.2 do not require a high cardinal threshold (simply $\mu > \mathrm{LS}(\mathbf{K})$ to make sense of saturated models) but we do need extra stability assumptions above μ. Such stability assumption can be replaced by a smaller range of stability plus more no-order-property. Except for transferring stability in a cardinal to superstability, all other properties are equivalent to each other up to a jump to the successor cardinal.

In the original list inside [GV17], (λ, ξ)-solvability was considered for $\lambda > \xi$, which they showed to be an equivalent definition of superstability, with a huge jump of cardinal from no long splitting chains to solvability. Further developments in [Vas17d] indicate that such solvability has downward transfer properties which seem too strong to be called superstability. We propose a variation where $\lambda = \xi$ and will prove its equivalence with no long splitting chains in the same cardinal above μ^+ (under continuity of nonsplitting and stability). At K_μ, we demand $(<\mu)$-tameness for the equivalence to hold, up to a jump to the successor cardinal.

Theorem 5.1.4. *Let $\mu > \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ have a monster model, be $(<\mu)$-tame, stable in μ.*

1. *[SV99] If there is $\lambda > \mu$ such that $\mathbf{K}$ is (λ, μ)-solvable, then it is μ-superstable;*

2. *[GV17, Corollary 5.5] If μ is high enough and $\mathbf{K}$ is μ-superstable, then there is some $\lambda \geq \mu$ and some $\lambda' < \lambda$ such that $\mathbf{K}$ is (λ, λ')-solvable;*

3. *(Proposition 5.6.24) If $\mathbf{K}$ has continuity of μ-nonsplitting, then it is μ-superstable iff it is (μ^+, μ^+)-solvable.*

Meanwhile, [Vas18c, Corollary 4.24] showed that stability in a tail is also an equivalent definition of superstability, but the starting cardinal of superstability $(\lambda'(\mathbf{K}))^+ + \chi_1$ is only bounded by the Hanf number of μ. Since we assume continuity of nonsplitting, we can obtain μ-superstability by assuming stability in unboundedly many cardinals below μ, and enough stability above μ.

Theorem 5.1.5. *Let $\mu > \mathrm{LS}(\mathbf{K})$ with cofinality $\aleph_0$, $\mathbf{K}$ have a monster model, be μ-tame, stable in both μ and unboundedly many cardinals below μ.*

1. *[Vas18c, Corollary 4.14] If $\mu \geq (\lambda'(\mathbf{K}))^+ + \chi_1$, then $\mathbf{K}$ is μ-superstable;*

2. *(Proposition 5.7.5) If $\mathbf{K}$ has continuity of μ-nonsplitting, then there is $\lambda < h(\mu)$ such that if $\mathbf{K}$ is stable in $[\mu, \lambda)$, then it is μ-superstable.*

As the final item of the list, we prove that the boundedness of the U-rank (with respect to μ-nonforking for limit models in K_μ ordered by universal extensions) is equivalent to μ-superstability (Corollary 5.7.14). We will need to extend our nonforking to longer types, using results from [BV17b]. Then we can quote a lot of known results from [BG17], [BGKV16] and [GMA21]. Our strategy of extending frames contrasts with [Vas16a] which used a complicated axiomatic framework and drew technical results from [She09a, III].

Here we directly construct a type-full good μ-frame from nonforking and the known results apply (which are independent of the technical ones in [Vas16a, She09a]).

Theorem 5.1.6. *Let $\mu \geq \mathrm{LS}(\mathbf{K})$ be regular, $\mathbf{K}$ have a monster model, be μ-tame, stable in μ and have continuity of μ-nonsplitting. Let $U(\cdot)$ be the U-rank induced by μ-nonforking restricted to limit models in K_μ ordered by $\leq_u$. The following are equivalent:*

1. *$\mathbf{K}$ is μ-superstable;*

2. *$U(p) < \infty$ for all $p \in \mathrm{gS}(M)$ and limit model $M \in K_\mu$.*

In Section 5.2, we will state our global assumptions; define limit models, skeletons and good frames. In Section 5.3, we will review useful properties of nonsplitting with miscellaneous improvements. In Section 5.4, we will use μ-nonforking to construct our good frame over the skeleton of $(\mu, \geq \chi)$-limit models ordered by $\leq_u$, except for two changes: the local character of the frame will be χ in place of $\aleph_0$, while symmetry properties will be proven in Section 5.5 under extra stability assumptions. In Section 5.6, we will generalize known superstability results using the symmetry properties. In particular we guarantee that the union of μ^+-saturated models is saturated, provided that we have extra stability, continuity of nonsplitting and the chain being long enough. In Section 5.7, we will consider two characterizations of superstability, stability in a tail and the boundedness of the U-rank. We will prove the main theorems in Section 5.8 and state two applications there.

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5.2 PRELIMINARIES

Throughout this paper, we assume the following:

Assumption 5.2.1.

1. $\mathbf{K}$ is an AEC with AP, JEP and NMM.

2. $\mathbf{K}$ is stable in some $\mu \geq \mathrm{LS}(\mathbf{K})$.

3. $\mathbf{K}$ is μ-tame.

4. $\mathbf{K}$ satisfies continuity of μ-nonsplitting.

5. $\chi \leq \mu$ where χ is the minimum local character cardinal of μ-nonsplitting (see Definition 5.3.10).

AP stands for amalgamation property, JEP for joint embedding property and NMM for no maximal model. They allow the construction of a monster model. Given a model $M \in K$, we write $\mathrm{gS}(M)$ the set of Galois types over M (the ambient model does not matter because of AP).

Definition 5.2.2. Let λ be an infinite cardinal.

1. $\alpha \geq 2$ be an ordinal, $\mathbf{K}$ is $(<\alpha)$-*stable in* λ if for any $\|M\| = \lambda$, $|\,\mathrm{gS}^{<\alpha}(M)| \leq \lambda$. We omit α if $\alpha = 2$.

2. $\mathbf{K}$ is λ-*tame* if for any $N \in K$, any $p \neq q \in \mathrm{gS}(N)$, there is $M \leq N$ of size λ such that $p \restriction M \neq q \restriction M$.

We will define continuity of μ-nonsplitting in Definition 5.3.5.

Definition 5.2.3. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$ be a cardinal and $\alpha, \beta < \lambda^+$ be regular. Let $M \leq N$ and $\|M\| = \lambda$.

1. N is *universal over* M ($M <_u N$) if $M < N$ and for any $\|N'\| = \|N\|$, there is $f : N' \underset{M}{\longrightarrow} N$.

2. N is (λ, α)-*limit over* M if $\|N\| = \lambda$ and there exists $\langle M_i : i \leq \alpha \rangle \subseteq K_\lambda$ increasing and continuous such that $M_0 = M$, $M_\alpha = N$ and M_{i+1} is universal over M_i for $i < \alpha$. We call α the *length* of N.

3. N is (λ, α)-*limit* if there exists $\|M'\| = \lambda$ such that N is (λ, α)-limit over M'.

4. N is $(\lambda, \geq \beta)$-*limit* (*over* M) if there exists $\alpha \geq \beta$ such that (2) (resp. (3)) holds.

5. N is (λ, λ^+)-*limit (over* M*)* if $\|N\| = \lambda^+$ and we replace α by λ^+ in (2) (resp. (3)).

6. Let $\lambda_1 \leq \lambda_2$, then N is $([\lambda_1, \lambda_2], \geq \beta)$-*limit* (*over* M) if there exists $\lambda \in [\lambda_1, \lambda_2]$ such that N is $(\lambda, \geq \beta)$-limit (over M).

7. If $\lambda > \mathrm{LS}(\mathbf{K})$, we say M is λ-*saturated* if for any $M' \leq M$, $\|M'\| < \lambda$, $M \vDash \mathrm{gS}(M')$.

8. M is *saturated* if it is $\|M\|$-saturated.

In general, we do not know limit models or saturated models are closed under chains, so they do not necessary form an AEC. We adapt [Vas16a, Definition 5.3] to capture such behaviours.

Definition 5.2.4. An abstract class $\mathbf{K_1}$ is a μ-*skeleton* of $\mathbf{K}$ if the following is satisfied:

1. $\mathbf{K_1}$ is a sub-AC of $\mathbf{K}_\mu$: $K_1 \subseteq K_\mu$ and for any $M, N \in K_1$, $M \leq_{\mathbf{K_1}} N$ implies $M \leq_{\mathbf{K}} N$.

2. For any $M \in K_\mu$, there is $M' \in K_1$ such that $M \leq_{\mathbf{K}} M'$.

3. Let α be an ordinal and $\langle M_i : i < \alpha \rangle$ be $\leq_{\mathbf{K}}$-increasing in K_1. There exists $N \in K_1$ such that for all $i < \alpha$, $M_i \leq_{\mathbf{K_1}} N$ (the original definition requires strict inequality but it is immaterial under NMM).

We say $\mathbf{K_1}$ is a $(\geq \mu)$-*skeleton* of $\mathbf{K}$ if the above items hold for $K_{\geq \mu}$ in place of K_μ.

By [She09a, II Claim 1.16], limit models in μ with $\leq_{\mathbf{K}}$ form a μ-skeleton of $\mathbf{K}$. Similarly let $\alpha < \mu^+$ be regular, then $(\geq \mu, \geq \alpha)$-limits form a $(\geq \mu)$-skeleton of $\mathbf{K}$.

On the other hand, good frames were developed by Shelah [She09a] for AECs in a range of cardinals. [Vas16a] defined good frames over a coherent abstract class. We specialize the abstract class to a skeleton of an AEC.

Definition 5.2.5. Let $\mathbf{K}$ be an AEC and $\mathbf{K_1}$ be a μ-skeleton of $\mathbf{K}$. We say a nonforking relation is a *good* μ-*frame over the skeleton of* $\mathbf{K_1}$ if the following holds:

1. The nonforking relation is a binary relation between a type $p \in \mathrm{gS}(N)$ and a model $M \leq_{\mathbf{K_1}} N$. We say p does not fork over M if the relation holds between p and M. Otherwise we say p forks over M.

2. Invariance: if $f \in \mathrm{Aut}(\mathfrak{C})$ and p does not fork over M, then $f(p)$ does not fork over $f[M]$.

3. Monotonicity: if $p \in \mathrm{gS}(N)$ does not fork over M and $M \leq_{\mathbf{K_1}} M' \leq_{\mathbf{K_1}} N$ for some $M' \in K_1$, then $p \restriction M'$ does not fork over M while p itself does not fork over M'.

4. Existence: if $M \in K_1$ and $p \in \mathrm{gS}(M)$, then p does not fork over M.

5. Extension: if $M \leq_{\mathbf{K_1}} N \leq_{\mathbf{K_1}} N'$ and $p \in \mathrm{gS}(N)$ does not fork over M, then there is $q \in \mathrm{gS}(N')$ such that $q \supseteq p$ and q does not fork over M.

6. Uniqueness: if $p, q \in \mathrm{gS}(N)$ do not fork over M and $p \restriction M = q \restriction M$, then $p = q$.

7. Transitivity: if $M_0 \leq_{\mathbf{K_1}} M_1 \leq_{\mathbf{K_1}} M_2$, $p \in \mathrm{gS}(M_2)$ does not fork over M_1, $p \restriction M_1$ does not fork over M_0, then p does not fork over M_0.

8. Local character $\aleph_0$: if δ is an ordinal of cofinality $\geq \aleph_0$, $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathbf{K_1}}$-increasing and continuous, then there is $i < \delta$ such that p does not fork over M_i.

9. Continuity: Let δ is a limit ordinal and $\langle M_i : i \leq \delta \rangle$ be $\leq_{\mathbf{K_1}}$-increasing and continuous. If for all $1 \leq i < \delta$, $p_i \in \mathrm{gS}(M_i)$ does not fork over M_0 and $p_{i+1} \supseteq p_i$, then p_δ does not fork over M_0.

10. Symmetry: let $M \leq_{\mathbf{K_1}} N$, $b \in |N|$, $\mathrm{gtp}(b/M)$ do not fork over M, $\mathrm{gtp}(a/N)$ do not fork over M. There is $N_a \geq_{\mathbf{K_1}} M$ such that $\mathrm{gtp}(b/N_a)$ do not fork over M.

If the above holds for a $(\geq \mu)$-skeleton $\mathbf{K_1}$, then we say the nonforking relation is a *good* $(\geq \mu)$*-frame over the skeleton* $\mathbf{K_1}$. If $\mathbf{K_1}$ is itself an AEC (in μ), then we omit "skeleton". Let $\alpha < \mu^+$ be regular. We say a nonforking relation has local character α if we replace "$\aleph_0$" in item (8) by α.

Remark 5.2.6. 1. In this paper, $\mathbf{K_1}$ will be the $(\mu, \geq \alpha)$-limit models for some $\alpha < \mu^+$, with $\leq_{\mathbf{K_1}} = \leq_u$ (the latter is in $\mathbf{K}$).

2. In Fact 5.7.20, we will draw results of a good frame over longer types, where we allow the types in the above definition to be of arbitrary length. Extension property will

have an extra clause that allows extension of a shorter type to a longer one that still does not fork over the same base.

3. Some of the properties of a good frame imply or simply one another. Instead of using a minimalistic formulation (for example in [Vas18a, Definition 17.1]), we keep all the properties because sometimes it is easier to deduce a certain property first.

5.3 PROPERTIES OF NONSPLITTING

Let $p \in \mathrm{gS}(N)$, $f : N \to N'$, we write $f(p) := \mathrm{gtp}(f^+(d)/f(N))$ where f^+ extends f to include some $d \vDash p$ in its domain.

Proposition 5.3.1. *Such f^+ exists by AP and $f(p)$ is independent of the choice of f^+.*

Proof. Pick $a \in N_1 \geq$ realizing p, use AP to obtain $f_1^+ : a \mapsto c$ extending f (enlarge N_1 if necessary so that $f_1^+(N_1)$ contains $f(N)$).

$$\begin{tikzcd}
 & b \in N_3 & b \in N_2 \arrow[l] \arrow[rr, "f_2^+"] & & d \in f_2^+(N_2) \\
a \in N_1' \arrow[ur, "\cong"] \arrow[rrr, dotted, "f_1^{++}"] & & & c \in f_1^{++}(N_1') & \\
 & a \in N_1 \arrow[ul] \arrow[uu, "g"] \arrow[rr, "f_1^+"] & & c \in f_1^+(N_1) \arrow[u] & \\
 & & N \arrow[ul] \arrow[uuu] \arrow[rr, "f"] & & f(N) \arrow[ul] \arrow[uuu]
\end{tikzcd}$$

Suppose $b \in N_2$ realizes p and there is $f_2^+ : b \mapsto d$ extending f. Extend N_2 so that f_2^+ is an isomorphism. We need to find $h : d \mapsto c$ which fixes $f(N)$. Since $a, b \vDash p$, by AP there is $N_3 \ni b$ and $g : N_1 \xrightarrow[N]{} N_3$ that maps a to b. Extend g to an isomorphism $N_1' \cong_N N_3 \geq N_2$. By AP again, obtain f_1^{++} of domain N_1' extending f_1^+. Therefore, $d \in f(N_2^+)$ and $f_1^{++} \circ g^{-1} \circ \mathrm{id}_{N_2} \circ (f_2^+)^{-1}(d) = c$. Hence we can take $h := f_1^{++} \circ g^{-1} \circ \mathrm{id}_{N_2} \circ (f_2^+)^{-1} : f_2^+(N_2) \xrightarrow[f(N)]{} f_1^{++}(N_1')$. □

Definition 5.3.2. Let $M, N \in K$, $p \in \mathrm{gS}(N)$. p *μ-splits over M* if there exists N_1, N_2 of size μ such that $M \leq N_1, N_2 \leq N$ and $f : N_1 \cong_M N_2$ such that $f(p) \upharpoonright N_2 \neq p \upharpoonright N_2$.

Proposition 5.3.3 (Monotonicity of nonsplitting)**.** *Let $M, N \in K_\mu$, $p \in \mathrm{gS}(N)$ do not μ-split over M. For any M_1, N_1 with $M \leq M_1 \leq N_1 \leq N$, we have $p \upharpoonright N_1$ does not μ-split over M_1.*

Proposition 5.3.4. *Let $M, N \in K$, $M \in K_\mu$ and $p \in \mathrm{gS}(N)$. p μ-splits over M iff p $(\geq \mu)$-splits over M (the witnesses N_1, N_2 can be in $K_{\geq\mu}$).*

Proof. We sketch the backward direction: pick $N_1, N_2 \in K_{\geq\mu}$ witnessing p $(\geq \mu)$-splits over M. By μ-tameness and Löwenheim-Skolem axiom, we may assume $N_1, N_2 \in K_\mu$. □

Definition 5.3.5. Let χ be a regular cardinal.

1. A chain $\langle M_i : i \leq \delta\rangle$ is *u-increasing* if $M_{i+1} >_u M_i$ for all $i < \delta$.
2. **K** satisfies *continuity of μ-nonsplitting* if for any limit ordinal δ, $\langle M_i : i \leq \delta\rangle \subseteq K_\mu$ u-increasing and continuous, $p \in \mathrm{gS}(M_\delta)$,
$$p \upharpoonright M_i \text{ does not } \mu\text{-split over } M_0 \text{ for } i < \delta \Rightarrow p \text{ does not } \mu\text{-split over } M_0.$$
3. **K** has *χ-weak local character of μ-nonsplitting* if for any limit ordinal $\delta \geq \chi$, $\langle M_i : i \leq \delta\rangle \subseteq K_\mu$ u-increasing and continuous, $p \in \mathrm{gS}(M_\delta)$, there is $i < \delta$ such that $p \upharpoonright M_{i+1}$ does not μ-split over M_i.
4. **K** has *χ-local character of μ-nonsplitting* if the conclusion in (3) becomes: p does not μ-split over M_i.

We call any δ that satisfies (3) or (4) a *(weak) local character cardinal*.

Remark 5.3.6. When defining the continuity of nonsplitting, we can weaken the statement by removing the assumption that p exists and replacing $p \upharpoonright M_i$ by p_i increasing. This is because we can use [Bon14a, Proposition 5.2] to recover p. In details, we can use the weaker version of continuity and weak uniqueness (Proposition 5.3.12) to argue that the p_i's form a coherent sequence. p can be defined as the direct limit of the p_i's.

The following lemma connects the three properties of μ-nonsplitting:

Lemma 5.3.7. [BGVV17, Lemma 11(1)] *If μ is regular, $\mathbf{K}$ satisfies continuity of μ-nonsplitting and has χ-weak local character of μ-nonsplitting, then it has χ-local character of μ-nonsplitting.*

Proof. Let δ be a limit ordinal of cofinality $\geq \chi$, $\langle M_i : i \leq \delta \rangle$ u-increasing and continuous. Suppose $p \in \mathrm{gS}(M_\delta)$ splits over M_i for all $i < \delta$. Define $i_0 := 0$. By δ regular and continuity of μ-nonsplitting, build an increasing and continuous sequence of indices $\langle i_k : k < \delta \rangle$ such that $p \upharpoonright M_{i_{k+1}}$ μ-splits over M_{i_k}. Notice that $M_{i_{k+1}} >_u M_{i_k}$. Then applying χ-weak local character to $\langle M_{i_k} : k < \delta \rangle$ yields a contradiction. $\square$

From stability (even without continuity of nonsplitting), it is always possible to obtain weak local character of nonsplitting. Shelah sketched the proof and alluded to the first-order analog, so we give details here.

Lemma 5.3.8. [She99, Claim 3.3(2)] *If $\mathbf{K}$ is stable in μ (which is in Assumption 5.2.1), then for some $\chi \leq \mu$, it has weak χ-local character of μ-nonsplitting.*

Proof. Pick $\chi \leq \mu$ minimum such that $2^\chi > \mu$. Suppose we have $\langle M_i : i \leq \chi \rangle$ u-increasing and continuous and $d \vDash p \in \mathrm{gS}(M_\chi)$ such that for all $i < \chi$, $p \upharpoonright M_{i+1}$ μ-splits over $p \upharpoonright M_i$. Then for $i < \chi$, we have N_i^1 and N_i^2 of size μ, $M_i \leq N_i^1, N_i^2 \leq M_{i+1}$, $f_i : N_i^1 \cong_{M_i} N_i^2$ and $f_i(p) \upharpoonright N_i^2 \neq p \upharpoonright N_i^2$. We build $\langle M_i' : i \leq \chi \rangle$ and $\langle h_\eta : M_{l(\eta)} \underset{M_0}{\longrightarrow} M'_{l(\eta)} \,|\, \eta \in 2^{\leq \chi} \rangle$ both increasing and continuous with the following requirements:

1. $h_{\langle\rangle} := \mathrm{id}_{M_0}$ and $M_0' := M_0$.

2. For $\eta \in 2^{<\chi}$, $h_{\eta \frown 0} \upharpoonright N^2_{l(\eta)} = h_{\eta \frown 1} \upharpoonright N^2_{l(\eta)}$.

$$\begin{tikzcd}
 & M_{hf_i} \arrow[r, "g_1"] & M'_{i+1} \\
M_{i+1} \arrow[ur, "g_0"] \arrow[rr, "h_{\nu\frown 0}"] & & M^{**} \arrow[u] \\
N_i^1 \arrow[u] \arrow[r, "f_i", "\cong"'] & N_i^2 \arrow[ul] \arrow[r, "h"] & M^* \arrow[u] \arrow[uul] \\
 & M_i \arrow[ul] \arrow[u] \arrow[r, "h_\nu"] & M_i' \arrow[u]
\end{tikzcd}$$

We specify the successor step: suppose $l(\nu) = i$ and h_ν has been constructed. By AP, obtain

1. $h : N_i^2 \to M^* \geq M_i'$ with $h \supseteq h_\nu$.
2. $h_{\nu^\frown 0} : M_{i+1} \to M^{**} \geq M^*$ with $h_{\nu^\frown 0} \supseteq h$.
3. $g_0 : M_{i+1} \to M_{hf_i} \geq M^*$ with $g_0 \supseteq h \circ f_i$.
4. $g_1 : M_{hf_i} \to M_{i+1}' \geq M^{**}$ with $g_1 \circ g_0 = h_{\nu^\frown 0}$.

Define $h_{\nu^\frown 1} := g_1 \circ g_0 : M_{i+1} \to M_{i+1}'$. By diagram chasing, $h_{\nu^\frown 1} \restriction M_i = g_1 \circ g_0 \restriction M_i = g_1 \circ h \circ f_i \restriction M_i = g_1 \circ h \restriction M_i = h \restriction M_i = h_\nu \restriction M_i$. On the other hand, $h_{\nu^\frown 0} \restriction M_i = h \restriction M_i = h_\nu \restriction M_i$. Therefore the maps are increasing. Now $h_{\nu^\frown 1} \restriction N_i^2 = g_1 \circ g_0 \restriction N_i^2 = h_{\nu^\frown 0} \restriction N_i^2$ by item (4) in our construction.

For $\eta \in 2^\chi$, extend h_η so that its range includes M_χ' and its domain includes d. We show that $\{\mathrm{gtp}(h_\eta(d)/M_\chi') : \eta \in 2^\chi\}$ are pairwise distinct. For any $\eta \neq \nu \in 2^\chi$, pick the minimum $i < \chi$ such that $\eta[i] \neq \nu[i]$. Without loss of generality, assume $\eta[i] = 0$, $\nu[i] = 1$. Using the diagram above (see the comment before Proposition 5.3.1),

$$\begin{aligned}
\mathrm{gtp}(h_\eta(d)/M_\chi') &\supseteq \mathrm{gtp}(h_\eta(d)/h(N_i^2)) \\
&= h(\mathrm{gtp}(d/N_i^2)) \\
&\neq h \circ f_i(\mathrm{gtp}(d/N_i^1)) \\
&= g_1 \circ h \circ f_i(\mathrm{gtp}(d/N_i^1)) \\
&\subseteq \mathrm{gtp}(h_\nu(d)/M_\chi')
\end{aligned}$$

This contradicts the stability in μ. □

Proposition 5.3.9. *If μ is regular, then for some $\chi \leq \mu$, $\mathbf{K}$ has the χ-local character of μ-nonsplitting.*

Proof. By Lemma 5.3.8, $\mathbf{K}$ has μ-weak local character of μ-nonsplitting. By Lemma 5.3.7 (together with continuity of μ-nonsplitting in Assumption 5.2.1), $\mathbf{K}$ has μ-local character of μ-nonsplitting. Hence χ exists and $\chi \leq \mu$. □

From now on, we fix

Definition 5.3.10. χ is the minimum local character cardinal of μ-nonsplitting. $\chi \leq \mu$ if either μ is regular (by the previous proposition), or μ is greater than some regular stability cardinal ξ where $\mathbf{K}$ has continuity of ξ-nonsplitting and is ξ-tame (by Lemma 5.6.7).

Remark 5.3.11. Without continuity of nonsplitting, it is not clear whether there can be gaps between the local character cardinals: Definition 5.3.5(4) might hold for $\delta = \aleph_0$ and $\delta = \aleph_2$ but not $\delta = \aleph_1$. In that case defining χ as the *minimum* local character cardinal might not be useful. Similar obstacles form when we only know a particular λ is a local character cardinal but not necessary those above λ.

Meanwhile, weak local character cardinals close upwards and we can eliminate the above situation by assuming continuity of nonsplitting: if we know χ is the minimum local character cardinal, then it is also a weak local character cardinal, so are all regular cardinals between $[\chi, \mu^+)$. By the proof of Lemma 5.3.7, the regular cardinals between $[\chi, \mu^+)$ are all local character cardinals.

We now state the existence, extension, weak uniqueness and weak transitivity properties of μ-nonsplitting. The original proof for weak uniqueness assumes $\|M\| = \mu$ but it is not necessary; while that for extension and for weak transitivity assume all models are in K_μ; but under tameness we can just require $\|M\| = \|N\|$.

Proposition 5.3.12. *Let $M_0 <_u M \leq N$ where $\|M_0\| = \mu$.*

1. [She99, Claim 3.3(1)] *(Existence) If $p \in \mathrm{gS}(N)$, there is $N_0 \leq N$ of size μ such that p does not μ-split over N_0.*

2. [GV06b, Theorem 6.2] *(Weak uniqueness) If $p, q \in \mathrm{gS}(N)$ both do not μ-split over M_0, and $p \restriction M = q \restriction M$, then $p = q$.*

3. [GV06b, Theorem 6.1] *(Extension) Suppose $\|M\| = \|N\|$. For any $p \in \mathrm{gS}(M)$ that does not μ-split over M_0, there is $q \in \mathrm{gS}(N)$ extending p such that q does not μ-split over M_0.*

4. [Vas16b, Proposition 3.7] *(Weak transitivity) Suppose* $\|M\| = \|N\|$. *Let* $M^* \leq M_0$ *and* $p \in \mathrm{gS}(N)$. *If* p *does not* μ*-split over* M_0 *while* $p \restriction M$ *does not* μ*-split over* M^*, *then* p *does not* μ*-split over* M^*.

Proof. 1. We skip the proof, which has the same spirit as that of Lemma 5.3.8.

2. By stability in μ, we may assume that $\|M\| = \mu$. Suppose $p \neq q$, by tameness in μ we may find $M' \in K_\mu$ such that $M \leq M' \leq N$ and $p \restriction M' \neq q \restriction M'$. By $M_0 <_u M$ and $M_0 < N$, we can find $f : M' \underset{M_0}{\longrightarrow} M$. Using nonsplitting twice, we have $p \restriction f(M') = f(p)$ and $q \restriction f(M') = f(q)$. But $f(M') \leq M$ implies $p \restriction f(M') = q \restriction f(M')$. Hence $f(p) = f(q)$ and $p = q$.

3. By universality of M, find $f : N \underset{M_0}{\longrightarrow} M$. We can set $q := f^{-1}(p \restriction f(N))$.

4. Let $q := p \restriction M$. By extension, obtain $q' \supseteq q$ in $\mathrm{gS}(N)$ such that q' does not μ-split over M^*. Now $p \restriction M = q \restriction M = q' \restriction M$ and both p, q' do not μ-split over M_0 (for q' use monotonicity, see Proposition 5.3.3). By weak uniqueness, $p = q'$ and the latter does not μ-split over M^*.

□

Transitivity does not hold in general for μ-nonsplitting. The following example is sketched in [Bal09, Example 19.3].

Example 5.3.13. Let T be the first-order theory of a single equivalence relation E with infinitely many equivalence classes and each class is infinite. Let $M \leq N$ where N contains (representatives of) two more classes than M. Let d be an element. Then $\mathrm{tp}(d/N)$ splits over M iff dEa for some element $a \in N$ but $\neg dEb$ for any $b \in M$. Meanwhile, suppose $M_0 \leq M$ both of size μ, then $M_0 <_u M$ iff M contains μ-many new classes and each class extends μ many elements. Now require $M_0 <_u M$ while N contains only an extra class than M, say witnessed by d, then $\mathrm{tp}(d/N)$ cannot split over M. Also $\mathrm{tp}(d/M)$ does not split over M_0 because d is not equivalent to any elements from M. Finally $\mathrm{tp}(d/N)$ splits over M_0 because it contains two more classes than M_0 (one must be from M).

The same argument does not work if also $M <_u N$ because N would contain two more classes than M and they will witness $\mathrm{tp}(d/N)$ splits over M. Baldwin originally assigned it as [Bal09, Exercise 12.9] but later [Bal18] retracted the claim.

Question 5.3.14. When models are ordered by $\leq_u$,

1. does uniqueness of μ-nonsplitting hold? Namely, let $M <_u N$ both in K_μ, $p, q \in \mathrm{gS}(N)$ both do not μ-split over M, $p \restriction M = q \restriction M$, then $p = q$.

2. does transitivity of μ-nonsplitting hold? Namely, let $M_0 <_u M <_u N$ all in K_μ, $p \in \mathrm{gS}(N)$ does not μ-split over M and $p \restriction M$ does not μ-split over M_0, then p does not μ-split over M_0.

In Assumption 5.2.1, we assumed continuity of μ-nonsplitting. One way to obtain it is to assume superstability which is stronger. Another way is to assume ω-type locality.

Definition 5.3.15. 1. [Gro02, Definition 7.12] Let $\lambda \geq \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ is λ-*superstable* if it is stable in λ and has $\aleph_0$-local character of λ-nonsplitting.

2. [Bal09, Definition 11.4] Types in $\mathbf{K}$ are ω-*local* if: for any limit ordinal α, $\langle M_i : i \leq \alpha \rangle$ increasing and continuous, $p, q \in \mathrm{gS}(M_\alpha)$ and $p \restriction M_i = q \restriction M_i$ for all $i < \alpha$, then $p = q$.

Proposition 5.3.16. *Let* $\mathbf{K}$ *satisfy Assumption 5.2.1 except for the continuity of* μ*-nonsplitting. It will satisfy the continuity of* μ*-nonsplitting if either*

1. $\mathbf{K}$ *is* μ*-superstable; or*

2. Types in $\mathbf{K}$ *are* ω*-local.*

Proof. For (1), it suffices to prove that for any regular $\lambda \geq \aleph_0$, λ-local character implies continuity of μ-nonsplitting over chains of cofinality $\geq \lambda$. Let $\langle M_i : i \leq \lambda \rangle$ be u-increasing and continuous. Suppose $p \in \mathrm{gS}(M_\lambda)$ satisfies $p \restriction M_i$ does not μ-split over M_0 for all $i < \lambda$. By λ-local character, p does not μ-split over some M_i. If $i = 0$ we are done. Otherwise, we have $M_0 <_u M_i <_u M_{i+1} <_u M_\lambda$. By assmption, $p \restriction M_{i+1}$ does not μ-split over M_0. By weak transitivity (Proposition 5.3.12), p does not μ-split over M_0 as desired.

For (2), let $\langle M_i : i \leq \lambda \rangle$ and p as above. By assumption $p \upharpoonright M_1$ does not μ-split over M_0 and $M_1 >_u M_0$. By extension (Proposition 5.3.12), there is $q \supseteq p \upharpoonright M_1$ in $\mathrm{gS}(M_\lambda)$ such that q does not μ-split over M_0. By monotonicity, for $2 \leq i < \lambda$, $q \upharpoonright M_i$ does not μ-split over M_0. Now $(q \upharpoonright M_i) \upharpoonright M_1 = p \upharpoonright M_1 = (p \upharpoonright M_i) \upharpoonright M_1$, we can use weak uniqueness (Proposition 5.3.12) to inductively show that $q \upharpoonright M_i = p \upharpoonright M_i$ for all $i < \lambda$. By ω-locality, $p = q$ and the latter does not μ-split over M_0 as desired. $\square$

Once we have continuity of μ-nonsplitting in K_μ, it automatically works for $K_{\geq\mu}$:

Proposition 5.3.17. *Let δ be a limit ordinal, $\langle M_i : i \leq \delta \rangle \subseteq K_{\geq\mu}$ be u-increasing and continuous, $p \in \mathrm{gS}(M_\delta)$. If for all $i < \delta$, $p \upharpoonright M_i$ does not μ-split over M_0, then p also does not μ-split over M_0.*

Proof. The statement is vacuous when $M_0 \in K_{>\mu}$ so we assume $M_0 \in K_\mu$. By cofinality argument we may also assume $\mathrm{cf}(\delta) \leq \mu$. Suppose p μ-splits over M_0 and pick witnesses N^a and N^b of size μ. Using stability, define another u-increasing and continuous chain $\langle N_i : i \leq \delta \rangle \subseteq K_\mu$ such that:

1. For $i \leq \delta$, $N_i \leq M_i$.

2. N_δ contains N^a and N^b.

3. $N_0 := M_0$.

4. For $i \leq \delta$, $|N_i| \supseteq |M_i| \cap (|N^a| \cup |N^b|)$.

By assumption each $p \upharpoonright M_i$ does not μ-split over M_0, so by monotonicity $p \upharpoonright N_i$ does not μ-split over $N_0 = M_0$. By continuity of μ-nonsplitting, $p \upharpoonright N_\delta$ does not μ-split over N_0, contradicting item (2) above. $\square$

5.4 GOOD FRAME OVER ($\geq \chi$)-LIMIT MODELS EXCEPT SYMMETRY

As seen in Proposition 5.3.12, μ-nonsplitting only satisfies weak transitivity but not transitivity, which is a key property of a good frame. We will adapt [Vas16b, Definitions 3.8, 4.2] to define nonforking from nonsplitting to solve this problem.

Definition 5.4.1. Let $M \leq N$ in $K_{\geq \mu}$ and $p \in \mathrm{gS}(N)$.

1. *p (explicitly) does not μ-fork over (M_0, M)* if $M_0 \in K_\mu$, $M_0 <_u M$ and p does not μ-split over M_0.

2. *p does not μ-fork over M* if there exists M_0 satisfying (1).

We call M_0 the *witness* to μ-nonforking over M.

The main difficulty of the above definition is that different μ-nonforkings over M may have different witnesses. For extension, the original approach in [Vas16b] was to work in μ^+-saturated models. Later [VV17, Proposition 5.1] replaced it by superstability in an interval, which works for $K_{\geq \mu}$. We weaken the assumption to stability in an interval and continuity of μ-nonsplitting, and use a direct limit argument similar to that of [Bon14a, Theorem 5.3].

Proposition 5.4.2 (Extension)**.** *Let $M \leq N \leq N'$ in $K_{\geq \mu}$. If $\mathbf{K}$ is stable in $[\|N\|, \|N'\|]$ and $p \in \mathrm{gS}(N)$ does not μ-fork over M, then there is $q \supseteq p$ in $\mathrm{gS}(N')$ such that q does not μ-fork over M.*

Proof. Since p does not μ-fork over M, we can find witness $M_0 \in K_\mu$ such that $M_0 <_u M$ and p does not μ-split over M_0. If $\|N\| = \|N'\|$, we can use extension of nonsplitting (Proposition 5.3.12) to obtain (the unique) $q \in \mathrm{gS}(N')$ extending p which does not μ-split over M_0. By definition q does not μ-fork over M.

If $\|N\| < \|N'\|$, first we assume $N' = \bigcup\{N_i : i \leq \alpha\}$ u-increasing and continuous where $N_0 = N$, $N_\alpha = N'$ for some α. We will define a coherent sequence $\langle p_i : i \leq \alpha\rangle$ such that p_i is a nonsplitting extension of p in $\mathrm{gS}(N_i)$. The first paragraph gives the successor step. For limit step $\delta \leq \alpha$, we take the direct limit to obtain an extension p_δ of $\langle p_i : i < \delta\rangle$. Since all previous p_i does not μ-split over M_0, by Proposition 5.3.17, p_δ also does not μ-split over M_0. After the construction has finished, we obtain $q := p_\alpha$ a nonsplitting extension of p in $\mathrm{gS}(N')$. Since $M_0 <_u M \leq N'$, we still have q does not μ-fork over M.

In the general case where $N \leq N'$, extend $N' \leq N''$ so that $\|N''\| = \|N'\|$ and N'' contains a limit model over N of size $\|N'\|$. The construction is possible by stability in

$[\|N\|, \|N'\|]$. Then we can extend p to a nonforking $q'' \in \mathrm{gS}(N'')$ and use monotonicity to obtain the desired q. $\square$

Corollary 5.4.3. *Let $M_0 <_u M \leq N'$ with $M_0 \in K_\mu$. If $\mathbf{K}$ is stable in $[\|M\|, \|N'\|]$ and $p \in \mathrm{gS}(M)$ does not μ-split over M_0, then there is $q \supseteq p$ in $\mathrm{gS}(N')$ such that q does not μ-split over M_0.*

Proof. Run through the exact same proof as in Proposition 5.4.2, where $M = N$ and M_0 is given in the hypothesis. $\square$

For continuity, the original approach in [Vas16b, Lemma 4.12] was to deduce it from superstability (which we do not assume) and transitivity. Transitivity there was obtained from extension and uniqueness, and uniqueness was proved in [Vas16b, Lemma 5.3] for μ^+-saturated models only (or assuming superstability in [Vas17c, Lemma 2.12]). Our new argument uses weak transitivity and continuity of μ-nonsplitting to show that continuity of μ-nonforking holds for a universally increasing chain in K_μ. The case in $K_{\geq\mu}$ will be proved after we have developed transitivity and local character of nonforking.

Proposition 5.4.4 (Continuity 1)**.** *Let $\delta < \mu^+$ be a limit ordinal and $\langle M_i : i \leq \delta\rangle \subseteq K_\mu$ be u-increasing and continuous. Let $p \in \mathrm{gS}(M_\delta)$ satisfy $p \upharpoonright M_i$ does not μ-fork over M_0 for all $1 \leq i < \delta$. Then p also does not μ-fork over M_0.*

Proof. For $1 \leq i < \delta$, since $p \upharpoonright M_i$ does not μ-fork over M_0, we can find $M^i <_u M_0$ of size μ such that $p \upharpoonright M_i$ does not μ-split over M^i. By monotonicity of nonsplitting, $p \upharpoonright M_i$ does not μ-split over M_0. By continuity of μ-nonsplitting, p does not μ-split over M_0. Since $M^1 <_u M_0 <_u M_1 <_u M_\delta$, by weak transitivity (Proposition 5.3.12) p does not μ-split over M^1. (By a similar argument, it does not μ-split over other M^i.) By definition p does not μ-fork over M_0. $\square$

We now show uniqueness of nonforking in K_μ, by generalizing the argument in [Vas17c]. Instead of superstability, we stick to our Assumption 5.2.1. Fact 2.9 in that paper will be replaced by our Proposition 5.4.2. The requirement that M_0, M_1 be limit models is removed.

Proposition 5.4.5 (Uniqueness 1). *Let $M_0 \leq M_1$ in K_μ and $p_0, p_1 \in \mathrm{gS}(M_1)$ both do not μ-fork over M_0. If in addition $p_{\langle\rangle} := p_0 \restriction M_0 = p_1 \restriction M_0$, then $p_0 = p_1$.*

Proof. Suppose the proposition is false. Let $N_0 <_u M_0$ and $N_1 <_u M_0$ such that p_0 does not μ-split over N_0 while p_1 does not μ-split over N_1 (necessarily $N_0 \neq N_1$ by weak uniqueness of nonsplitting). We will build a u-increasing and continuous $\langle M_i : i \leq \mu \rangle \subseteq K_\mu$ and a coherent $\langle p_\eta \in \mathrm{gS}(M_{l(\eta)}) : \eta \in 2^{\leq \mu} \rangle$ such that for all $\nu \in 2^{<\mu}$, $p_{\nu^\frown 0}$ and $p_{\nu^\frown 1}$ are distinct nonforking extensions of p_ν. If done $\{p_\eta : \eta \in 2^\mu\}$ will contradict stability in μ.

The base case is given by the assumption. For successor case, suppose M_i and $\{p_\eta : \eta \in 2^i\}$ have been constructed for some $1 \leq i < \mu$. Define M'_{i+1} to be a (μ, ω)-limit over M_i. Fix $\eta \in 2^i$, we will define $p_{\eta^\frown 0}, p_{\eta^\frown 1} \in \mathrm{gS}(M'_{i+1})$ nonforking extensions of p_η (nonsplitting will be witnessed by different models; otherwise weak uniqueness of nonsplitting applies). Since p_η does not μ-fork over M_0, we can find $N_\eta <_u M_0$ such that p_η does not μ-split over N_η. Pick $p_\eta^+ \in \mathrm{gS}(M'_{i+1})$ a nonsplitting extension of p_η. On the other hand, obtain $N'_\eta <_u N^* <_u M_0$ such that N^* is a (μ, ω)-limit over N'_η and $N'_\eta >_u N_\eta$. By uniqueness of limit models over N_η of the same length, there is $f : M'_{i+1} \cong_{N'_\eta} N^*$.

$$\begin{tikzcd}
& & & & p_0 \arrow[d, dotted, no head] & & & & p_{\eta^\frown 0} \arrow[d, dotted, no head] \\
N_\eta \arrow[r, "u"] & N'_\eta \arrow[r, "(\mu,\omega)"] & N^* \arrow[r, "u"] & M_0 \arrow[r] & M_1 \arrow[d, dotted, no head] & \cdots & M_i \arrow[r, "(\mu,\omega)"] & M'_{i+1} \arrow[r] & M_{i+1} \arrow[d, dotted, no head] \arrow[llllll, dashed, bend left, "f"] \\
& & & & p_1 & & & & p_{\eta^\frown 1}
\end{tikzcd}$$

By invariance of nonsplitting, $f(p_\eta^+)$ does not μ-split over N_η. By monotonicity of nonsplitting, p_η, and hence $p_\eta \restriction N^*$ does not μ-split over N_η. $f(p_\eta^+) \restriction N'_\eta = p_\eta^+ \restriction N'_\eta = (p_\eta \restriction N^*) \restriction N'_\eta$. By weak uniqueness of μ-nonsplitting, $f(p_\eta^+) = p_\eta \restriction N^*$. Since $p_\eta \restriction N^*$ has two nonforking extensions $p_0 \neq p_1 \in \mathrm{gS}(M_1)$ where $M_1 >_u N^*$, we can obtain their isomorphic copies $p_{\eta^\frown 0} \neq p_{\eta^\frown 1} \in \mathrm{gS}(M_{i+1})$ for some $M_{i+1} >_u M'_{i+1}$. They still do not μ-fork over M_0 because M_0 is fixed (actually $p_{\eta^\frown i}$ does not μ-split over $N_i <_u M_0$). Ensure coherence at the end.

For limit case, let $\eta \in 2^\delta$ for some limit ordinal $\delta \leq \mu$. Define $p_\eta \in \mathrm{gS}(M_\delta)$ to be the direct limit of $\langle p_{\eta \restriction i} : i < \delta \rangle$. By Proposition 5.4.4, p_η does not μ-fork over M_0.

$\square$

Corollary 5.4.6 (Uniqueness 2)**.** *Let $M \leq N$ in $K_{\geq \mu}$ and $p, q \in \mathrm{gS}(N)$ both do not μ-fork over M. If in addition $p \restriction M = q \restriction M$, then $p = q$.*

Proof. Proposition 5.4.5 takes care of the case $M, N \in K_\mu$. Suppose the corollary is false, then $p \neq q$ and there exist $N^p, N^q <_u M$ such that p does not μ-fork over N^p and q does not μ-fork over N^q. We have two cases:

1. Suppose $M \in K_\mu$ but $N \in K_{>\mu}$. By tameness obtain $N' \in K_\mu$ such that $M \leq N' \leq N$ and $p \restriction N' \neq q \restriction N'$. Together with $p \restriction M = q \restriction M$, it contradicts Proposition 5.4.5.

2. Suppose $M \in K_{>\mu}$. Obtain $M^p, M^q \leq M$ of size μ that are universal over N^p and N^q respectively. By Löwenheim-Skolem axiom, pick $M' \leq M$ of size μ containing M^p and M^q. Thus M' is universal over both N^p and N^q, and $p \restriction M' = q \restriction M'$. Since $p \neq q$, tameness gives some $N' \in K_\mu$, $M' \leq N' \leq N$ such that $p \restriction N' \neq q \restriction N'$, which contradicts Proposition 5.4.5.

$\square$

Remark 5.4.7. The strategy of case (2) cannot be applied to Proposition 5.4.5 because M' might coincide with M and we do not have enough room to invoke weak uniqueness of nonsplitting. This calls for a specific proof in Proposition 5.4.5. Similarly, we cannot simply invoke weak uniqueness of nonsplitting to prove case (2) because we do not know if M is also universal over M'.

Corollary 5.4.8 (Transitivity)**.** *Let $M_0 \leq M_1 \leq M_2$ be in $K_{\geq \mu}$, $p \in \mathrm{gS}(M_2)$. If $\mathbf{K}$ is stable in $[\|M_1\|, \|M_2\|]$, p does not μ-fork over M_1 and $p \restriction M_1$ does not μ-fork over M_0, then p does not μ-fork over M_0.*

Proof. By Proposition 5.4.2, obtain $q \supseteq p \restriction M_1$ a nonforking extension in $\mathrm{gS}(M_2)$. Both q and p do not fork over M_1 and $q \restriction M_1 = p \restriction M_1$. By Corollary 5.4.6, $p = q$, but q does not μ-fork over M_0. $\square$

For local character, we imitate [Vas16b, Lemma 4.11] which handled the case of μ^+-saturated models ordered by $\leq_{\mathbf{K}}$ instead of $<_u$. That proof originates from [She09a, II Claim 2.11(5)].

Proposition 5.4.9 (Local character)**.** *Let $\delta \geq \chi$ be regular, $\langle M_i : i \leq \delta \rangle \subseteq K_{\geq \mu}$ u-increasing and continuous, $p \in \mathrm{gS}(M_\delta)$. There is $i < \delta$ such that p does not μ-fork over M_i.*

Proof. If $\delta \geq \mu^+$, then by existence of nonsplitting (Proposition 5.3.12) and monotonicity, there is $j < \delta$ such thtat p does not μ-split over M_j. As M_{j+1} is universal over M_j, p does not μ-fork over M_{j+1}.

If $\chi \leq \delta \leq \mu$ and suppose the conclusion fails, then we can build

1. $\langle N_i : i \leq \delta \rangle \subseteq K_\mu$ u-increasing and continuous;
2. $\langle N_i' : i \leq \delta \rangle \subseteq K_\mu$ increasing and continuous;
3. $N_0 = N_0' \leq M_0$ be any model in K_μ;
4. For all $i < \delta$, $N_i \leq M_i$ and $N_i \leq N_i' \leq M_\delta$.
5. For all $i < \delta$, $\bigcup_{j \leq i}(|N_j'| \cap |M_{i+1}|) \subseteq |N_{i+1}|$
6. For all $j < \delta$, $p \restriction N_{j+1}'$ μ-splits over N_j.

We specify the successor step of N_i': suppose N_i has been constructed. Since p μ-forks over M_i, hence over N_i. Thus (N_{i-1}, N_i) cannot witness nonforking, so there is $N_i' \in K_\mu$ with $N_i \leq N_i' \leq M_\delta$ such that $p \restriction N_i'$ μ-splits over N_{i-1}. After the construction, by monotonicity $p \restriction N_\delta \supseteq p \restriction N_i'$ μ-splits over N_{i-1} for each successor i, contradicting χ-local character of μ-nonsplitting. $\square$

In Section 6, we will need the original form of [Vas16b, Lemma 4.11], whose proof is similar to Proposition 5.4.9. We write the statement here for comparison.

Fact 5.4.10. Let $\delta \geq \chi$ be regular, $\langle M_i : i \leq \delta \rangle$ be an increasing and continuous chain of μ^+-saturated models, $p \in \mathrm{gS}(M_\delta)$. There is $i < \delta$ such that p does not μ-fork over M_i.

We now show the promised continuity of nonforking. In [Vas16b, Lemma 4.12], the chain must be of length $\geq \chi$. We do not have the restriction here because we have continuity of nonsplitting in Assumption 5.2.1.

Proposition 5.4.11 (Continuity 2)**.** *Let $\delta < \mu^+$ be regular, $\langle M_i : i \leq \delta \rangle \subseteq K_{\geq \mu}$ u-increasing and continuous, and $\mathbf{K}$ is stable in $[\|M_1\|, \|M_\delta\|)$. Let $p \in \mathrm{gS}(M_\delta)$ satisfy $p \restriction M_i$ does not μ-fork over M_0 for all $1 \leq i < \delta$. Then p also does not μ-fork over M_0.*

Proof. If $\delta \geq \chi$, by Proposition 5.4.9 there is $i < \delta$ such that p does not μ-fork over M_i. By Corollary 5.4.8, p does not μ-fork over M_0.

If $\delta < \chi \leq \mu$, we have two cases: (1) $M_0 \in K_\mu$: then for $1 \leq i < \delta$, $p \restriction M_i$ does not μ-split over M_0. By Proposition 5.3.17, p does not μ-split over M_0, so p does not μ-fork over M_1. By Corollary 5.4.8, p does not μ-fork over M_0. (2) $M_0 \in K_{>\mu}$: for $1 \leq i < \delta$, let $N_i <_u M_0$ witness $p \restriction M_i$ does not μ-fork over M_0. By Löwenheim-Skolem axiom, there is $N \in K_\mu$ (here we need $\delta \leq \mu$) such that $N_i <_u N \leq M_0$ for all i. Apply case (1) with N replacing M_0. $\square$

Existence is more tricky because nonforking requires the base to be universal over the witness of nonsplitting. The second part of the proof is based on [Vas16b, Lemma 4.9].

Proposition 5.4.12 (Existence)**.** *Let M be a $(\geq \mu, \geq \chi)$-limit model, $p \in \mathrm{gS}(M)$. Then p does not μ-fork over M. Alternatively M can be a μ^+-saturated model.*

Proof. The first part is immediate from Proposition 5.4.9. For the second part, apply existence of nonsplitting Proposition 5.3.12 to obtain $N \in K_\mu$, $N \leq M$ such that p does not μ-split over N. By model-homogeneity, M is universal over N, hence p does not μ-fork over M. $\square$

Corollary 5.4.13. *There exists a good μ-frame over the μ-skeleton of $(\mu, \geq \chi)$-limit models ordered by $\leq_u$, except for symmetry and local character χ in place of $\aleph_0$.*

Proof. Define nonforking as in Definition 5.4.1(2). Invariance and monotonicity are immediate. Existence is by Proposition 5.4.12, χ-local character is by Proposition 5.4.9, extension is by Proposition 5.4.2, uniqueness is by Proposition 5.4.5, continuity is by Proposition 5.4.4. $\square$

Remark 5.4.14. 1. We do not expect $\aleph_0$-local character because there are strictly stable AECs. For the same reason we restrict models to be $(\mu, \geq \chi)$-limit to guarantee existence property.

2. Let $\lambda \geq \mu$. Our frame extends to $([\mu, \lambda], \geq \chi)$-limit models if we assume stability in $[\mu, \lambda]$. However [Vas16b] has already developed μ-nonforking for μ^+-saturated models *ordered by* $\leq$, and we will see in Corollary 5.6.2(2) that under extra stability assumptions, $(> \mu, \geq \chi)$-limit models are automatically μ^+-saturated, so the interesting part is K_μ here.

3. We will see in Corollary 5.5.13(2) that symmetry also holds if we have enough stability.

Since we have built an approximation of a good frame in Corollary 5.4.13, one might ask if it is canonical. We first observe the following fact (Assumption 5.2.1 is not needed):

Fact 5.4.15. [Vas18a, Theorem 14.1] Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Suppose $\mathbf{K}$ is λ-superstable and there is an independence relation over the limit models (ordered by $\leq$) in K_λ, satisfying invariance, monotonicity, universal local character, uniqueness and extension. Let $M \leq N$ be limit models in K_λ and $p \in \mathrm{gS}(N)$. Then p is independent over M iff p does not λ-fork over M.

Its proof has the advantage that it does not require the independence relation to be for longer types as in [BGKV16, Corollary 5.19]. However, it still uses the following lemma from [BGKV16, Lemma 4.2]:

Lemma 5.4.16. *Suppose there is an independence relation over models in K_μ ordered by $\leq$. If it satisfies invariance, monotonicity and uniqueness, then the relation is extended by μ-nonsplitting.*

Proof. Suppose $M \leq N$ in K_μ, $p \in \mathrm{gS}(N)$ is independent over M. For any $N_1, N_2 \in K_\mu$ with $M \leq N_1, N_2 \leq N$, and any $f : N_1 \cong_M N_2$. We need to show that $f(p) \restriction N_2 = p \restriction N_2$. By monotonicity, $p \restriction N_1$ and $p \restriction N_2$ do not depend on M. By invariance, $f(p) \restriction N_2$ is independent over M. By uniqueness and the fact that f fixes M, we have $f(p) \restriction N_2 = p \restriction N_2$. $\square$

In the above proof, it utilizes the assumption that the independence relation is for models ordered by $\leq$, so it makes sense to talk about $p \restriction N_i$ is independent over M for $i = 1, 2$. To generalize Fact 5.4.15 to our frame in Corollary 5.4.13, one way is to assume the independence relation to be for models ordered by $\leq$, and with universal local character χ. But since we defined our frame to be for models ordered by $\leq_u$, we want to keep the weaker assumption that the arbitrary independence relation is also for models ordered by $\leq_u$. Thus we cannot directly invoke Lemma 5.4.16, where the N_i's are not necessarily universal over M. To circumvent this, we adapt the lemma by allowing more room:

Lemma 5.4.17. *Let $M <_u N <_u N'$ all in K_μ, $p \in \mathrm{gS}(N')$. If $p \restriction N$ μ-splits over M, then p also μ-splits over M with witnesses universal over M. Namely, there are $N_1', N_2' \leq N'$ such that $N_1' >_u M$, $N_2' >_u M$ and there is $f' : N_1' \cong_M N_2'$ with $f(p) \restriction N_2' \neq p \restriction N_2'$.*

Proof. By assumption, there are $N_1, N_2 \in K_\mu$ such that $M \leq N_1, N_2 \leq N$ and there is $f : N_1 \cong_M N_2$ such that $f(p \restriction N) \restriction N_2 \neq p \restriction N_2$. Extend f to an isomorphism $\tilde{f}$ of codomain N, and let $N^* \geq N_1$ be the domain of $\tilde{f}$. Since $N >_u M$, by invariance $N^* >_u M$. On the other hand, $N' >_u N$, then $N' >_u N_1$ and there is $g : N^* \xrightarrow[N_1]{} N'$. Let the image of g be N^{**}

In the diagram below, we use dashed arrows to indicate isomorphisms. Solid arrows indicate $\leq$.

$$\begin{tikzcd}
 & & N^* \arrow[dd, dashed, "\tilde{f}"] \arrow[dr, dashed, "g"] & & \\
 & N_1 \arrow[ur] \arrow[rr] \arrow[d, dashed, "f"] \arrow[dr] & & N^{**} \arrow[dl, dashed, "\tilde{f}\circ g^{-1}"'] \arrow[dr] & \\
M \arrow[ur] \arrow[r] & N_2 \arrow[r] & N \arrow[rr] & & N'
\end{tikzcd}$$

Since $\tilde{f} \circ g^{-1} : N^{**} \cong_M N$ and $M <_u N^{**}, N \leq N'$, we consider $\tilde{f} \circ g^{-1}(p) \restriction N$ and $p \restriction N$.

$$\begin{aligned}
\tilde{f} \circ g^{-1}(p) \restriction N &\geq [\tilde{f} \circ g^{-1}(p)] \restriction N_2 \\
&= \tilde{f}([g^{-1}(p)] \restriction N_1) \restriction N_2 \quad \text{as } \tilde{f}^{-1}[N_2] = N_1 \\
&= \tilde{f}(p \restriction N_1) \restriction N_2 \quad \text{as } g \text{ fixes } N_1 \\
&= f(p \restriction N_1) \restriction N_2 \quad \text{as } \tilde{f} \text{ extends } f \\
&= f(p \restriction N) \restriction N_2 \quad \text{as } f^{-1}[N_2] = N_1 \leq N \\
p \restriction N &\geq p \restriction N_2
\end{aligned}$$

Since $f(p \restriction N) \restriction N_2 \neq p \restriction N_2$, $\tilde{f} \circ g^{-1}(p) \restriction N \neq p \restriction N$ and we can take $N_1' := N^{**}$, $N_2' := N$, $f' := \tilde{f} \circ g^{-1}$ in the statement of the lemma. $\square$

Now we can prove a canonicity result for our frame. In order to apply Lemma 5.4.17, we will need to enlarge N to a universal extension in order to have more room. This procedure is absent in the original forward direction of Fact 5.4.15 but is similar to the backward direction (to get q below).

Proposition 5.4.18. *Suppose there is an independence relation over the $(\mu, \geq \chi)$-limit models ordered by $\leq_u$ satisfying invariance, monotonicity, local character χ, uniqueness and extension. Let $M <_u N$ be $(\mu, \geq \chi)$-limit models and $p \in \mathrm{gS}(N)$. Then p is independent over M iff p does not μ-fork over M.*

Proof. Suppose p is independent over M. By assumption M is a (μ, δ)-limit for some regular $\delta \in [\chi, \mu^+)$. Resolve $M = \bigcup_{i<\delta} M_i$ such that all M_i are also (μ, δ)-limit. By local character, $p \restriction M$ is independent over M_i for some $i < \delta$. Since the independence relation satisfies uniqueness and extension, by the proof of Corollary 5.4.8 it also satisfies transitivity. Therefore p is independent over M_i. Let $N' >_u N$. By extension, there is $p' \in \mathrm{gS}(N')$ independent over M_i and $p' \supseteq p$. Now suppose p μ-splits over M_i, by Lemma 5.4.17 p' μ-splits over M_i with universal witnesses, contradicting Lemma 5.4.16 (where $\leq$ is replaced by $<_u$ where). As a result, p does not μ-split over M_i. Since $M_i <_u M$, p does not μ-fork over M.

Conversely suppose p does not μ-fork over M. By local character and monotonicity, $p \restriction M$ is independent over M. By extension, obtain $q \in \mathrm{gS}(N)$ independent over M and $q \supseteq p$. From the forward direction, q does not μ-fork over M. By Proposition 5.4.5, $p = q$ so invariance gives q independent over M. $\square$

To conclude this section, we show that the existence of a frame similar to Corollary 5.4.13 is sufficient to obtain local character of nonsplitting. Continuity of μ-nonsplitting and μ-tameness in Assumption 5.2.1 are not needed.

Proposition 5.4.19. *Let $\delta < \mu^+$ be regular. Suppose there is an independence relation over the $(\mu, \geq \delta)$-limit models ordered by $\leq_u$ satisfying invariance, monotonicity, local character δ, uniqueness and extension. Then $\mathbf{K}$ has δ-local character of μ-nonsplitting.*

Proof. Let $\langle M_i : i \leq \delta \rangle$ be u-increasing and continuous, $p \in \mathrm{gS}(M_\delta)$. There is $i < \delta$ such that p is independent over M_i. By the forward direction of Proposition 5.4.18 (local character of nonsplitting is not used), p does not μ-split over M_i. $\square$

5.5 LOCAL SYMMETRY

Tower analysis was used in [Van16a, Theorem 3] to connect a notion of μ-symmetry and reduced towers. Combining with [GVV16], superstability and μ-symmetry imply the uniqueness of limit models. [VV17, Lemma 4.6] observed that a weaker form of μ-symmetry is sufficient to deduce one direction of [Van16a, Theorem 3], and enough superstability implies the weaker form of μ-symmetry. Therefore enough superstability already implies the uniqueness of limit models [VV17, Corollary 1.4]. Meanwhile, [BV15a] localized the notion of μ-symmetry and deduced the uniqueness of limit models of length $\geq \chi$. We will imitate the above argument and replace the hypothesis of local symmetry by sufficient stability. As a corollary we will obtain symmetry property of nonforking. The uniqueness of limit models will be discussed in the next section.

The following is based on [BV15a, Definition 10]. They restricted M_0 to be exactly (μ, δ)-limit over N but they should mean $(\mu, \geq \delta)$ for the proofs to go through. We will use $\delta := \chi$ in this paper.

Definition 5.5.1. Let $\delta < \mu^+$ be a limit ordinal. $\mathbf{K}$ has *(μ, δ)-symmetry for μ-nonsplitting* if for any $M, M_0, N \in K_\mu$, elements a, b with

1. $a \in M - M_0$;
2. $M_0 <_u M$ and M_0 is $(\mu, \geq \delta)$-limit over N;
3. $\mathrm{gtp}(a/M_0)$ does not μ-split over N;
4. $\mathrm{gtp}(b/M)$ does not μ-split over M_0,

then there is $M^b \in K_\mu$ universal over M_0 and containing b such that $\mathrm{gtp}(a/M^b)$ does not μ-split over N. We will abbreviate (μ, δ)-symmetry for μ-nonsplitting as *(μ, δ)-symmetry.*

Now we localize the hierarchy of symmetry properties in [VV17, Definition 4.3]. The first two items will be important in our improvement of [BV15a].

Definition 5.5.2. Let $\delta < \mu^+$ be a limit ordinal. In the following items, we always let $a \in M - M_0$, $M_0 <_u M$, M_0 be $(\mu, \geq \delta)$-limit over N and b be an element. In the conclusion, $M^b \in K_\mu$ universal over M_0 and containing b is guaranteed to exist.

1. $\mathbf{K}$ has *uniform (μ, δ)-symmetry*: If $\mathrm{gtp}(b/M)$ does not μ-split over M_0, $\mathrm{gtp}(a/M_0)$ does not μ-fork over (N, M_0), then $\mathrm{gtp}(a/M^b)$ does not μ-fork over (N, M_0).
2. $\mathbf{K}$ has *weak uniform (μ, δ)-symmetry*: If $\mathrm{gtp}(b/M)$ does not μ-fork over M_0, $\mathrm{gtp}(a/M_0)$ does not μ-fork over (N, M_0), then $\mathrm{gtp}(a/M^b)$ does not μ-fork over (N, M_0).
3. $\mathbf{K}$ has *nonuniform (μ, δ)-symmetry*: If $\mathrm{gtp}(b/M)$ does not μ-split over M_0, $\mathrm{gtp}(a/M_0)$ does not μ-fork over M_0, then $\mathrm{gtp}(a/M^b)$ does not μ-fork over M_0.
4. $\mathbf{K}$ has *weak nonuniform (μ, δ)-symmetry*: If $\mathrm{gtp}(b/M)$ does not μ-fork over M_0, $\mathrm{gtp}(a/M_0)$ does not μ-fork over M_0, then $\mathrm{gtp}(a/M^b)$ does not μ-fork over M_0.

The following results generalize [VV17, Section 4] which assumes superstability and works with full symmetry properties.

Proposition 5.5.3. *Let $\delta < \mu^+$ be a limit ordinal. (μ, δ)-symmetry is equivalent to uniform (μ, δ)-symmetry. Both imply nonuniform (μ, δ)-symmetry and weak uniform (μ, δ)-symmetry. Nonuniform (μ, δ)-symmetry implies weak nonuniform (μ, δ)-symmetry.*

Proof. In the definition of the symmetry properties, we always have $N <_u M_0$, so the following are equivalent:

- $\mathrm{gtp}(a/M_0)$ does not μ-fork over (N, M_0);
- $\mathrm{gtp}(a/M_0)$ does not μ-split over N.

Similarly, the following are equivalent:

- $\mathrm{gtp}(a/M^b)$ does not μ-fork over (N, M_0);
- $\mathrm{gtp}(a/M^b)$ does not μ-split over N.

Therefore, (μ, δ)-symmetry is equivalent to uniform (μ, δ)-symmetry.

Uniform (μ, δ)-symmetry implies weak uniform (μ, δ)-symmetry because nonforking over M_0 is a stronger assumption than nonsplitting over M_0. Uniform (μ, δ)-symmetry implies nonuniform (μ, δ)-symmetry because the latter does not require the witness to nonforking be the same, so its conclusion is weaker. Nonuniform (μ, δ)-symmetry implies weak nonuniform (μ, δ)-symmetry because nonforking over M_0 is a stronger assumption than nonsplitting over M_0. $\square$

The following result modifies the proof of [BV15a] which involves a lot of tower analysis. We will only mention the modifications and refer the readers to the original proof.

Proposition 5.5.4. *Let $\delta < \mu^+$ be a limit ordinal. If $\delta \geq \chi$, then weak uniform (μ, δ)-symmetry implies uniform (μ, δ)-symmetry.*

Proof sketch. [BV15a, Theorems 18, Proposition 19] establish that (μ, δ)-symmetry is equivalent to continuity of reduced towers at $\geq \delta$. We will show that the backward direction only requires weak uniform (μ, δ)-symmetry. Then using the equivalence twice we deduce that weak uniform (μ, δ)-symmetry implies (μ, δ)-symmetry. By the previous proposition, it is equivalent to uniform (μ, δ)-symmetry.

There are three places in [BV15a, Theorems 18] which use (μ,δ)-symmetry. In the first two paragraphs of page 11:

1. By χ-local character, there is a successor $i^* < \delta$ such that $\mathrm{gtp}(b/M^\delta_\delta)$ does not μ-split over $M^{i^*}_{i^*}$.

2. For any $j < \delta$, M^δ_δ is universal over M^j_j.

3. For any $j < \delta$, $\mathrm{gtp}(a_j/M^j_j)$ does not μ-split over N_j.

4. For any successor $j < \delta$, M^j_j is $(\mu, \geq \delta)$-limit over M^{j-1}_{j-1} and over N_j.

Let $j^* := i^* + 1$ which is still a successor ordinal less than δ. Combining (1) and (4), we have $\mathrm{gtp}(b/M^\delta_\delta)$ does not μ-fork over $M^{j^*}_{j^*}$. Combining (3) and (4), $\mathrm{gtp}(a_{j^*}/M^{j^*}_{j^*})$ does not μ-fork over $M^{j^*}_{j^*}$). Moreover, (2) gives M^δ_δ is universal over $M^{j^*}_{j^*}$. Together with (4) and weak uniform (μ,δ)-symmetry, we can find M^b $(\mu, \geq \delta)$-limit over $M^{j^*}_{j^*}$ and containing b such that $\mathrm{gtp}(a/M^b)$ does not μ-fork over $(N_{j^*}, M^{j^*}_{j^*})$. In other words, $\mathrm{gtp}(a/M^b)$ does not μ-split over N_{j^*} and so the original argument goes through with i^* replaced by j^*.

In "Case 2" on page 12:

a. $\mathrm{gtp}(b/\bigcup_{l<\alpha} M^l_l)$ does not μ-split over $M^{i^*}_{i^*}$.

b. $i^* + 2 \leq k < \alpha$ and $\mathrm{gtp}(a_k/M^{k+1}_k)$ does not μ-split over N_k.

c. M^{k+1}_k is universal over $M^{i^*}_{i^*}$.

d. $\bigcup_{l<\alpha} M^l_l$ is universal over M^{k+1}_k. M^{k+1}_k is $(\mu, \geq \delta)$-limit over N_k.

Combining (a) and (c), $\mathrm{gtp}(b/\bigcup_{l<\alpha} M^l_l)$ does not μ-fork over M^{k+1}_k. (b) gives $\mathrm{gtp}(a_k/M^{k+1}_k)$ does not μ-fork over (N_k, M^{k+1}_k). Together with (d) and weak uniform (μ,δ)-symmetry, we can find M^b_k $(\mu, \geq \delta)$-limit over M^{k+1}_k and containing b such that $\mathrm{gtp}(a_k/M^b_k)$ does not μ-fork over (N_k, M^{k+1}_k) so the proof goes through (we do not change index this time).

Before "Case 1" on page 11, they refer the successor case to the original proof of [Van16a, Theorem 3] which also uses (μ,δ)-symmetry. But the idea from the previous case applies equally. $\square$

In [Vas17c, Corollary 2.18], it was shown that under superstability, weak nonuniform μ-symmetry implies weak uniform μ-symmetry. We generalize this as:

Proposition 5.5.5. *Let $\delta < \mu^+$ be a limit ordinal. Weak nonuniform (μ, δ)-symmetry implies weak uniform (μ, δ)-symmetry.*

Proof. Using the notation in Definition 5.5.2, we assume $\mathrm{gtp}(b/M)$ does not μ-fork over M_0 and $\mathrm{gtp}(a/M_0)$ does not μ-fork over (N, M_0). By weak nonuniform (μ, δ)-symmetry, we can find M^b such that $\mathrm{gtp}(a/M^b)$ does not μ-fork over M_0. Since $\mathrm{gtp}(a/M_0)$ does not μ-fork over (N, M_0), by extension of nonsplitting (Proposition 5.3.12), there is a' such that $\mathrm{gtp}(a/M_0) = \mathrm{gtp}(a'/M_0)$ and $\mathrm{gtp}(a'/M^b)$ does not μ-split over N. Now both $\mathrm{gtp}(a/M^b)$ and $\mathrm{gtp}(a'/M^b)$ do not μ-fork over M_0 and they agree on the restriction of M_0. By uniqueness of nonforking (Proposition 5.4.5), $\mathrm{gtp}(a/M^b) = \mathrm{gtp}(a'/M^b)$ and hence $\mathrm{gtp}(a/M^b)$ does not μ-split over N. In other words, it does not μ-fork over (N, M_0) as desired. □

Corollary 5.5.6. *The following are equivalent:*

0. *(μ, χ)-symmetry for μ-nonsplitting;*
1. *Uniform (μ, χ)-symmetry;*
2. *Weak uniform (μ, χ)-symmetry;*
3. *Nonuniform (μ, χ)-symmetry;*
4. *Weak nonuniform (μ, χ)-symmetry.*

Proof. By Proposition 5.5.3, (0) and (1) are equivalent, (1) implies (2) and (3) while (3) implies (4). By Proposition 5.5.4 (this is where we need χ instead of a general δ), (2) implies (1). By Proposition 5.5.5, (4) implies (2). □

The following adapts [VV17, Lemma 5.6] and fills in some gaps. In particular we need μ-tameness (in Assumption 5.2.1) and stability in $\|N_\alpha\|$ for the proof to go through. It is not clear how to remove μ-tameness which they do not assume.

Lemma 5.5.7. *Let $M_0 \in K_\mu$, $N_\alpha \in K_{\geq \mu}$ with $M_0 \leq N_\alpha$, $b, b_\beta \in |N_\alpha|$, a_α be an element. If* **K** *is stable in $\|N_\alpha\|$, $\mathrm{gtp}(a_\alpha/N_\alpha)$ does not μ-fork over M_0 and $\mathrm{gtp}(b/M_0) = \mathrm{gtp}(b_\beta/M_0)$, then $\mathrm{gtp}(a_\alpha b/M_0) = \mathrm{gtp}(a_\alpha b_\beta/M_0)$.*

Proof. Let $M^* <_u M_0$ witness that $\mathrm{gtp}(a_\alpha/N_\alpha)$ does not μ-fork over (M^*, M_0). By extension (Corollary 5.4.3) and weak uniqueness of nonsplitting (Proposition 5.3.12(2)), we can extend N_α to $N^* >_u N_\alpha$ such that $\mathrm{gtp}(a_\alpha/N^*)$ does not μ-split over M^*. As $\mathrm{gtp}(b/M_0) = \mathrm{gtp}(b_\beta/M_0)$ and $N^* >_u N_\alpha$, there is $f : N_\alpha \underset{M_0}{\longrightarrow} N^*$ such that $f(b) = b_\beta$. As $\mathrm{gtp}(a_\alpha/N^*)$ does not μ-split over M^*, by Proposition 5.3.4 $\mathrm{gtp}(f(a_\alpha)/f(N_\alpha)) = \mathrm{gtp}(a_\alpha/f(N_\alpha))$. Hence there is $g \in \mathrm{Aut}_{f(N_\alpha)}(\mathfrak{C})$ such that $g(f(a_\alpha)) = a_\alpha$. Then

$$\mathrm{gtp}(a_\alpha b/M_0) = \mathrm{gtp}(f(a_\alpha)f(b)/M_0) = \mathrm{gtp}(g(f(a_\alpha))f(b)/M_0) = \mathrm{gtp}(a_\alpha b_\beta/M_0).$$

□

Remark 5.5.8. By swapping the dummy variables, we have the following formulation: Let $M_0 \in K_\mu$, $N'_\beta \in K_{\geq\mu}$ with $M_0 \leq N'_\beta$, $a, a_\alpha \in |N'_\beta|$, b_β be an element. If **K** is stable in $\|N'_\beta\|$, $\mathrm{gtp}(b_\beta/N'_\beta)$ does not μ-fork over M_0 and $\mathrm{gtp}(a/M_0) = \mathrm{gtp}(a_\alpha/M_0)$, then $\mathrm{gtp}(ab_\beta/M_0) = \mathrm{gtp}(a_\alpha b_\beta/M_0)$.

The following adapts [VV17, Lemma 5.7] which assumes superstability in $[\mu, \lambda)$. When we write the *μ-order property*, we mean tuples that witness order property have length μ.

Proposition 5.5.9. *Let $\lambda \geq \mu$ be a cardinal. If* **K** *is stable in $[\mu, \lambda)$ and fails (μ, χ)-symmetry, then it has the μ-order property of length λ.*

Proof. By Corollary 5.5.6(2)⇒(0), **K** fails weak uniform (μ, χ)-symmetry. So there are $N, M_0, M \in K_\mu$ and elements a, b such that

- $a \in M - M_0$, $M_0 <_u M$ and M_0 is $(\mu, \geq \chi)$-limit over N;
- $\mathrm{gtp}(b/M)$ does not μ-fork over M_0;
- $\mathrm{gtp}(a/M_0)$ does not μ-fork over (N, M_0);
- There is no $M^b \in K_\mu$ universal over M_0 containing b such that $\mathrm{gtp}(a/M^b)$ does not μ-fork over (N, M_0).

Build $\langle a_\alpha, b_\alpha, N_\alpha, N'_\alpha : \alpha < \lambda \rangle$ such that:

1. $N_\alpha, N'_\alpha \in K_{\mu+|\alpha|}$;

2. $b \in |N_0|$ and N_0 is universal over M;

3. $N_\alpha <_u N'_\alpha <_u N_{\alpha+1}$;

4. $a_\alpha \in |N'_\alpha|$ and $\mathrm{gtp}(a_\alpha/M_0) = \mathrm{gtp}(a/M_0)$;

5. $b_\alpha \in |N_{\alpha+1}|$ and $\mathrm{gtp}(b_\alpha/M) = \mathrm{gtp}(b/M)$;

6. $\mathrm{gtp}(a_\alpha/N_\alpha)$ does not μ-fork over (N, M_0);

7. $\mathrm{gtp}(b_\alpha/N'_\alpha)$ does not μ-fork over M_0.

N_0 is specified in (2). We specify the successor step: suppose N_α has been constructed , by Corollary 5.4.3 there is a_α such that $\mathrm{gtp}(a_\alpha/N_\alpha)$ extends $\mathrm{gtp}(a/M_0)$ and does not μ-fork over (N, M_0). Build any N'_α universal over N_α containing a_α. By Proposition 5.4.2 again, there is b_α such that $\mathrm{gtp}(b_\alpha/N'_\alpha)$ extends $\mathrm{gtp}(b/M)$ and does not μ-fork over M_0. Build $N_{\alpha+1}$ universal over N'_α containing b_α. Notice that stability is used to guarantee the existence of N_α, N'_α and the extension of types.

After the construction, we have the following properties for $\alpha, \beta < \lambda$:

a. $\mathrm{gtp}(a_\alpha b/M_0) \neq \mathrm{gtp}(ab/M_0)$;

b. $\mathrm{gtp}(ab_\beta/M_0) = \mathrm{gtp}(ab/M_0)$;

c. If $\beta < \alpha$, $\mathrm{gtp}(ab/M_0) \neq \mathrm{gtp}(a_\alpha b_\beta/M_0)$;

d. If $\beta \geq \alpha$, $\mathrm{gtp}(ab/M_0) = \mathrm{gtp}(a_\alpha b_\beta/M_0)$.

Suppose (a) is false. By invariance and the choice of a, b, M_0, N there is no $M' \in K_\mu$ universal over M_0 containing b such that $\mathrm{gtp}(a_\alpha/M')$ does not μ-fork over (N, M_0). This contradicts $M' := N_\alpha$ and item (6) in the construction. (b) is true because of item (5) of the construction and $a \in |M|$. For (c), items (5), (6) and Lemma 5.5.7 (with the exact same notations) imply $\mathrm{gtp}(a_\alpha b_\beta/M_0) = \mathrm{gtp}(a_\alpha b/M_0)$ which is not equal to $\mathrm{gtp}(ab/M_0)$ by (a). For (d), items (4), (7) and Remark 5.5.8 imply $\mathrm{gtp}(a_\alpha b_\beta/M_0) = \mathrm{gtp}(ab_\beta/M_0)$ which is equal to $\mathrm{gtp}(ab/M_0)$ by (b).

To finish the proof, let d enumerate M_0, and for $\alpha < \lambda$, $c_\alpha := a_\alpha b_\alpha d$. By (c) and (d) above, $\langle c_\alpha : \alpha < \lambda \rangle$ witnesses the μ-order property of length λ. $\square$

Remark 5.5.10. When proving (d), we used Remark 5.5.8 which requires $\mathrm{gtp}(b_\beta/N'_\beta)$ nonforking over M_0, and this is from extending $\mathrm{gtp}(b/M)$ nonforking over M_0. This called for the failure of weak uniform (μ, χ)-symmetry instead of just (μ, χ)-symmetry. (In the original proof, they claimed the same for (c) in place of (d), which should be a typo.)

Question 5.5.11. Is it possible to weaken the stability assumption in Proposition 5.5.9?

Fact 5.5.12. For any infinite cardinal λ, $h(\lambda) := \beth_{(2^\lambda)^+}$. When we write the μ-stable, we mean stability of tuples of length μ.

1. [She99, Claim 4.6] If $\mathbf{K}$ does not have the μ-order property, then there is $\lambda < h(\mu)$ such that $\mathbf{K}$ does not have the μ-order property of length λ.
2. [BGKV16, Fact 5.13] If $\mathbf{K}$ is μ-stable (in some cardinal $\geq \mu$), then it does not have the μ-order property.
3. If $\mathbf{K}$ is stable in some $\lambda = \lambda^\mu$, then $\mathbf{K}$ is μ-stable in λ.
4. [GV06b, Corollary 6.4] If $\mathbf{K}$ is stable and tame in μ (these are in Assumption 5.2.1), then it is stable in all $\lambda = \lambda^\mu$. In particular it is stable in 2^μ.
5. For some $\lambda < h(\mu)$, $\mathbf{K}$ does not have the μ-order property of length λ.

Proof. For (1) and (2), see also Proposition 3.3.4 for a proof sketch. (3) is an immediate corollary of [Bon17, Theorem 3.1], see Theorem 3.2.2 for a proof. We show (5): by (4) $\mathbf{K}$ is stable in 2^μ. By (3) it is μ-stable in 2^μ. Combining with (2) and (1) gives the conclusion. $\square$

Corollary 5.5.13. *There is $\lambda < h(\mu)$ such that if $\mathbf{K}$ is stable in $[\mu, \lambda)$, then*

1. *$\mathbf{K}$ has (μ, χ)-symmetry;*

2. *the frame in Corollary 5.4.13 satisfies symmetry.*

Proof. 1. By Fact 5.5.12(5), there is $\lambda < h(\mu)$ such that $\mathbf{K}$ does not have the μ-order property of length λ. By the contrapositive of Proposition 5.5.9, $\mathbf{K}$ has (μ, χ)-symmetry.

2. By (1) and Proposition 5.5.3, $\mathbf{K}$ has weak nonuniform (μ, χ)-symmetry. Compared to symmetry in a good frame, weak nonuniform (μ, χ)-symmetry has the extra assumption that $\mathrm{gtp}(a/M_0)$ does not μ-fork over M_0, but this is always true by Proposition 5.4.12.

$\square$

Remark 5.5.14. From the proof of Corollary 5.5.13(2), we see that if the frame in Corollary 5.4.13 (which is defined for $(\mu, \geq \chi)$-limits) has symmetry, then weak nonuniform (μ, χ)-symmetry, and hence all the other ones in Corollary 5.5.6 hold.

5.6 SYMMETRY AND SATURATED MODELS

As mentioned in the previous section, [VV17, Corollary 1.4] deduced symmetry from superstability and obtained the uniqueness of limit models. It is natural to localize such argument, which was partially done in

Fact 5.6.1. [BV15a, Theorem 20] Assume $\mathbf{K}$ has (μ, χ)-symmetry (together with Assumption 5.2.1). Then it has the uniqueness of $(\mu, \geq \chi)$-limit models: let $M_0, M_1, M_2 \in K_\mu$. If both M_1 and M_2 are $(\mu, \geq \chi)$-limit over M_0, then $M_1 \cong_{M_0} M_2$.

In the original proof of the above fact, they did not assume tameness. However, we will need tameness when we remove the symmetry assumption (see also the discussion before Lemma 5.5.7).

Corollary 5.6.2. *There is $\lambda < h(\mu)$ such that if $\mathbf{K}$ is stable in $[\mu, \lambda)$, then*

1. *$\mathbf{K}$ has the uniqueness of $(\mu, \geq \chi)$-limit models.*

2. *if also $\mu > \mathrm{LS}(\mathbf{K})$, any $(\mu, \geq \chi)$-limit model is saturated.*

Proof. 1. By Corollary 5.5.13(1), $\mathbf{K}$ has (μ, χ)-symmetry. Apply Fact 5.6.1.

2. Suppose μ is regular. Since $\chi \leq \mu$, any $(\mu, \geq \chi)$-limit is isomorphic to a (μ, μ)-limit, which is saturated. Suppose μ is singular. Let M be a $(\mu, \geq \chi)$-limit model. We show that it is δ-saturated for any regular $\delta < \mu$. Since $\delta + \chi$ is a regular cardinal in $[\chi, \mu^+)$, M is also $(\mu, \delta + \chi)$-limit, which implies it is $(\delta + \chi)$-saturated.

□

Before stating a remark, we quote a fact in order to compare Vasey's results with ours (but we will not use that fact in our paper). Continuity of μ-nonsplitting in Assumption 5.2.1 is not needed.

Fact 5.6.3. [BV17a, Theorems 5.15] Let $\chi_0 \geq 2^\mu$ be such that $\mathbf{K}$ does not have the μ-order property of length χ_0^+, define $\chi_1 := (2^{2^{\chi_0}})^{+3}$, and let $\xi \geq \chi_1$. If $\mathbf{K}$ is stable in unboundedly many cardinals $< \xi$, then any increasing chain of ξ-saturated models of length $\geq \chi$ is ξ-saturated.

Remark 5.6.4. We assumed enough stability to get a *local* result: the same μ was considered throughout. In contrast, [Vas18c, Theorems 6.3, 11.7] are *eventual*: Fact 5.6.3 was heavily used. Some of the hypotheses there require unboundedly many (H_1-closed) stability cardinals.

Now we turn to an AEC version of Harnik's Theorem. [Vas18c, Lemma 11.9] improved [Van16b, Theorem 1] and showed that:

Fact 5.6.5. Let $\mathbf{K}$ be μ-tame with a monster model. Let $\xi \geq \mu^+$. Suppose

1. $\mathbf{K}$ is stable in μ and ξ;
2. $\langle M_i : i < \delta \rangle$ is an increasing chain of ξ-saturated models;
3. $\text{cf}(\delta) \geq \chi$;
4. (ξ, δ)-limit models are saturated,

then $\bigcup_{i<\delta} M_i$ is ξ-saturated.

We remove the assumption of (4) by assuming more stability and continuity of non-splitting. Our proof is based on [Vas18c, Lemma 11.9] which have some omissions. For comparison, we write down all the assumptions.

Proposition 5.6.6. *Let* $\mathbf{K}$ *be an AEC with a monster model. Assume* $\mathbf{K}$ *is* μ*-tame, stable in* μ *and has* χ*-local charcter of* μ*-nonsplitting. Let* $\xi \geq \mu^+$*. There is* $\lambda < h(\xi)$ *such that if*

1. $\mathbf{K}$ *is stable in* $[\xi, \lambda)$*,*
2. $\langle M_i : i < \delta \rangle$ *is an increasing chain of* ξ*-saturated models;*
3. $\mathrm{cf}(\delta) \geq \chi$*;*
4. *Continuity of* μ*-nonsplitting and of* ξ*-nonsplitting holds,*

then $\bigcup_{i<\delta} M_i$ *is* ξ*-saturated.*

Before proving the proposition, we need to justify that the local character χ (Definition 5.3.10), which was defined for K_μ, also applies to K_ξ. In other words, we need to show that K_ξ has local character of nonsplitting (at most) χ. (Vasey usually cited this fact as [Vas16b, Section 4], by which he should mean an adaptation of [Vas16b, Lemma 4.11].)

Lemma 5.6.7 (Local character transfer)**.** *If* $\mathbf{K}$ *is stable in some* $\xi \geq \mu$*, then it has* χ*-local character of* ξ*-nonsplitting.*

Proof. Let $\langle M_i : i \leq \delta \rangle$ be u-increasing and continuous in K_ξ, $p \in \mathrm{gS}(M_\delta)$. By Proposition 5.4.9, there is $i < \delta$ such that p does not μ-fork over M_i. By definition of nonforking, there is $N <_u M_i$ of size μ such that p does not μ-split over N. Suppose p ξ-splits over M_i then it also ξ-splits over N. By μ-tameness, it μ-splits over N, contradiction. □

Proof of Proposition 5.6.6. Let $\delta \geq \chi$ be regular. If $\delta \geq \xi$ we can use a cofinality argument. So we assume $\delta < \xi$. Let $M_\delta := \bigcup_{i<\delta} M_i$ and $N \in K_{<\xi}$, $N \leq M_\delta$, $p \in \mathrm{gS}(N)$. Without loss of generality, we may assume for $i \leq \delta$, $M_i \in K_\xi$: given a saturated $M^* \in K_{\geq\xi^+}$ and some $\tilde{N} \leq M^*$ of size $\leq \xi$, we can close $\tilde{N}$ into a (ξ, χ)-limit N^*. By ξ-model-homogeneity of M^*, we may assume $N^* \leq M^*$. By Lemma 5.6.7 and Corollary 5.6.2(2), any $(\xi, \geq \chi)$-limits are saturated, so N^* is saturated. Therefore we can recursively shrink each M_i to a saturated model in K_ξ while still containing the same intersection with $\tilde{N}$.

Extend p to a type in $\mathrm{gS}(M_\delta)$. By Fact 5.4.10, there is $i < \delta$ such that p does not μ-fork over M_i. By reindexing assume $i = 0$ and let $M_0^0 \in K_\mu$ witness the nonforking. Obtain $N_0 \in K_\mu$ such that $M_0^0 <_u N_0 \leq M_0$. Define $\mu' := \mu + \delta$, we build $\langle N_i : 1 \leq i \leq \delta \rangle$ increasing and continuous in $K_{\mu'}$ such that $N_0 \leq N_1 \leq N \leq N_\delta$ and for $i \leq \delta$, $N_i \leq M_i$. Now we construct

1. $\langle M_i^*, f_{i,j} : i \leq j < \delta \rangle$ an increasing and continuous directed system;
2. For $i < \delta$, $M_i^* \in K_\xi$, $N_i \leq M_i^* \leq M_i$;
3. For $i < \delta$, $f_{i,i+1} : M_i^* \underset{N_i}{\longrightarrow} M_{i+1}^*$;
4. $M_0^* := M_0$. For $i < \delta$, $f_{i,i+1}[M_i^*] <_u M_{i+1}^*$.

$$\begin{tikzcd}
K_\xi & M_0 \arrow[r] & M_1 \arrow[r] & \cdots \arrow[r] & M_\delta \arrow[r] & \tilde{M} \\
K_\xi & M_0^* \arrow[u] \arrow[r, "f_{0,1}"] & M_1^* \arrow[u] \arrow[rrr, dotted, "f_{1,\delta}"] & & & M_\delta^* \arrow[u] \\
K'_\mu & & N_1 \arrow[u] \arrow[r] & \cdots \arrow[r] & N_\delta \arrow[ur] & \\
K_\mu & M_0^0 <_u N_0 \arrow[uu] \arrow[ur] & & & &
\end{tikzcd}$$

At limit stage $i < \delta$, take direct limit M_i^* which contains N_i. Since $\|N_i\| < \xi$ and M_i is model-homogeneous, we may assume M_i^* is inside M_i. Suppose M_i^* is constructed for some $i < \delta$, obtain the amalgam M_{i+1}^{**} of M_i^* and N_{i+1} over N_i. Since $\|N_{i+1}\| < \xi$ and M_{i+1} is model-homogeneous, we may embed the amalgam into M_{i+1}. Call the image of the amalgam M_{i+1}^*. After the construction, take one more direct limit to obtain $(M_\delta^*, f_{i,\delta})_{i<\delta}$ (but this time we do not know if $M_\delta^* \leq M_\delta$). By item (4) above, we have that M_δ^* is a (ξ, δ)-limit, hence saturated.

We will work in a local monster model, namely we find a saturated $\tilde{M} \in K_\xi$ such that

a. $\tilde{M}$ contains M_δ and M_δ^*;

b. For $i < \delta$, $f_{i,\delta}$ can be extended to $f^*_{i,\delta} \in \mathrm{Aut}(\tilde{M})$;

c. For $i < \delta$, $f^*_{i,\delta}[N_\delta] \leq M^*_\delta$.

(c) is possible because M^*_δ is universal over $f_{i,\delta}[M^*_i]$. Finally, we define $N^* \leq M^*_\delta$ of size μ' containing $\bigcup_{i<\delta} f^*_{i,\delta}[N_\delta]$. By model-homogeneity of M^*_δ, we build $M^{**} \in K_\xi$ saturated such that $N^* \leq M^{**} <_u M^*_\delta$.

By Proposition 5.4.2, extend p to $q \in \mathrm{gS}(\tilde{M})$ nonforking over N_0 (here we need $N_0 \in K_\mu$ or else we have to assume more stability). Since $M^*_\delta >_u M^{**}$, we can find $b_\delta \in M^*_\delta$ such that $b_\delta \vDash q \restriction M^{**}$. Since M^*_δ is a direct limit of the M^*_i's, there is $i < \delta$ such that $f_{i,\delta}(b) = b_\delta$. As $b \in M^*_i \subseteq M_i \leq M_\delta$, it suffices to show that $b \vDash q \restriction (f^*_{i,\delta})^{-1}[M^{**}]$, because $N \leq N_\delta \leq (f^*_{i,\delta})^{-1}[N^*] \leq (f^*_{i,\delta})^{-1}[M^{**}]$. In the following diagram, dotted arrows refer to $\leq$ or $<_u$ between models, while the dashed equal sign is our goal.

$$\begin{tikzcd}
& q \in \mathrm{gS}(\tilde{M}) & & p \in \mathrm{gS}(M_\delta) \arrow[ll] \\
q \restriction M^*_\delta \arrow[ur] & b_\delta \in M^*_\delta \arrow[u, dotted] & b \in M^*_i \arrow[l, "f^*_{i,\delta}"'] & \\
q \restriction M^{**} \arrow[u] \arrow[r, equal] & \mathrm{gtp}(b_\delta/M^{**}) \arrow[u, dotted, "u"] & \mathrm{gtp}(b/(f^*_{i,\delta})^{-1}[M^{**}]) \arrow[l, "f^*_{i,\delta}"'] \arrow[r, dashed, equal] & q \restriction (f^*_{i,\delta})^{-1}[M^{**}] \arrow[uull, bend right] \\
q \restriction N_0 \arrow[u] \arrow[r, equal] & \mathrm{gtp}(b_\delta/N_0) \arrow[u] \arrow[r, equal] & \mathrm{gtp}(b/N_0) & p \restriction N \arrow[u] \arrow[uuu, bend right]
\end{tikzcd}$$

Since $q \restriction M^{**} = \mathrm{gtp}(b_\delta/M^{**})$ does not μ-fork over N_0 and $f^*_{i,\delta}$ fixes $N_i \geq N_0$, by invariance $\mathrm{gtp}(b/(f^*_{i,\delta})^{-1}[M^{**}])$ does not μ-fork N_0. By monotonicity, q and hence $q \restriction (f^*_{i,\delta})^{-1}[M^{**}]$ does not μ-fork over N_0. By invariance again, $\mathrm{gtp}(b/N_0) = \mathrm{gtp}(b_\delta/N_0) = q \restriction N_0$. By Corollary 5.4.6, $q \restriction (f^*_{i,\delta})^{-1}[M^{**}] = \mathrm{gtp}(b/(f^*_{i,\delta})^{-1}[M^{**}])$ as desired. $\square$

Remark 5.6.8. 1. In Proposition 5.6.6, the assumption of stability in $[\xi, \lambda)$ is to guarantee local symmetry from no ξ-order property of length λ. We can relax the stability assumption if we have the stronger assumption of no ξ-order property. Namely, if $\mathbf{K}$ does not have ξ-order property of length ζ where $\zeta > \xi$, then we can simply assume stability in $[\xi, \zeta)$.

2. We compare our approach with Vasey's. To satisfy hypothesis (4) in Fact 5.6.5, he used Fact 5.6.1 which requires (ξ, χ)-symmetry and continuity of nonsplitting [Vas18c, Theorem 11.11(1)]. Meanwhile he obtained the equivalence of (ξ, χ)-symmetry $\Leftrightarrow$ the increasing union of saturated models of length $\geq \chi$ in K_{ξ^+} is saturated (see Fact 5.6.15). By Fact 5.6.3, the latter is true for large enough ξ. In short, he raised the cardinal threshold while we assumed more stability. More curiously, both our stability assumption and his cardinal threshold are linked to no order property.

   A comparison table can be found below. For $\xi \geq \mu$, we abbreviate the increasing union of saturated models of length $\geq \chi$ in K_ξ is saturated by "Union(ξ)".

Our approach	Vasey's approach
For $\xi \geq \mu^+$ and	For large enough ξ,
Enough stability $\big([\mu, h(\xi)) \text{ suffices}\big)$	$\Rightarrow$ Union(ξ^+) (Fact 5.6.3)
$\Rightarrow$$(\xi, \chi)$-symmetry (Corollary 5.5.13(1))	$\Rightarrow$ (ξ, χ)-symmetry (Fact 5.6.15)
$\Rightarrow$Saturation of $(\xi, \geq \chi)$-limits (Corollary 5.6.2(2))	$\Rightarrow$Saturation of $(\xi, \geq \chi)$-limits (Fact 5.6.1)
$\Rightarrow$Union(ξ) (Proposition 5.6.6)	$\Rightarrow$Union(ξ) (Fact 5.6.5)

Observation 5.6.9. The $[\xi, \lambda)$ stability assumption in Proposition 5.6.6 can be replaced by (ξ, χ)-symmetry, because we can directly apply Fact 5.6.1 instead of using extra stability to invoke Corollary 5.6.2. This applies to other results in the paper.

We now recover two known results with different proofs. The original proof for [Vas16a, Proposition 10.10] is extremely abstract so we supplement a direct argument. (Here we already assumed a monster model which implies no maximal models everywhere. Alternatively, one can adapt the proof of [Bon14a, Theorem 7.1] without using symmetry to transfer no maximal models upward.) On the other hand, since we have generalized the arguments in [VV17], we can specialize them to $\chi = \aleph_0$ and recover [VV17, Corollary 6.10] (see below). In their approach, [Van16b, Theorem 22] was cited for the successor case of λ and the limit case was proven by inductive hypothesis. We provide a uniform argument to both cases for closure under chains, and fill in the computation of the Löwenheim-Skolem number for the successor case, which they glossed over.

The following facts do not require continuity of nonsplitting.

Fact 5.6.10. 1. [BKV06, Theorem 1] Let $\xi \geq \mu$. If $\mathbf{K}$ is stable in ξ, then it is also stable in ξ^{+n} for all $n < \omega$.

2. [Vas16b, Theorem 5.5] Let $\xi_0 \geq \mu$ while δ be regular, $\langle \xi_i : i < \delta \rangle$ be strictly increasing stability cardinals. If $\mathbf{K}$ has δ-local character of ξ_0-nonsplitting, then $\sup_{i<\delta} \xi_i$ is also a stability cardinal. In particular, if $\mathbf{K}$ is ξ-superstable for some $\xi \geq \mu$, then it is stable in all $\lambda \geq \xi$.

Corollary 5.6.11. *1.* [Vas16a, Proposition 10.10] *Let $\xi \geq \mu$. If $\mathbf{K}$ is ξ-superstable, then it is superstable in all $\zeta \geq \xi$.*

2. [VV17, Corollary 6.10] *Let $\mathbf{K}$ be μ-superstable and $\xi \geq \mu^+$, then $\mathbf{K}^{\xi\text{-sat}}$ the class of ξ-saturated models in $\mathbf{K}$ forms an AEC with Löwenheim-Skolem number ξ.*

Proof. 1. Combine Fact 5.6.10(2) and Lemma 5.6.7.

2. By (1) and Proposition 5.3.16, we have continuity of ξ-nonsplitting and stability in $[\xi, \infty)$. By Proposition 5.6.6, $\mathbf{K}^{\xi\text{-sat}}$ is closed under chains. We show that the Löwenheim-Skolem number is ξ: let A be a subset of a ξ-saturated model M. We need to find a ξ-saturated $N \leq M$ of size $\xi + |A|$ containing A.

Consider the case where ξ is regular : then we construct $\langle N_i : i \leq \xi \rangle$ increasing and continuous such that for $1 \leq i < \xi$,

- N_0 contains A;
- $N_i \in K_{\xi+|A|}$ is ξ-saturated;
- If $N^* \leq N_i$ is of size less than ξ, then N_{i+1} realizes all types over N^*.

The construction is possible by stability in $\xi + |A|$ (implied by μ-superstability): M is ξ-saturated so it has witnesses to all types over N^*, but those types can be extended to be over $N_i \in K_{\xi+|A|}$. By stability we can restrict to $(\xi + |A|)$-many witnesses that work for all such N^*. Now $N_\xi \leq M$ is ξ-saturated by a cofinality argument. Also, it has size $\xi + |A|$.

For the singular case, write $\xi = \bigcup_{i<\mathrm{cf}(\xi)} \xi_i$ where the ξ_i's form an increasing chain of regular cardinals with $\mu^+ \leq \xi_i < \xi$. By the inductive hypothesis that $\mathrm{LS}(\mathbf{K}^{\xi_i\text{-}\mathbf{sat}}) = \xi_i$, we can build $\langle N_i : i \leq \mathrm{cf}(\xi)\rangle$ increasing and continuous such that N_0 contains A, $N_i \in K_{\xi_i+|A|}$ is ξ_i-saturated. Since each $\mathbf{K}^{\xi_i\text{-}\mathbf{sat}}$ is closed under chains, N_ξ is ξ-saturated and has size $\xi + |A|$.

$\square$

It is natural to ask if there are converses to our results. In particular what are the sufficient conditions to $\mathbf{K}$ having the χ-local character in K_ξ for some $\xi \geq \mu$. [Vas18c, Lemma 4.12] gave one useful criterion which we adapt below. The original statement did not cover the case $\delta = \xi$ below and such omission affects the rest of his results. In particular [Vas18c, Theorem 4.11] should only apply to singular μ there. Our result covers regular cardinals because we assume stability and continuity of nonsplitting. Only in [Vas18c, Section 11] did he start to assume continuity of nonsplitting and in [Vas18c, Theorem 12.1] did he take care of the regular case by under extra assumptions.

We state the full assumptions in the following proposition.

Proposition 5.6.12. *Let $\mu \geq \mathrm{LS}(\mathbf{K})$. Suppose $\mathbf{K}$ has a monster model, is μ-tame and stable in some $\xi \geq \mu^+$. Let $\delta < \xi^+$ be regular, $\langle M_i : i \leq \delta\rangle$ be u-increasing and continuous in K_ξ and $p \in \mathrm{gS}(M_\delta)$. There is $i < \delta$ such that p does not ξ-split over M_i if one of the following holds:*

1. *$\delta = \xi$ (so ξ is regular), $\mathbf{K}$ has continuity of ξ-nonsplitting;*
2. *$\delta < \xi$ and M_δ is $(\mu + \delta)^+$-saturated.*

Proof. The first case is by Proposition 5.3.9 (with ξ in place of μ). We consider the second case $\delta < \xi$. Suppose the conclusion is false, then for $i < \delta$, there exist

1. $N_i^1, N_i^2 \in K_\xi$ with $M_i \leq N_i^1, N_i^2 \leq M_\delta$;
2. $f_i : N_i^1 \cong_{M_i} N_i^2$ with $f_i(p \restriction N_i^1) \neq p \restriction N_i^2$;

3. $M_i^1 \leq N_i^1$ and $M_i^2 \leq N_i^2$ such that $f_i[M_i^1] \cong M_i^2$ and $f_i(p \upharpoonright M_i^1) \neq p \upharpoonright M_i^2$.

Let $N \leq M_\delta$ of size $\mu + \delta$ containing M_i^1 and M_i^2 for all $i < \delta$. Since M_δ is $(\mu+\delta)^+$-saturated, there is $b \in |M_\delta|$ realizing $p \upharpoonright N$. Then there is $i < \delta$ such that $b \in |M_i|$. Since f_i fixes M_i, it also fixes b. Thus

$$f_i(p \upharpoonright M_i^1) = \text{gtp}(f_i(b)/M_i^2) = \text{gtp}(b/M_i^2) = p \upharpoonright M_i^2,$$

contradicting item (3) above. $\square$

Corollary 5.6.13. *Suppose $\xi \geq \mu^+$ and $\delta < \xi^+$ be regular. If $\mathbf{K}$ is stable in ξ, has continuity of ξ-nonsplitting and has unique $(\xi, \geq \delta)$-limit models, then it has δ-local character in K_ξ. If in addition K_ξ has unique limit models, then it is ξ-superstable.*

Proof. Let $\delta' \geq \delta$ be regular and $\langle M_i : i \leq \delta' \rangle \subseteq K_\xi$ be u-increasing and continuous, $p \in \text{gS}(M_{\delta'})$. By the proof of Corollary 5.6.2(2), $M_{\delta'}$ is saturated. By Proposition 5.6.12, there is $i < \delta'$ such that p does not ξ-split over M_i. $\square$

Remark 5.6.14. As before, our result is local. [GV17, Theorem 3.18] proved a similar result which is eventual: they managed to guarantee superstability after $\beth_\omega(\chi_0)$ where $\mathbf{K}$ has no order property of length χ_0.

Vasey [Vas18c, Fact 11.6] also made another observation that connects saturated models and symmetry. In the original statement, he omitted writing continuity of nonsplitting in the hypothesis and did not give a proof sketch, so we give more details here (Assumption 5.2.1 applies). As in the discussion before Definition 5.5.1, we consider the tail of regular cardinals $\delta' \geq \delta$ in place of a fixed $\delta' = \delta$ to match our notations.

Fact 5.6.15. Let $\delta < \mu^+$ be regular. If for any $\delta' \in [\delta, \mu^+)$ regular, any $\langle M_i : i < \delta' \rangle$ increasing chain of saturated models in K_{μ^+} has a saturated union, then $\mathbf{K}$ has (μ, δ)-symmetry.

Proof. In [Van16a, Theorem 2], it was shown that if the above fact holds for any $\delta < \mu^+$, then any reduced tower is continuous at all $\delta < \mu^+$. We can localize this argument to show that if the above fact holds for a specific $\delta < \mu^+$, then any reduced tower is continuous at $\geq \delta$. By [BV15a, Proposition 19], $\mathbf{K}$ has (μ, δ)-symmetry. $\square$

Corollary 5.6.16. *Let $\delta < \mu^+$ be regular. If for any $\delta' \in [\delta, \mu^+)$ regular, any $\langle M_i : i < \delta' \rangle$ increasing chain of saturated models in K_{μ^+} has a saturated union, then* $\mathbf{K}$ *has uniqueness of $(\mu, \geq \delta)$-limit models.*

Proof. Combine Fact 5.6.15 and Fact 5.6.1. $\square$

Question 5.6.17. Is there an analog of Fact 5.6.15 and Corollary 5.6.16 where "μ^+" is replaced by a general $\xi \geq \mu^+$?

We look at superlimits and solvability before ending this section. The following localizes [SV18a, Definition 2.1], which is more natural than [Vas18c, Definition 6.2].

Definition 5.6.18. Let $\xi \geq \mu$. $M \in K_\xi$ is a *χ-superlimit* if M is universal in K_ξ, not maximal, and for any regular δ with $\chi \leq \delta < \xi^+$, $\langle M_i : i < \delta \rangle$ increasing such that $M_i \cong M$ for all $i < \delta$, then $\bigcup_{i<\delta} M_i \cong M$. M is called a *superlimit* if it is a $\aleph_0$-superlimit.

Proposition 5.6.19. *Let* $\mathbf{K}$ *have continuity of ξ-nonsplitting for some $\xi \geq \mu^+$. There is $\lambda < h(\xi)$ such that if* $\mathbf{K}$ *is stable in $[\xi, \lambda)$, then it has a saturated χ-superlimit in K_ξ.*

Proof. By Corollary 5.6.2(2) and Lemma 5.6.7, any $(\xi, \geq \chi)$-limit M is saturated (hence universal in K_ξ). Let δ be regular, $\chi \leq \delta < \xi^+$, $\langle M_i : i < \delta \rangle$ increasing such that $M_i \cong M$ for all $i < \delta$. Then all M_i are saturated in K_ξ. By Proposition 5.6.6, $\bigcup_{i<\delta} M_i$ is also saturated, hence isomorphic to M. $\square$

Remark 5.6.20. The specific χ-superlimit built above is saturated. Under the same assumptions, it is true for all χ-superlimits (Lemma 5.6.23).

The following connects superlimit models with *solvability* (see [GV17, Definition 2.17] for a definition).

Fact 5.6.21. [GV17, Lemma 2.19] Let $\lambda \geq \xi$. The following are equivalent:

1. $\mathbf{K}$ is (λ, ξ)-solvable.
2. There exists an AEC $\mathbf{K}'$ in $\mathrm{L}(\mathbf{K}') \supseteq \mathrm{L}(\mathbf{K})$ such that $\mathrm{LS}(\mathbf{K}') \leq \xi$, $\mathbf{K}'$ has arbitarily large models and for any $M \in K'_\lambda$, $M \restriction \mathrm{L}(\mathbf{K})$ is a superlimit in $\mathbf{K}$.

In [GV17, Theorem 4.9], they showed that (λ, ξ)-solvability is *eventually* (in λ) equivalent to other criteria of superstability (modulo a jump of $\beth_{\omega+2}$). Also, λ is required to be greater than ξ. We propose that a better formulation of superstability which has $\lambda = \xi$. The case $\lambda > \xi$ should be a stronger condition because it allows downward transfer (see [Vas17d, Corollary 5.1] for more development on this). Our result proceeds with a series of lemmas.

The next lemma generalizes [GV17, Fact 2.8(5)] (which is based on [Dru13]).

Lemma 5.6.22. *Let $\xi \geq \mu^+$ and M be a saturated model in K_ξ. M is a χ-superlimit iff for any regular δ with $\chi \leq \delta < \xi^+$, any increasing chain of saturated models in K_ξ of length δ has a saturated union.*

Proof. Immediate from the definition of a χ-superlimit. Notice that we need $\delta < \xi^+$ to make sure that the chain of saturated models have a union in K_ξ. □

The following lemma generalizes [Dru13, Theorem 2.3.11].

Lemma 5.6.23. *Let $\xi > \mathrm{LS}(\mathbf{K})$. If M is a χ-superlimit in K_ξ, then M is saturated.*

Proof. We show that M is a (ξ, δ)-limit for regular $\delta \in [\chi, \xi^+)$. If done, the argument in Corollary 5.6.2(2) shows that it is saturated. Construct $\langle M_i, N_i : i < \delta\rangle$ in K_ξ such that $M_0 := M \cong M_i <_u N_i < M_{i+1}$ for $i < \delta$. Suppose N_i is constructed, by universality N_i embeds inside M so we can build M_{i+1}, an isomorphic copy of M over N_i. To construct M_i for limit i, we embed the union of previous N_i inside M and repeat the above process. By the property of a χ-superlimit, $M \cong \bigcup_{i<\delta} M_i = \bigcup_{i<\delta} N_i$ which is a (ξ, δ)-limit. □

Proposition 5.6.24. *If $\mu > \mathrm{LS}(\mathbf{K})$ and $\mathbf{K}$ is $(< \mu)$-tame, then it is μ-superstable iff it is (μ^+, μ^+)-solvable.*

Proof. Suppose $\mathbf{K}$ is μ-superstable. By Lemma 5.6.23 with $\xi = \mu^+$, superlimits in K_ξ are saturated. By Corollary 5.6.11(2), ξ-saturated models are closed under chains. By Lemma 5.6.22, saturated models in K_ξ are superlimits. Therefore, saturated models and superlimits coincide in K_ξ. By Fact 5.6.21, we can define $\mathrm{L}(\mathbf{K}') := \mathrm{L}(\mathbf{K})$ and $\mathbf{K}'$ to be the class of ξ-saturated models. By Corollary 5.6.11(2) again, it is an AEC with $\mathrm{LS}(\mathbf{K}') = \xi$.

Suppose $\mathbf{K}$ is (μ^+, μ^+)-solvable. By Lemma 5.6.23 there is a saturated superlimit in K_{μ^+}, which witnesses the union of saturated models in K_{μ^+} is μ^+-saturated. By Corollary 5.6.16, it has uniqueness of limit models in K_μ. By $(< \mu)$-tameness and the proof of Corollary 5.6.13 (replace "ξ" there by μ and "μ^+" there by $\mathrm{LS}(\mathbf{K})^+$), it is μ-superstable. $\square$

Remark 5.6.25. One might want to generalize the argument to strictly stable AECs. In that case the statement of Fact 5.6.21(2) should naturally be for a χ-AEC instead of an AEC, but we do not know how to prove that saturated models are closed under χ-directed systems (a similar obstacle is in [BGL$^+$16, Remark 2.3(4)]). On top of that, the equivalence in Fact 5.6.21 is not clear in that case because we do not have a first-order presentation theorem on χ-AECs to extract an Ehrenfeucht-Mostowski blueprint (but we do have a $(< \mu)$-ary presentation theorem, see [BGL$^+$16, Theorem 3.2] or Theorem 4.5.6).

5.7 STABILITY IN A TAIL AND U-RANK

In this section we look at two characterizations of superstability. For convenience we follow [Vas18c, Section 4] to define some cardinals:

Definition 5.7.1.

1. $\lambda(\mathbf{K})$ stands for the first stability cardinal above $\mathrm{LS}(\mathbf{K})$.
2. $\chi(\mathbf{K})$ stands for the least regular cardinal δ such that $\mathbf{K}$ has δ-local character of ξ-nonsplitting for some stability cardinal $\xi \geq \mathrm{LS}(\mathbf{K})$.
3. $\lambda'(\mathbf{K})$ stands for the minimum stability cardinal ξ such that for any stability cardinal $\xi' \geq \xi$, $\mathbf{K}$ has $\chi(\mathbf{K})$-local character of ξ'-nonsplitting.

Observation 5.7.2.

1. By Assumption 5.2.1, $\lambda(\mathbf{K}) \leq \mu$.
2. By Definition 5.3.10 (see also the remark after it), $\chi(\mathbf{K}) \leq \chi$.
3. By Lemma 5.6.7, we can equivalently define $\lambda'(\mathbf{K})$ as the minimum stability cardinal ξ such that $\mathbf{K}$ has $\chi(\mathbf{K})$-local character of ξ-nonsplitting.
4. $\mathbf{K}$ is eventually superstable (ξ-superstable for large enough ξ) iff $\chi(\mathbf{K}) = \aleph_0$.

Currently we do not have a nice bound of $\lambda'(\mathbf{K})$ so the cardinal threshold might be very high if we invoke $\lambda'(\mathbf{K})$ or $\chi(\mathbf{K})$. Vasey built upon [She99] and spent several sections to derive:

Fact 5.7.3. [Vas18c, Theorem 11.3(2)] Suppose $\mathbf{K}$ has continuity of ξ-nonsplitting for all stability cardinal ξ, then $\lambda'(\mathbf{K}) < h(\lambda(\mathbf{K}))$.

We can now state Vasey's characterization that superstability is equivalent to stability in a tail of cardinals. Since continuity of μ-nonsplitting is not assumed there, item (1) only holds for singular ξ. Also, the original formulation wrote $\lambda'(\mathbf{K})$ instead of $(\lambda'(\mathbf{K}))^+$ but the proof did not go through.

Fact 5.7.4. Let $\mathbf{K}$ be $\mathrm{LS}(\mathbf{K})$-tame with a monster model.

1. [Vas18c, Corollary 4.14] Let χ_1 as in Fact 5.6.3, $\xi \geq (\lambda'(\mathbf{K}))^+ + \chi_1$ be singular, $\mathbf{K}$ be stable in unboundedly many cardinal $< \xi$. $\mathbf{K}$ is stable in ξ iff $\mathrm{cf}(\xi) \geq \chi(\mathbf{K})$.

2. [Vas18c, Corollary 4.24] $\chi(\mathbf{K}) = \aleph_0$ iff $\mathbf{K}$ is stable in a tail of cardinals.

We prove a simpler and local analog to Fact 5.7.4. Rather than looking at the whole tail of cardinals (more accurately the class of singular cardinals with all possible cofinalities) after a potentially high threshold, we directly look for the next $\omega + 1$ many cardinals of μ and verify that $\mathbf{K}$ has enough stability, continuity of nonsplitting and symmetry in those cardinals. Symmetry will be guaranteed by more stability.

Proposition 5.7.5. *There is $\lambda < h(\mu^{+\omega})$ such that if $\mathbf{K}$ is stable in $[\mu, \lambda)$ and has continuity of $\mu^{+\omega}$-nonsplitting, then it is $\mu^{+\omega}$-superstable.*

Proof. Obtain λ from Corollary 5.6.2(2) and suppose $\mathbf{K}$ is stable in $[\mu, \lambda)$ and has continuity of $\mu^{+\omega}$. The conclusion of Corollary 5.6.2(2) (which uses stability in $\mu^{+\omega}$ and continuity of $\mu^{+\omega}$-nonsplitting) gives a saturated model M of size $\mu^{+\omega}$. We show that is a $(\mu^{+\omega}, \omega)$-limit: by stability in $[\mu, \mu^{+\omega})$, build $\langle M_n : n \leq \omega \rangle \subseteq K_{<\mu^{+\omega}}$ u-increasing and continuous such that for $n < \omega$, $M_n \in K_{\mu^{+n}}$ and $M_\omega = M$. On the other hand, by stability in $\mu^{+\omega}$, build $\langle N_i : i \leq \omega \rangle \subseteq K_{\mu^{+\omega}}$ u-increasing and continuous such that $M_0 \leq N_0$. By a back-and-forth argument, $M \cong_{M_0} N_\omega$ and the latter is a $(\mu^{+\omega}, \omega)$-limit. By uniqueness of limit models of the same cofinality, any $(\mu^{+\omega}, \omega)$-limit is saturated.

By Proposition 5.6.12(2) where $\xi = \mu^{+\omega}$, $\delta = \aleph_0$, $\mathbf{K}$ has $\aleph_0$-local character of $\mu^{+\omega}$-nonsplitting. Together with stability in $\mu^{+\omega}$, we know that $\mathbf{K}$ is superstable in $\mu^{+\omega}$. $\square$

We state a more general form of the above proposition:

Corollary 5.7.6. *Let δ be a regular cardinal. There is $\lambda < h(\mu^{+\delta})$ such that if $\mathbf{K}$ is stable in $[\mu, \lambda)$ and has continuity of $\mu^{+\delta}$-nonsplitting, then it has δ-local character of $\mu^{+\delta}$-nonsplitting. Stability in $[\mu, \lambda)$ can be replaced by stability in $[\mu^{+\delta}, \lambda)$ and unboundedly many cardinals below $\mu^{+\delta}$.*

Proof. Replace "ω" by δ in Proposition 5.7.5. Notice that unboundedly stability many cardinals below $\mu^{+\delta}$ are sufficient to build $\langle M_i : i < \delta\rangle \subseteq K_{<\mu^{+\delta}}$ u-increasing. $\square$

Remark 5.7.7.
1. A missing case of Proposition 5.7.5 is perhaps the regular cardinal $\aleph_0$. In [BKV06, Theorem 2], it was shown that if $\mathbf{K}$ has ω-locality, $\aleph_0$-tameness and stability in $\aleph_0$, then $\mathbf{K}$ is stable everywhere. The original proof used a tree argument of height ω. We provide an alternative proof using our general tools: by ω-locality and Proposition 5.3.16(2), $\mathbf{K}$ has continuity of $\aleph_0$-nonsplitting. By Proposition 5.3.9, $\mathbf{K}$ has $\aleph_0$-local character of $\aleph_0$-nonsplitting. By Corollary 5.6.11(1), it is (super)stable everywhere.

2. Our proof strategy of Proposition 5.7.5 is similar to that of [Vas18c, Theorem 4.11] but we use different tools. Both assume stability in $\mu^{+\omega}$ and unboundedly many cardinals in $\mu^{+\omega}$. To obtain a saturated model, Vasey raised the threshold of μ so that the union of μ^{+n}-saturated models is μ^{+n}-saturated (see Fact 5.6.3). Then he used [Vas18c, Theorem 4.13] that models in $K_{\mu^{+\omega}}$ can be closed to a μ^{+n}-saturated model. These two give a saturated model in $K_{\mu^{+\omega}}$. In contrast, we bypass such gap by using the uniqueness of long enough limit models in $K_{\mu^{+\omega}}$, this immediately gives us a saturated model in $K_{\mu^{+\omega}}$. After that, Vasey and our approaches converge: the saturated model is a $(\mu^{+\omega}, \omega)$-limit and Proposition 5.6.12 gives $\aleph_0$-local character of $\mu^{+\omega}$-nonsplitting.

Question 5.7.8. 1. Perhaps under extra assumptions, is it possible to obtain a tighter bound of $\lambda'(\mathbf{K})$ in terms of $\lambda(\mathbf{K})$ than in Fact 5.7.3?

2. Let ξ_1, ξ_2 be stability cardinals. Is there any relationship between continuity of ξ_1-nonsplitting and continuity of ξ_2-nonsplitting? Similarly, can one say anything about continuity of ξ_1-nonsplitting if for unboundedly many stability cardinal $\xi < \xi_1$, $\mathbf{K}$ has continuity of ξ-nonsplitting? A positive answer might help improve Proposition 5.7.5.

In [BG17, Section 7], Boney and Grossberg developed a U-rank for an independence relation over types of arbitrary length. Until Fact 5.7.16, we specify that we only need an independence relation over 1-types for the proofs to go through.

Definition 5.7.9. [BG17, Definition 7.2] Let $\mathbf{K}$ have a monster model and an independence relation over types of length one. U is a class function that maps each Galois type (of length one) in the monster model to an ordinal or ∞, such that for any $M \in K$, $p \in \mathrm{gS}(M)$,

1. $U(p) \geq 0$;
2. For limit ordinal α, $U(p) \geq \alpha$ iff $U(p) \geq \beta$ for all $\beta < \alpha$;
3. For an ordinal β, $U(p) \geq \beta + 1$ iff there is $M' \geq M$, $\|M'\| = \|M\|$ and $p' \in \mathrm{gS}(M')$ such that p' is a forking (in the sense of the given independence relation) extension of p and $U(p') \geq \beta$;
4. For an ordinal α, $U(p) = \alpha$ iff $U(p) \geq \alpha$ but $U(p) \not\geq \alpha + 1$;
5. $U(p) = \infty$ iff $U(p) \geq \alpha$ for all ordinals α.

Through a series of lemmas, they managed to obtain the following fact (Assumption 5.2.1 is not needed).

Fact 5.7.10. [BG17, Theorem 7.9] Let $\mathbf{K}$ have a monster model and an independence relation over types of length one. Suppose the independence relation satisfies invariance and monotonicity. Let $M \in K$ and $p \in \mathrm{gS}(M)$. The following are equivalent:

1. $U(p) = \infty$;

2. There is $\langle p_n : n < \omega \rangle$ such that $p_0 = p$ and for $n < \omega$, the domain of p_n has size $\|M\|$, and p_{n+1} is a forking extension of p_n.

The original proof proceeds with a lemma followed by the theorem statement. Since the proof of the lemma omitted some details, and that the lemma and the theorem made reference to each other, we straighten the proof as follows:

Lemma 5.7.11. *(2)$\Rightarrow$(1) holds in Fact 5.7.10.*

Proof. By induction on each ordinal α, we show that for each α, for each $n < \omega$, $U(p_n) \geq \alpha$. The base case $\alpha = 0$ is by the definition of U. The limit case follows from the inductive hypothesis. Suppose we have proven the case α, then for each $n < \omega$, inductive hypothesis gives $U(p_{n+1}) \geq \alpha$. By the definition of U, $U(p_n) \geq \alpha + 1$. $\square$

Lemma 5.7.12. *Let* $\mathbf{K}$ *have a monster model and an independence relation over types of length one. Suppose the independence relation satisfies invariance and monotonicity. Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. There is an ordinal $\alpha_\lambda < (2^\lambda)^+$ such that for $M \in K_\lambda$, $p \in \mathrm{gS}(M)$, if $U(p) \geq \alpha_\lambda$ then $U(p) = \infty$.*

Proof. By invariance, there are at most 2^λ many U-ranks of types over models of size λ. It suffices to show that there is no gap in the U-rank: if β is an ordinal, $N \in K_\lambda$, $q \in \mathrm{gS}(N)$ with $\beta < U(q) < \infty$, then there is a forking extension q' of q (with domain of size λ) such that $U(q') = \beta$. Otherwise pick a counterexample $q \in \mathrm{gS}(N)$. Since $U(q) \geq \beta + 1$, there is a forking extension q_1 of q such that $U(q_1) \geq \beta$. As $U(q_1)$ cannot be β, $U(q_1) \geq \beta + 1$. Using monotonicity of forking, we can inductively build $\langle q_n : n < \omega \rangle$ with $q_0 := q$ and for $n < \omega$, q_{n+1} is a forking extension of q_n. By Lemma 5.7.11, $U(q_0) = U(q) = \infty$, contradicting the assumption on $U(q)$. $\square$

Lemma 5.7.13. *Let* $\mathbf{K}$ *have a monster model and an independence relation over types of length one. Suppose the independence relation satisfies invariance and monotonicity. Then (1)$\Rightarrow$(2) in Fact 5.7.10 holds.*

Proof. Let $\lambda = \|M\|$, α_λ as in Lemma 5.7.12 and $p_0 := p$. Define $\langle p_n : n < \omega \rangle$ inductively such that $U(p_n) = \infty$. The base case is by assumption on p. Suppose p_n is constructed with $U(p_n) = \infty$, then in particular $U(p_n) \geq \alpha_\lambda + 1$. By definition of U, there is a forking extension p_{n+1} of p_n (with domain of size λ) such that $U(p_{n+1}) \geq \alpha_\lambda$. By Lemma 5.7.12 again, $U(p_{n+1}) = \infty$. $\square$

Proof of Fact 5.7.10. Combine Lemma 5.7.11 and Lemma 5.7.13. $\square$

We have now arrived at an alternative characterization of superstability. At the end of [GV17, Section 6], they suggested the use of coheir and show that superstability implies bounded U-rank. Since we cannot verify the claim, we use instead μ-nonforking as the independence relation to characterize superstability as bounded U-rank for limit models in K_μ.

Corollary 5.7.14. *Under Assumption 5.2.1, restrict μ-nonforking to limit models in K_μ ordered by $\leq_u$. Then $\mathbf{K}$ is μ-superstable iff $U(p) < \infty$ for all $p \in \mathrm{gS}(M)$ and limit model $M \in K_\mu$.*

Proof. By Fact 5.7.10, we need to show μ-superstability is equivalent to the negation of criterion (2) there. By continuity of μ-nonforking (Proposition 5.4.4) and the proof of Lemma 5.3.7, it suffices to prove that μ-superstability is equivalent to μ-nonforking having local character $\aleph_0$ (under AP it is always possible to extend an omega-chain of types). The forward direction is given by Proposition 5.4.9 and the backward direction is given by Proposition 5.4.2, Proposition 5.4.5 and Proposition 5.4.19. $\square$

We look at one more result of U-rank, which shows the equivalence of being a nonforking extension and having the same U-rank (Fact 5.7.16). The extra assumption of LS($\mathbf{K}$)-witness property for singletons was pointed out by [GMA21, Lemma 8.8] to allow the proof of monotonicity of U-rank [BG17, Lemma 7.3] to go through. We will adapt their definition of LS($\mathbf{K}$)-witness property for singletons because our nonforking is originally defined for model-domains while their independence relations assume set-domains (another approach is perhaps to work in the closure (Definition 5.7.17) of nonforking, but we will not pursue it here).

Definition 5.7.15.

1. Let λ be a cardinal. An independence relation $\downarrow$ has the λ-witness property if the following holds: let a be a singleton and $M, N \in K$. If for any M' with $M \leq M' \leq N$, $\|M'\| \leq \|M\| + \lambda$, we have $a \underset{M}{\downarrow} M'$, then $a \underset{M}{\downarrow} N$.

2. An independence relation satisfies left transitivity if the following holds: let A be a set, $M_0 \leq M_1 \leq N$ with $A \underset{M_1}{\downarrow} N$ and $M_1 \underset{M_0}{\downarrow} N$, then $A \underset{M_0}{\downarrow} N$.

Fact 5.7.16. [BG17, Theorem 7.7] Let $\mathbf{K}$ have a monster model and an independence relation over types of arbitrary length. Suppose the independence relation satisfies: invariance, monotonicity, left transitivity, existence, extension, uniqueness, symmetry and LS($\mathbf{K}$)-witness property for singletons. For any $p \in \mathrm{gS}(M_0)$, any $q \in \mathrm{gS}(M_1)$ extending p such that both $U(p), U(q) < \infty$, then

$$U(p) = U(q) \Leftrightarrow q \text{ is a nonforking extension of } p$$

We notice a gap in [BG17, Lemma 7.6] which Fact 5.7.16 depends on (readers can skip to Fact 5.7.20 if they simply use Fact 5.7.16 as a blackbox; we will also give an alternative proof that does not depend on the lemma). As usual, their definition of independence relations assume that the domain contains the base: if we write $A \underset{M}{\downarrow} N$, we assume $M \leq N$. In the proof of [BG17, Lemma 7.6], they applied monotonicity to obtain $N_2 c \underset{\bar{N}_0}{\downarrow} N_1$. However, $\bar{N}_0 \not\leq N_1$ because $c \in \bar{N}_0 - N_1$ might happen. We will rewrite the proof in Proposition 5.7.19 using the idea of a closure of an independence relation, and drawing results from [BGKV16].

Definition 5.7.17. [BGKV16, Definition 3.4] $\bar{\downarrow}$ is a closure of an independence relation $\downarrow$ if it satisfies the following properties:

1. $\bar{\downarrow}$ is defined on triples of the form (A, M, B) where $M \in K$, A and B are sets of elements. We allow $M \not\subseteq B$.

2. Invariance: if $f \in \mathrm{Aut}(\mathfrak{C})$ and $A \underset{M}{\bar{\downarrow}} B$, then $f[A] \underset{f[M]}{\bar{\downarrow}} f[B]$;

3. Monotonicity: if $A \underset{M}{\bar{\downarrow}} B$, $A' \subseteq A$, $B' \subseteq B$, then $A' \underset{M}{\bar{\downarrow}} B'$;

4. Base monotonicity: if $A\underset{M}{\overline{\downarrow}}B$ and $M \leq M' \subseteq M \cup B$, then $A\underset{M'}{\overline{\downarrow}}B$.

The minimal closure of $\downarrow$ (which is the smallest closure of $\downarrow$) is defined by: $A\underset{M}{\overline{\downarrow}}C$ iff there is $N \geq M$, $N \supseteq C$ such that $A \underset{M}{\downarrow} N$.

We quote the following lemma without proof.

Lemma 5.7.18. [BGKV16, Lemmas 5.1, 5.3, 5.4] *Let $\downarrow$ be an independence relation for types of arbitrary length, $\overline{\downarrow}$ be the minimal closure of $\downarrow$.*

1. *$\downarrow$ has symmetry iff $\overline{\downarrow}$ has symmetry.*

2. *Suppose $\downarrow$ has extension. Then $\downarrow$ has left transitivity iff $\overline{\downarrow}$ does.*

3. *$\downarrow$ has extension iff $\overline{\downarrow}$ has extension.*

Proposition 5.7.19. *Under the same hypothesis as Fact 5.7.10, let $N_0 \leq N_1 \leq \bar{N}_1$; $N_0 \leq \bar{N}_0 \leq \bar{N}_1$; $N_0 \leq N_2$; $c \in |\bar{N}_0|$. If*

$$N_1 \underset{N_0}{\downarrow} \bar{N}_0 \text{ and } N_2 \underset{\bar{N}_0}{\downarrow} \bar{N}_1$$

then there is some N_3 extending both N_1 and N_2 such that

$$c \underset{N_2}{\downarrow} N_3.$$

Proof. We write $\overline{\downarrow}$ to mean the minmal closure of the given independence relation $\downarrow$. By symmetry twice on $N_2 \underset{\bar{N}_0}{\downarrow} \bar{N}_1$, there is $\bar{N}_2$ containing c and extending $\bar{N}_0, N_2$ such that $\bar{N}_2 \underset{\bar{N}_0}{\downarrow} \bar{N}_1$. By definition of the minimal closure,

$$\bar{N}_2\underset{\bar{N}_0}{\overline{\downarrow}}N_1.$$

On the other hand, by symmetry (and monotonicity) on $N_1 \underset{N_0}{\downarrow} \bar{N}_0$, $\bar{N}_0 \underset{N_0}{\downarrow} N_1$. Then $\bar{N}_0\underset{N_0}{\overline{\downarrow}}N_1$. Applying Lemma 5.7.18(2) to the last two closure independence, we have $N_2c\underset{N_0}{\overline{\downarrow}}N_1$. By Lemma 5.7.18(1), there is $N_3' \geq N_2$ and containing c such that $N_1\underset{N_0}{\overline{\downarrow}}N_3'$. By definition of the minimal closure, $N_1 \underset{N_0}{\downarrow} N_3'$. (Here we return to the original proof.) By base monotonicity, $N_1 \underset{N_2}{\downarrow} N_3'$. By symmetry, there is N_3 extending N_1 and N_2 such that $N_3' \underset{N_2}{\downarrow} N_3$. By monotonicity, $c \underset{N_2}{\downarrow} N_3$ as desired. □

Back to Fact 5.7.16, we would like to know if there are any examples of independence relations that satisfy its hypotheses. The approach in [BG17] is to consider coheir [BG17, Definition 3.2], assuming tameness, shortness, no weak order property and that coheir satisfies extension. More developments of coheir can be found in [Vas16a] but the framework there is too abstract to handle.

Another natural candidate is μ-nonforking. One obstacle is that the hypotheses in Fact 5.7.16 require the independence relation to be over types of arbitrary length, while we have defined it for singletons only. Another obstacles is that if we extend our frame to longer types, we might not necessarily guarantee type-fullness (existence holds for all nonalgebraic types), so we cannot invoke Fact 5.7.16. To resolve these, we use the following fact to extend our frame to types of arbitrary length, while acknowledging that the new frame might not be type-full. Then we give an alternative proof to Fact 5.7.16 that does not use existence.

We state the full assumptions of the following facts.

Fact 5.7.20. Let $\mathbf{K}$ have a monster model, $\lambda \geq \mathrm{LS}(\mathbf{K})$.

1. [BV17b, Theorem 1.1] Suppose $\mathbf{K}$ is λ-tame and there is a good $(\geq \lambda)$-frame perhaps except the symmetry property. Then the frame can be extended to a (perhaps non-type-full) good frame for types of arbitrary length and satisfying symmetry.

2. [BGKV16, Lemma 5.9] Let $\downarrow$ be an independence relation for types of arbitrary length. Suppose $\downarrow$ satisfies symmetry and right transitivity, then it satisfies left transitivity.

Remark 5.7.21.

1. Fact 5.7.20(1) is achieved by *independent sequences*. If we simply build nonforking from nonsplitting for longer types, then some of the results in this paper do not generalize (for example stability of μ-types in K_μ immediately fails). One would need extra assumptions (say shortness) and to build the frame in higher cardinals. See also [Vas17e, Appendix A].

2. Another known approach to get a type-full frame for longer types is via Shelah's NF. Vasey [Vas16a, Sections 11, 12] showed that with shortness (which we do not assume

in this paper), one can extend a nice enough frame by NF, which is type-full.

Under μ-superstability, we can derive an independence relation that satisfies all the hypotheses of Fact 5.7.16 except for existence for longer types. We will use Assumption 5.2.1.

Proposition 5.7.22. *Let $\mathbf{K}$ be μ-superstable. Let $\mathbf{K}'$ be the AEC of the limit models in $K_{\geq\mu}$ ordered by $\leq_u$. Then μ-nonforking restricted to $\mathbf{K}'$ can be extended to a (perhaps non-type-full) good frame for types of arbitrary length. Also it satisfies left transitivity and μ-witness property for singletons.*

Proof. By Corollary 5.4.13 and Remark 5.4.14(2), μ-nonforking restricted to $\mathbf{K}'$ forms a good $(\geq \mu)$-frame perhaps except symmetry (it actually satisfies symmetry by Corollary 5.5.13(2) but we do not need this result here). $\mathbf{K}'$ is also μ-tame because $\mathbf{K}$ is μ-tame under Assumption 5.2.1 and we can extend a model in K_μ to a limit model which is in K'. By Fact 5.7.20(1), μ-nonforking can be extended to a good $(\geq \mu)$-frame for types of arbitrary length.

Since the extended frame enjoys symmetry and right transitivity, by Fact 5.7.20(2) it satisfies left transitivity. We check the μ-witness property for singletons: let $M \leq_u N$ both in K', $p \in \mathrm{gS}(N)$. Suppose for any M' with $M \leq_u M' \leq_u N$, $\|M'\| \leq \|M\| + \mu = \|M\|$, we have $p \restriction M'$ does not μ-fork over M. We need to show that p does not μ-fork over M. Without loss of generality assume $\|N\| > \|M\|$. By existence of μ-nonsplitting (Proposition 5.3.12), there is $N' \in K_\mu$, $N' \leq N$ such that p does not μ-split over N'. As N is saturated (replace "μ" by $\|N\|$ in Corollary 5.6.2(2)), we can obtain $N'' \in K'_{\|M\|}$ such that $N' <_u N'' <_u N$ and $M \leq_u N''$. By definition p does not μ-fork over N''. Since $p \restriction N''$ does not μ-fork over M by assumption, Corollary 5.4.8 guaratees that p does not μ-fork over M. $\square$

For comparison purposes, we reproduce the original proof of Fact 5.7.16 that uses existence for longer types. Then we give an alternative proof that bypasses it, so that we can utilize the frame in Proposition 5.7.22.

Original proof of Fact 5.7.16. The forward direction is by definition of U-rank. For the backward direction, we show that for any ordinal α, $U(p) \geq \alpha$ iff $U(q) \geq \alpha$. It suffices to consider the successor case: if $U(q) \geq \alpha + 1$, then it has a forking extension $q' \in \mathrm{gS}(M_2)$ of rank $\geq \alpha$, with $\|M_2\| = \|M_1\|$. By monotonicity of nonforking, q' is also a forking extension of p. However, $\|M\|$ might not be the same as $\|M_2\|$ (this was pointed out by [GMA21]). We claim that there must be some $p' \in \mathrm{gS}(M')$ such that

- $\|M'\| = \|M\|$;
- $p \leq p' \leq q'$; and
- p' is a forking extension of p.

Otherwise, every such p' satisfying the first two requirements must be a nonforking extension of p. By $\mathrm{LS}(\mathbf{K})$-witness property, q' is also a nonforking extension of p, contradiction. Since $U(q') \geq \alpha$, by inductive hypothesis $U(p') \geq \alpha$, and hence $U(p) \geq \alpha + 1$.

If $U(p) \geq \alpha + 1$, by definition there is $p' \in \mathrm{gS}(M_2)$ such that $\|M_2\| = \|M\|$ and p' is a forking extension of p of rank $\geq \alpha$. We claim that we can choose p' and M_2 so that there is $q' \in \mathrm{gS}(M_3)$ with

- q' extends p and p';
- M_3 extends M_1 and M_2;
- q' is a nonforking extension of p'.

Assume that such p' and M_2 are chosen, we show that q' is a forking extension of q: otherwise by transitivity, q' is a nonforking extension of p, and by monotonicity p' is also a nonforking extension of p, contradiction. Now q' is a nonforking extension of p', so by inductive hypothesis $U(q') = U(p') \geq \alpha$. On the other hand, q' is a forking extension of q, so by definition $U(q) \geq U(q') + 1 \geq \alpha + 1$ as desired.

It remains to guarantee such p' and M_2 above exist. Let d realizes q and d' realizes p'. Since both p' and q extends p, there is $f \in \mathrm{Aut}_{M_0}(\mathfrak{C})$ such that $f(d') = d$. Since $\mathrm{gtp}(d/M_1)$ does not fork over M_0, by symmetry there is $\bar{M}_0$ containing M_0 and d such that

$\mathrm{gtp}(M_1/\bar{M}_0)$ does not fork over M_0. Let $\bar{M}_1$ extends both $\bar{M}_0$ and M_1 (possible because we work in $\mathfrak{C}$). By *existence* $\mathrm{gtp}(f[M_2]/\bar{M}_0)$ does not fork over $\bar{M}_0$. By extension there is M_2^* such that $\mathrm{gtp}(M_2^*/\bar{M}_1)$ does not fork over $\bar{M}_0$ and $\mathrm{gtp}(M_2^*/\bar{M}_0) = \mathrm{gtp}(f[M_2]/\bar{M}_0)$. Hence there is $g \in \mathrm{Aut}_{\bar{M}_0}(\mathfrak{C})$ with $g[f[M_2]] = M_2^*$. We now invoke Proposition 5.7.19 where we substitute $N_0, N_1, \bar{N}_0, \bar{N}_1, N_2, c$ by $M_0, M_1, \bar{M}_0, \bar{M}_1, M_2^*, d$ respectively. Then we obtain some M_3 extending M_1 and M_2^* such that $\mathrm{gtp}(d/M_3)$ does not fork over M_2^*. $p' := \mathrm{gtp}(d/M_2^*)$ satisfies the requirements. $\square$

Alternative proof of Fact 5.7.16. In the original proof, the only place that uses existence for longer types is to guarantee $\mathrm{gtp}(f[M_2]/\bar{M}_0)$ does not fork over $\bar{M}_0$. Pick any $M_4 \leq \mathfrak{C}$ that extends both $f[M_2]$ and M_1. We will work in the minimal closure of the independence relation and use Lemma 5.7.18. From the original proof, we have obtained $\mathrm{gtp}(M_1/\bar{M}_0)$ does not fork over M_0. By monotonicity $\mathrm{gtp}(M_1/\bar{M}_0)$ does not fork over $\bar{M}_0$. By symmetry (for the minimal clsoure), $\mathrm{gtp}(\bar{M}_0/M_1)$ does not fork over $\bar{M}_0$. By extension (see [BGKV16, Definition 3.5]), there is M^* and $f \in \mathrm{Aut}_{\bar{M}_0 M_1}(\mathfrak{C})$ such that $\mathrm{gtp}(M^*/M_4)$ does not fork over $\bar{M}_0$ and $f[\bar{M}_0] = M^*$. Since f fixes $\bar{M}_0$, $M^* = \bar{M}_0$. Therefore, $\mathrm{gtp}(\bar{M}_0/M_4)$ does not fork over $\bar{M}_0$. By monotonicity, $\mathrm{gtp}(\bar{M}_0/f[M_2])$ does not fork over $\bar{M}_0$. Symmetry gives the desired result. $\square$

Corollary 5.7.23. *Let* $\mathbf{K}$ *be* μ*-superstable and* $\mathbf{K}'$ *be the AEC of the limit models in* $K_{\geq\mu}$ *ordered by* $\leq_u$*. Let* $\downarrow$ *be the extended frame from Proposition 5.7.22 and define the* U*-rank for* $\downarrow$*. For any* $M <_u M_1 \in K'$*,* $p \in \mathrm{gS}(M)$*, any* $q \in \mathrm{gS}(M_1)$ *extending* p *such that both* $U(p), U(q) < \infty$*, then*

$$U(p) = U(q) \Leftrightarrow q \textit{ is a nonforking extension of } p$$

Proof. Combine Fact 5.7.16 and Proposition 5.7.22. The alternative proof of Fact 5.7.16 (given before Proposition 5.7.22) shows that existence is not necessary. $\square$

5.8 THE MAIN THEOREMS AND APPLICATIONS

We summarize our results in two main theorems. The first one concerns stable AECs while the second one concerns superstable ones. Some of the following items allow $\mu \geq$

LS($\mathbf{K}$) but we assume $\mu > \mathrm{LS}(\mathbf{K})$ for a uniform statement. The proofs will come after the main theorems.

Main Theorem 5.8.1. *Let $\mathbf{K}$ be an AEC with a monster model, $\mu > \mathrm{LS}(\mathbf{K})$, $\delta \leq \mu$ both be regular. Suppose $\mathbf{K}$ is μ-tame, stable in μ and has continuity of μ-nonsplitting. The following statements are equivalent under extra assumptions specified after the list:*

1. *$\mathbf{K}$ has δ-local character of μ-nonsplitting;*
2. *There is a good frame over the skeleton of $(\mu, \geq \delta)$-limit models ordered by $\leq_u$, except for symmetry and local character δ in place of $\aleph_0$. In this case the frame is canonical;*
3. *$\mathbf{K}$ has uniqueness of $(\mu, \geq \delta)$-limit models;*
4. *For any increasing chain of μ^+-saturated models, if the length of the chain has cofinality $\geq \delta$, then the union is also μ^+-saturated;*
5. *K_{μ^+} has a δ-superlimit.*

(1) and (2) are equivalent. If $\mathbf{K}$ is $(< \mu)$-tame, then (3) implies (1). There is $\lambda_1 < h(\mu)$ such that if $\mathbf{K}$ is stable in $[\mu, \lambda_1)$, then (1) implies (3). Given any $\zeta \geq \mu^+$, stability in $[\mu, \lambda_1)$ can be replaced by stability in $[\mu, \zeta)$ plus no μ-order property of length ζ.

There is $\lambda_2 < h(\mu^+)$ such that if $\mathbf{K}$ is stable in $[\mu^+, \lambda_2)$ and has continuity of μ^+-nonsplitting, then (1) implies (4). Given any $\zeta \geq \mu^{++}$, stability in $[\mu^+, \lambda_2)$ can be replaced by stability in $[\mu^+, \zeta)$ plus no μ^+-order property of length ζ. Always (4) and (5) are equivalent and they imply (3).

The following diagram summarizes the implications in Main Theorem 5.8.1. Labels on the arrows indicate the extra assumptions needed, in addition to a monster model, μ-tameness, stability in μ and continuity of μ-nonsplitting. As in the theorem statement, whenever we require stability in the form $[\xi, \lambda)$, we can replace it by stability in $[\xi, \zeta)$ plus no ξ-order property of length ζ.

$$\begin{tikzcd}
& & (3) \arrow[dl, "(<\mu)\text{-tame}"] & & \\
(2) \arrow[r, Leftrightarrow] & (1) \arrow[ur, bend left, "\text{stable in }[\mu,\lambda_1)"] \arrow[rr, "\text{stable in }[\mu,\lambda_1)", "\text{cont. of }\mu^+\text{-nonsplitting}"'] & & (4) \arrow[ul, bend right] \arrow[r, Leftrightarrow] & (5)
\end{tikzcd}$$

Main Theorem 5.8.2. *Let* $\mathbf{K}$ *be an AEC with a monster model,* $\mu > \mathrm{LS}(\mathbf{K})$ *be regular. Suppose* $\mathbf{K}$ *is* μ*-tame, stable in* μ *and has continuity of* μ*-nonsplitting. The following statements are equivalent modulo* $(< \mu)$*-tameness and a jump in cardinal (specified after the list):*

1. $\mathbf{K}$ *has* $\aleph_0$*-local character of* μ*-nonsplitting;*
2. *There is a good frame over the limit models in* K_μ *ordered by* $\leq_u$*, except for symmetry. In this case the frame is canonical;*
3. K_μ *has uniqueness of limit models;*
4. *For any increasing chain of* μ^+*-saturated models, the union of the chain is also* μ^+*-saturated;*
5. K_{μ^+} *has a superlimit;*
6. $\mathbf{K}$ *is* (μ^+, μ^+)*-solvable;*
7. $\mathbf{K}$ *is stable in* $\geq \mu$ *and has continuity of* $\mu^{+\omega}$*-nonsplitting;*
8. U*-rank is bounded when* μ*-nonforking is restricted to the limit models in* K_μ *ordered by* $\leq_u$*.*

(1), (2) and (8) are equivalent and each of them implies (3) and (4). If $\mathbf{K}$ *is* $(< \mu)$*-tame, then (3) implies (1). Always (4) and (5) are equivalent and they imply (3). (1) implies (6) and (7) while (6) implies (4). (7) implies* $(1)_{\mu^{+\omega}}$*:* $\mathbf{K}$ *has* $\aleph_0$*-local character of* $\mu^{+\omega}$*-nonsplitting.*

The jump in cardinal is due to the lack of a precise bound on $\lambda'(\mathbf{K})$ in deducing (7)$\Rightarrow$(1) (see Question 5.7.8(1)). The following diagram summarizes the implications in Main Theorem 5.8.2. "$\mu^{+\omega}$" indicates the jump in cardinal.

$$
\begin{tikzcd}
 & & (3) \arrow[dl, "(<\mu)\text{-tame}"] & & \\
(2) \arrow[dr, Leftrightarrow] & & & & \\
(8) \arrow[r, Leftrightarrow] & (1) \arrow[uur, bend left] \arrow[rr] \arrow[d] \arrow[dr] & & (4) \arrow[uul, bend right] \arrow[r, Leftrightarrow] & (5) \\
 & (7) \arrow[u, bend left, "\mu^{+\omega}"] & (6) \arrow[ur] & &
\end{tikzcd}
$$

Proof of Main Theorem 5.8.1. (1) and (2) are equivalent by Corollary 5.4.13 and Proposition 5.4.19. The canonicity of the frame is by Proposition 5.4.18. Suppose (3) holds. Then the proof of Corollary 5.6.2(2) and Proposition 5.6.12(1) give (1).

Suppose (1) holds. Obtain $\lambda_1 = \lambda$ from Corollary 5.6.2 and take $\chi = \delta$. If $\mathbf{K}$ is stable in $[\mu, \lambda_1)$, then it has uniqueness of $(\mu, \geq \delta)$-limit models, so (3) holds. The alternative hypotheses of stability and no-order-property work because we can replace λ in the proof of Proposition 5.5.9 by ζ.

The direction of (1) to (4) is by Proposition 5.6.6. The alternative hypotheses work because we can replace λ in the proof of Proposition 5.5.9 by ζ. (4) and (5) are equivalent by Lemma 5.6.22 and Lemma 5.6.23. They imply (3) by Corollary 5.6.16. $\square$

For the proof of Main Theorem 5.8.2, we show the additional directions and refer the readers to the proof of Main Theorem 5.8.1 for the original directions.

Proof of Main Theorem 5.8.2. Compared to Main Theorem 5.8.1, we do not need the extra stability and continuity of nonsplitting assumptions because superstability already implies them (Corollary 5.6.11(1) and Proposition 5.3.16(1)). (1) and (8) are equivalent by Corollary 5.7.14. (1) implies (7) by Corollary 5.6.11(1) while (1) implies (6) by the forward direction of Proposition 5.6.24. (6) plus $(< \mu)$-tameness implies (4) by the proof of the backward direction of Proposition 5.6.24. (7) implies $(1)_{\mu^{+\omega}}$ by Proposition 5.7.5. $\square$

Remark 5.8.3. In [GV17, Corollary 5.5], they did not assume continuity of nonsplitting and showed that: if item (4) in Main Theorem 5.8.2 holds in some $\xi \geq \beth_\omega(\chi_0 + \mu)$ (see Fact 5.6.3 for the definition of χ_0), then every limit model in K_ξ is $\beth_\omega(\chi_0 + \mu)$-saturated.

This implies $\aleph_0$-local character of ξ-nonsplitting. Using [BV17b, Theorem 7.1], there is a $\lambda < h(\xi)$ such that (3) holds with μ replaced by λ. From hindsight, the last argument can be improved by quoting Corollary 5.6.11(3) instead and having $\lambda = \xi^+$. In comparison, our (4)$\Rightarrow$(3) allows (3) to still be in K_μ and does not have the high cardinal threshold.

Corollary 5.8.4. *Let $\xi > \mathrm{LS}(\mathbf{K})$ and $\mathbf{K}$ have a monster model, continuity of ξ-nonsplitting and be $(<\xi)$-tame. Then the following are equivalent:*

1. $\mathbf{K}$ *has uniqueness of limit models in K_ξ: for any $M_0, M_1, M_2 \in K_\xi$, if both M_1 and M_2 are limit over M_0, then $M_1 \cong_{M_0} M_2$;*

2. $\mathbf{K}$ *has uniqueness of limit models without base in K_ξ: any limit models in K_ξ are isomorphic.*

Proof. The forward direction is immediate and only requires JEP. For the backward direction, the proof of (3)$\Rightarrow$(1) in Main Theorem 5.8.2 goes through (JEP is needed) and we have ξ-superstability. By (1)$\Rightarrow$(3) in Main Theorem 5.8.2, it has uniqueness of limit models in K_ξ. $\square$

As applications, we present alternative proofs to the results in [MA20] and [SV18a] with stronger assumptions. In [MA20], limit models of abelian groups are studied.

Fact 5.8.5.

1. [MA20, Definition 3.1, Fact 3.2] Let $\mathbf{K}^{ab}$ be the class of abelian groups ordered by subgroup relation. Then $\mathbf{K}^{ab}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{ab}) = \aleph_0$, has a monster model and is $(<\aleph_0)$-tame.

2. [MA20, Fact 3.3(2)] $\mathbf{K}^{ab}$ is stable in all infinite cardinals.

3. [MA20, Corollary 3.8] $\mathbf{K}^{ab}$ has uniqueness of limit models in all infinite cardinals.

In the original proof of Fact 5.8.5(3), an explicit algebraic expression of limit models was obtained, so that limit models of the same cardinality are isomorphic to each other. In [MA20, Remark 3.9], it was remarked that [Vas18c] could be used to obtain uniqueness of limit models for high enough cardinals (above $\geq \beth_{(2^{\aleph_0})^+}$). We write down the exact argument using known results. Then we present another proof that covers lower cardinals using results in this paper (but not any algebraic description of limit models).

First proof of Fact 5.8.5(3). In Fact 5.7.4(1), pick $\xi \geq (\lambda'(\mathbf{K}))^+ + \chi_1$ with $\mathrm{cf}(\xi) = \aleph_0$. By Fact 5.8.5(2), $\mathbf{K}^{ab}$ is stable in ξ. So the conclusion of Fact 5.7.4(1) gives superstability in $\geq \lambda'(\mathbf{K}^{ab})$. By [VV17, Corollary 1.4] (which combines [VV17, Fact 2.16, Corollary 6.9]), $\mathbf{K}^{ab}$ has uniqueness of limit models in $K^{ab}_{\geq \lambda'(\mathbf{K}^{ab})}$. Notice that by Fact 5.7.3, $\lambda'(\mathbf{K}^{ab}) < h(\lambda(\mathbf{K}^{ab})) = h(\aleph_0) = \beth_{(2^{\aleph_0})^+}$, so we can guarantee uniqueness of limit models above $\beth_{(2^{\aleph_0})^+}$. $\square$

Second proof of Fact 5.8.5(3). By Fact 5.8.5(1)(2), $\mathbf{K}^{ab}$ is stable in $\aleph_0$ and is $(<\aleph_0)$-tame. The latter implies ω-locality. By Proposition 5.3.16(2), $\mathbf{K}^{ab}$ has continuity of $\aleph_0$-nonsplitting. By Remark 5.7.7(1), it is superstable in $\geq \aleph_0$. By Corollary 5.6.2(1) (or simply [VV17, Corollary 1.4]), it has uniqueness of limit models in all infinite cardinals. $\square$

We turn to look at a strictly stable AEC.

Fact 5.8.6.

1. [MA20, Definition 4.1, Facts 4.2, 4.5] Let $\mathbf{K}^{tf}$ be the class of torsion-free abelian groups ordered by pure subgroup relation. Then $\mathbf{K}^{tf}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{tf}) = \aleph_0$, has a monster model and is $(<\aleph_0)$-tame.
2. [MA20, Fact 4.7] $\mathbf{K}^{tf}$ is stable in λ iff $\lambda^{\aleph_0} = \lambda$. In particular $\mathbf{K}^{tf}$ is strictly stable.
3. [MA20, Corollary 4.18] Let $\lambda \geq \aleph_1$. $\mathbf{K}^{tf}$ has uniqueness of $(\lambda, \geq \aleph_1)$-limit models.
4. [MA20, Theorem 4.22] Let $\lambda \geq \aleph_0$. Any $(\lambda, \aleph_0)$-limit model in $\mathbf{K}^{tf}$ is not algebraically compact.
5. [MA20, Lemmas 4.10, 4.14] Let $\lambda \geq \aleph_1$. Any $(\lambda, \geq \aleph_1)$-limit model in $\mathbf{K}^{tf}$ is algebraically compact. Any two algebraically compact limit models in K^{tf}_λ are isomorphic.

The original proof of the second part of Fact 5.8.6(3) uses an explicit algebraic expression of algebraically compact groups [MA20, Fact 4.13]. Using the results of this paper, we give a weaker version but without using any algebraic expression of algebraically compact groups.

Proposition 5.8.7. *Assume CH. If for all stability cardinal $\lambda \geq \aleph_1$, $\mathbf{K}^{tf}$ does not have the λ-order property of length $\lambda^{+\omega}$, then for all such λ, it has uniqueness of $(\lambda, \geq \aleph_1)$-limit models.*

Proof. By CH and Fact 5.8.6(2), $\mathbf{K}^{tf}$ is stable in $\aleph_1$. By Fact 5.8.6(1), $\mathbf{K}^{tf}$ is $(<\aleph_0)$-tame, hence it has ω-locality. By Proposition 5.3.16(2), $\mathbf{K}^{tf}$ has continuity of $\aleph_1$-nonsplitting. Proposition 5.3.9 and Lemma 5.6.7 give $\aleph_1$-local character of λ-nonsplitting for all stability cardinals λ. By Fact 5.6.10(1), $\mathbf{K}^{tf}$ is stable in $[\lambda, \lambda^{+\omega})$. By Corollary 5.6.2(1) and Remark 5.6.8(1), $\mathbf{K}^{tf}$ has uniqueness of $(\lambda, \geq \aleph_1)$-limit models for all $\lambda \geq \aleph_1$. $\square$

Question 5.8.8. Is it true that $\mathbf{K}^{tf}$ does not have $\aleph_1$-order property of length $\aleph_\omega$?

For Fact 5.8.6(4), the original proof argued that uniqueness of limit models *eventually* leads to superstability for large enough λ (from an older result in [GV17]). Then a specific construction deals with small λ. In [MA20, Remark 4.23], it was noted that [Vas18c, Lemma 4.12] could deal with both cases of λ. We give a full proof here (the algebraic description of limit models is needed):

Proof of Fact 5.8.6(4). Let $\lambda \geq \aleph_0$ and M be a $(\lambda, \aleph_0)$-limit model. Then $\mathbf{K}^{tf}$ is stable in λ and by Fact 5.8.6(2) $\lambda > \aleph_0$. Suppose M is algebraically compact, by Fact 5.8.6(5) and Corollary 5.6.2(2) M is isomorphic to $(\lambda, \geq \aleph_1)$-limit models and is saturated. By Proposition 5.6.12(2) (where $\langle M_i : i \leq \aleph_0 \rangle$ witnesses that M is $(\lambda, \aleph_0)$-limit), $\aleph_0$-local character of λ-nonsplitting applies to M. Since M is arbitrary, $\mathbf{K}^{tf}$ has $\aleph_0$-local character of λ-nonsplitting, which implies stability in $\geq \lambda$ by Fact 5.6.10(2), contradicting Fact 5.8.6(2). $\square$

Remark 5.8.9. [Vas18c, Lemma 4.12] happened to work because we do not care about the case $\aleph_0$ (which is not stable) and we can always apply item (2) in Proposition 5.6.12.

In [SV18a], $\aleph_0$-stable AECs with $\aleph_0$-AP, $\aleph_0$-JEP and $\aleph_0$-NMM were studied. They built a superlimit model in $\aleph_0$ by connecting limit models with sequentially homogeneous models [SV18a, Theorem 4.4]. Then they defined splitting over finite sets where types have countable domains and obtained finite character assuming categoricity in $\aleph_0$ [SV18a, Fact 5.3]. This allowed them to build a good $\aleph_0$-frame over models generated by the superlimit. These methods are absent in our paper because we studied AECs with a general LS($\mathbf{K}$), and our splitting is defined for types over model-domains.

In [SV18a, Corollary 5.9], they showed the existence of a superlimit in $\aleph_1$ assuming weak $(<\aleph_0, \aleph_0)$-locality among other assumptions. We will strengthen the locality assumption to ω-locality, and work in a monster model to give an alternative proof. This allows us to bypass the machinery in [SV18a] that are sensitive to the cardinal $\aleph_0$, and the technical manipulation of symmetry in [SV18a, Section 3]. Also, our result extends to a general $\mathrm{LS}(\mathbf{K})$.

Proposition 5.8.10. *Let* $\mathbf{K}$ *is an* $\aleph_0$*-stable AEC with a monster model and has* ω*-locality. Then there is a superlimit in* $\aleph_1$*. In general, let* $\lambda \geq \mathrm{LS}(\mathbf{K})$*, and if* $\mathbf{K}$ *is stable in* λ *instead of* $\aleph_0$*, then it has a superlimit in* λ^+*.*

Proof. Apply Main Theorem 5.8.2(1)$\Rightarrow$(5) where $\mu = \mathrm{LS}(\mathbf{K})$ (that direction does not require $\mu > \mathrm{LS}(\mathbf{K})$). Notice that ω-locality implies $\mathrm{LS}(\mathbf{K})$-tameness. $\square$

Tracing our proof, we require global assumptions of a monster model and ω-locality in order to use our symmetry results, especially Proposition 5.5.9. We end this section with the following:

Question 5.8.11. Instead of global assumptions like monster model and no-order-property, is it possible to obtain local symmetry properties in Section 5.5 using more local assumptions?

CHAPTER 6

CATEGORICITY TRANSFER FOR TAME AECS WITH AMALGAMATION OVER SETS

ABSTRACT

Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame abstract elementary class and assume amalgamation over sets and arbitrarily large models. Suppose $\mathbf{K}$ is categorical in some $\mu > \mathrm{LS}(\mathbf{K})$, then it is categorical in all $\mu' \geq \mu$. At the cost of using amalgamation over sets instead of over models, our result removes the successor requirement of μ made by Grossberg-VanDieren [GV06a], and the primes requirement by Vasey [Vas17b]. As a corollary, we obtain an alternative proof of the upward categoricity transfer for first-order theories [Mor65a, She74]. In our construction, we simplify Vasey's results [Vas16a, Vas17e] to build a weakly successful frame. This allows us to use Shelah-Vasey's argument [SV18b] to obtain primes for sufficiently saturated models. If we replace the categoricity assumption by $\mathrm{LS}(\mathbf{K})$-superstability, $\mathbf{K}$ is already excellent for sufficiently saturated models. This sheds light on the investigation of the main gap theorem for uncountable first-order theories within ZFC.

6.1 INTRODUCTION

For first-order theories, we have the following categoricity theorems:

Theorem 6.1.1. *1.* [Mor65a] *Let T be a countable first-order theory. If T is categorical in some uncountable cardinal, then it is categorical in all uncountable cardinals.*

2. [She74] *Let T be a first-order theory. If T is categorical in some cardinal $> |T|$, then it is categorical in all cardinals $> |T|$.*

In the late seventies after Shelah completed his book [She90], he came up with a far reaching program: develop classification theory for non-elementary classes. Thus he titled his papers [She83a, She83b, She87] "Classification theory for non-elementary classes". In the summer of 1976, Shelah proposed as a test question for such a theory (which appeared in [She83a, Conjecture 2]):

Conjecture 6.1.2 (Categoricity conjecture for $L_{\omega_1,\omega}$)**.** Let ψ be a sentence of $L_{\omega_1,\omega}$ in a countable language. If ψ is categorical in some $\mu \geq \beth_{\omega_1}$, then ψ is categorical in all $\mu \geq \beth_{\omega_1}$.

In the second edition of his book [She90], the conjecture was generalized to:

Conjecture 6.1.3 (Categoricity conjecture for $L_{\lambda^+,\omega}$)**.** Let ψ be a sentence of $L_{\lambda^+,\omega}$ in a language of size λ. If ψ is categorical in some $\mu \geq \beth_{(2^\lambda)^+}$, then ψ is categorical in all $\mu \geq \beth_{(2^\lambda)^+}$.

In [She00, Section 6], Shelah stated that classification theory for abstract elementary classes (AECs) is the most important direction of model theory. He conjectured:

Conjecture 6.1.4 (Categoricity conjecture for AECs)**.** Let $\mathbf{K}$ be an AEC and $\lambda = \mathrm{LS}(\mathbf{K})$. The threshold for categoricity transfer is $\beth_{(2^\lambda)^+}$ (the Hanf number).

The importance of these conjectures is the structural theory that needs to be developed. The main concept of the previously-developed structural theory for first-order theories is *forking*: a canonical notion that generalizes *combinatorial geometries* (also called *matroids* when they are finitely generated).

In about 3000 pages of publications towards these conjectures indeed such a theory evolved (see the table at the end of this section for a partial list of results). We can divide the approaches into three types:

a. Assuming tameness and other model theoretic properties: Grossberg and VanDieren [GV06a, GV06c] extracted the notion of tameness from [She99] and derived categoricity transfer from a successor cardinal for tame AECs with a monster model. Many subsequent results were obtained by Boney and Vasey but the successor assumption from [GV06a] still could not be removed. Vasey [Vas18b] building upon Shelah's results, showed that categoricity transfer holds for AECs with amalgamation and primes (without starting from a successor cardinal) and managed to prove that the eventual categoricity conjecture is true for universal classes [Vas17e, Vas17f].

b. Assuming non-ZFC axioms and model theoretic properties: Shelah [She83a, She83b] showed that under WGCH, if a countable theory in $L_{\omega_1,\omega}$ is excellent and has few models in $\aleph_n$ for $n < \omega$, then categoricity transfers up from an uncountable cardinal. [She09a] also developed heavy machinery such as good frames to derive categoricity transfers. However many of his results have technical assumptions which are not easy to verify. Later Shelah and Vasey [SV18b] generalized the notion of excellence to AECs and derived categoricity transfers assuming WGCH and restricting the spectrum in an interval of cardinals. A few variations were given in [SV18b, Vas19] where they replaced the spectrum requirements by other model theoretic properties.

Meanwhile, Makkai and Shelah [MS90] proved that the eventual categoricity conjecture is true for an $L_{\kappa,\omega}$ theory starting at successor cardinals, where κ is strongly compact. Boney [Bon14b] showed that tameness holds for compact AECs (assuming the existence of strongly compact cardinals), thus by [GV06a, GV06c] the eventual categoricity is true starting at successor cardinals. Eventually [SV18b] used the excellence argument to remove the successor assumption.

c. Using specific constructions: Cheung [Che21] showed that given a free notion of amalgamation and the existence of prime models, the AEC behaves like strongly minimal theories, which allows one to manipulate the AEC algebraically.

Mazari-Armida [MA22] combined decomposition results from algebra and categoricity transfer from [Vas17b] to characterize algebraically the property of being categorical in a tail. In particular, let R be an associative ring with unity, he proved that the threshold of categoricity transfer is $(|R| + \aleph_0)^+$ for the class of locally pure-injective modules, flat modules and absolutely pure modules.

Espíndola [Esp22] used topos-theoretic argument to show that the eventual categoricity conjecture holds. However, there is no explicit bound to the threshold cardinal μ.

In this paper we follow approach (a) above and focus on AECs that have a monster model, satisfy amalgamtion over sets and tameness. In doing so we can remove the successor assumption in (2) in the table. In our proof, we rely heavily on many recent papers and replace the use of WGCH in (8) by amalgamation over sets to obtain excellence. Then using [SV18b] that excellence implies primes, we can invoke the categoricity transfer in (3). A main application of this result is the removal of the successor requirement in the categorical transfer in [MS90] (see also [She00, Question 6.14] for the problem statement).

Our work was motivated by a simple question: using the common model theoretic assumptions and techniques, can we recover the upward[1] categoricity transfer in Theorem 6.1.1? [Les00] and [HK11] have relevant results but they require $\mathrm{LS}(\mathbf{K}) = \aleph_0$ and several additional assumptions (say *simplicity*: there is a strong example by Shelah that in the context of homogeneous model theory, simplicity is not a consequence of $\aleph_0$-stability [HL02]). Such results might not be easy to check and generalize to uncountable $\mathrm{LS}(\mathbf{K})$. Meanwhile, Vasey [Vas18b, Section 4] adopted a hybrid approach where he quoted syntactic results from [She71, HS00] to conclude that a homogeneous diagram has primes and a nonforking relation over sets, and then combined it with the categoricity transfer for AECs with amalgamation and primes. In comparison, our result is cleaner because we do not invoke primeness or stability results from [She71, HS00]. The assumptions of tameness and amalgamation over sets are immediate to check.

When we show excellence, we only require tameness, amalgamation over sets, arbitrarily large models and superstability. This way of obtaining excellence does not use any non-ZFC axioms and might shed light on the main gap theorem for uncountable first-order theories: [GL05] used an axiomatic framework to obtain the abstract decomposition theorem, a key step to the main gap theorem. The results from [SV18b] provide us with a multidimensional independence relation, which satisfies some of the axioms in [GL05]. For future work, one may look at the axioms on regular types (see [GL05, Axioms 8-10]).

We now list some of the known results on categoricity transfer for AECs. The numbering is for reference only and is not chronological. We strengthen some of the assumptions to a monster model "$\mathfrak{C}$" for readability (unless they assumed a local frame). Here a monster model means amalgamation, joint embedding and no maximal models. We write "$\mathfrak{C}_{\mathrm{set}}$"

[1] *Downward* transfer is a much harder problem for AECs: the currently known transfer with common assumptions is down to the first Hanf number. Example 6.6.15 shows that the first categoricity cardinal can go up to the first Hanf number, but such example fails amalgamation and joint-embedding.

if we also require amalgamation over sets. We strengthen instances of WGCH in an interval of cardinals to full WGCH. Throughout we let $\lambda = \mathrm{LS}(\mathbf{K})$. Except for (5)(12), we assume that the categoricity cardinal $\mu < h(\lambda)$ (so we can omit the downward transfer to the first Hanf number $h(\lambda)$). Some of the results can be combined but we highlight the new parts. The key results of categoricity transfers within ZFC are (2), (3) and (4). By assuming amalgamation over sets, we remove the successor assumption in (2) and (4), while removing the prime triples assumption in (3).

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	Assumptions on $\mathbf{K}$	If $I(\mu, \mathbf{K}) = 1$ for some	Then $I(\mu', \mathbf{K}) = 1$ for all
	λ-tame, $\mathfrak{C}_{\text{set}}$	$\mu \geq \lambda^+$	$\mu' \geq \mu$ (Theorem 6.6.13)
1.	Homogeneous diagram with $\mathfrak{C}_{\text{set}}$	$\mu \geq \lvert T\rvert^+$	$\mu' \geq \mu$ [Vas18b, Theorem 4.22]
2.	λ-tame, $\mathfrak{C}$	successor $\mu \geq \lambda^+$	$\mu' \geq \mu$ [GV06a, Theorem 5.3]
3.	λ-tame, $\mathfrak{C}$, has primes	$\mu \geq \lambda^+$	$\mu' \geq \mu$ [Vas17b, Theorem 10.9]
4.	Has a type-full good $[\mu_1, \mu_2]$-frame where μ_2 is a successor $> \mu_1 \geq \lambda$	μ_1, μ_2 as on the left	$\mu' \in [\mu_1, \mu_2]$ [Vas17b, Theorem 6.14]
5.	$\lambda < \kappa$ for some strongly compact κ	successor $\mu \geq \kappa^+$	$\mu' \geq \mu$ [Bon14b, Theorem 7.4]
6.	Compact	$\mu \geq \lambda^+$	$\mu' \geq \mu$ [SV18b, Theorem 14.5]
7.	Excellent	$\mu \geq \lambda^+$	$\mu' \geq \mu$ [SV18b, Theorem 14.2]
8.	WGCH, has a $(< \omega)$-extendible categorical good μ_1-frame	$\mu_2 \geq \mu_1^+$	$\mu' \geq \mu_1^+$ [SV18b, Corollary 14.4]
9.	WGCH, $K_{\lambda^{++}} \neq \emptyset$ and for $n < \omega$, $I(\lambda^{+n}, \mathbf{K}) < \mu_{\text{unif}}(\lambda^{+n}, 2^{\lambda^{+(n-1)}})$	$\mu = \lambda, \lambda^+$	$\mu' \geq \lambda$ [SV18b, Theorem 14.11]
10.	WGCH, $\mathfrak{C}$	$\mu_1, \mu_2 \geq \lambda$	$\mu' \in [\mu_1, \mu_2]$ [Vas19, Lemma 9.5]
11.	WGCH, $\mathfrak{C}$	$\mu > \lambda^{+\omega}$	$\mu' \geq \mu$ [Vas19, Lemma 9.6]
12.	Universal class	$\mu \geq \beth_{h(\lambda)}$	$\mu' \geq \mu$ [Vas17f, Theorem 7.3]
13.	$\text{PC}_{\aleph_0}$, $\aleph_0$-tame, has primes, $2^{\aleph_0} < 2^{\aleph_1}$	$\mu = \aleph_1$	$\mu' \geq \mu$ [MAV18, Theorem 4.4]
14.	WGCH, $\text{PC}_{\aleph_0}$, $1 \leq I(\aleph_1, \mathbf{K}) < 2^{\aleph_1}$, and few models in $\aleph_n$	$\mu \geq \aleph_1$ and $\mu = \aleph_0$	$\mu' \geq \aleph_0$ [SV18b, Theorem 14.12]
15.	Atomic models of a countable first-order theory, WGCH, few models in $\aleph_n$	$\mu \geq \aleph_1$	[She83a, She83b]
16.	Universal $L_{\omega_1,\omega}$ sentence	Tail of $[\text{LS}(\mathbf{K}), \beth_\omega)$	$\mu' \geq \beth_\omega$ [Vas20, Corollary 5.10]
17.	Has prime and small models and a free notion of amalgamation	$\mu \geq \mu(\mathbf{K}) + \lambda$	$\mu' \geq \mu(\mathbf{K}) + \lambda + I(\lambda, \mathbf{K})^+$ [Che21, Theorem 5.7]
18.	The class of locally pure-injective modules/ flat modules/absolutely pure modules	$\mu \geq (\lvert R\rvert + \aleph_0)^+$	[MA22]
19.	None	(no explicit bound)	[Esp22]

6.2 PRELIMINARIES

In this section, we will define the main notions used in this paper (see Definition 4.2.2 for the definition of AECs). Relevant results will be discussed in the subsequent sections.

Definition 6.2.1. Let $\mathbf{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathbf{K})$. The functions f mentioned below will be $\mathbf{K}$-embeddings.

1. $\mathbf{K}$ has the *λ-amalgamation property* (λ-*AP*) if for any $M_0, M_1, M_2 \in K_\lambda$, $M_0 \leq_{\mathbf{K}} M_1$, $M_0 \leq_{\mathbf{K}} M_2$, there is $M_3 \in K_\lambda$ and $f : M_1 \xrightarrow[M_0]{} M_3$ such that $M_2 \leq_{\mathbf{K}} M_3$. $\mathbf{K}$ has the *amalgamation property* (*AP*) when the above is true without the cardinal restriction.

2. $\mathbf{K}$ has the *amalgamation property over set bases* (*AP over sets*) if for any $M_1, M_2 \in K$, any $A \subseteq |M_1| \cap |M_2|$, there is $M_3 \in K$ and $f : M_1 \xrightarrow[A]{} M_3$ such that $M_2 \leq_{\mathbf{K}} M_3$.

3. $\mathbf{K}$ has the *λ-joint embedding property* (λ-*JEP*) if for any $M_1, M_2 \in K$, there is $M_3 \in K_\lambda$ and $f : M_1 \to M_3$ such that $M_2 \leq_{\mathbf{K}} M_3$. $\mathbf{K}$ has the *joint embedding property* (*JEP*) when the above is true without the cardinal restriction.

4. $\mathbf{K}$ has *no maximal models* (*NMM*) if for any $M \in K$, there is $N \in K$ such that $M \leq_{\mathbf{K}} N$ but $M \neq N$.

5. $\mathbf{K}$ has *arbitrarily large models* (*AL*) if for any cardinal $\mu \geq \mathrm{LS}(\mathbf{K})$, there is $M \in K_{\geq \mu}$.

6. $\mathbf{K}$ has a *monster model* $\mathfrak{C}$ if it has *AP*, *JEP* and *NMM*.

7. $\mathbf{K}$ has $\mathfrak{C}_{\mathrm{set}}$ if it has *AP* over sets (which implies *JEP*) and *NMM*.

Remark 6.2.2. All the properties except for (2)(7) in the above definition hold in complete first-order theories because of the compactness theorem. (2)(7) hold if we also fix a monster model and they will only be used in Section 6.6. See also the discussions around [Bal09, Definition 4.34, Lemma 18.8].

Definition 6.2.3. Let $\alpha \geq 2$ be an ordinal.

1. We denote *Galois types* (*orbital types*) of length ($< \alpha$) as $\mathrm{gS}^{<\alpha}(\cdot)$ (see [Vas16c, Definition 2.16]; we will not need the precise definition in this paper). The argument

can be a set A in some model $M \in K$. In general $\mathrm{gS}^{<\alpha}(A) := \bigcup\{\mathrm{gS}^{<\alpha}(A; M) : M \in K, |M| \supseteq A\}$ (under AP, the choice of M does not matter).

2. $\mathbf{K}$ is $(<\alpha)$-*stable in* λ if for any set A in some model $M \in K$, $|A| \leq \lambda$, then $|\mathrm{gS}^{<\alpha}(A; M)| \leq \lambda$. We omit "$(<\alpha)$" if $\alpha = 2$, while we omit "in λ" if there exists such a $\lambda \geq \mathrm{LS}(\mathbf{K})$. Similarly $\mathbf{K}$ is α-*stable in* λ if for any such A and M above, we have $|\mathrm{gS}^{\alpha}(A)| \leq \lambda$.

The notion of *tameness* was introduced by Grossberg and VanDieren [GV06a] as an extra assumption to an AEC. Later Boney [Bon14b] introduced a dual property named *shortness*. Tameness is a locality property on the domain of types while shortness is a locality property on the tuples that realize the types.

Definition 6.2.4. Let κ be an infinite cardinal.

1. Let $p = \mathrm{gtp}(a/A, N)$ where $a = \langle a_i : i < \alpha\rangle$ may be infinite, $I \subseteq \alpha$, $A_0 \subseteq A$. We write $l(p) := l(a)$, $p \restriction A_0 := \mathrm{gtp}(a/A_0, N)$, $a^I = \langle a_i : i \in I\rangle$ and $p^I := \mathrm{gtp}(a^I/A, N)$.

2. K is $(<\kappa)$-*tame for* $(<\alpha)$-*types* if for any subset A in some model of K, any $p \neq q \in \mathrm{gS}^{<\alpha}(A)$, there is $A_0 \subseteq A$, $|A_0| < \kappa$ with $p \restriction A_0 \neq q \restriction A_0$. We omit $(<\alpha)$ if $\alpha = 2$.

3. K is $(<\kappa)$-*short* if for any $\alpha \geq 2$, any subset A in some model of K, $p \neq q \in \mathrm{gS}^{<\alpha}(A)$, there is $I \subseteq \alpha$, $|I| < \kappa$ with $p^I \neq q^I$.

4. κ-*tame* means $(<\kappa^+)$-tame. Similarly for shortness.

Remark 6.2.5. By [Vas16c, Corollary 3.18], $(<\kappa)$-shortness implies $(<\kappa)$-tameness. First-order theories are trivially $(<\aleph_0)$-short, while a theorem due to Boney shows that universal classes are also $(<\aleph_0)$-short [Vas17e, Theorem 3.7].

The notion of a good frame was introduced in [She09a, Chapter II]. The definition was extended for domains of sizes from an interval of cardinals (instead of a single cardinal) in [Vas16b] while for longer types in [BV17b]. We follow the notation in [BV17b] but specialize it in our context, where the types are always type-full (basic types coincide with nonalgebraic types) and we work inside a monster model. [Vas16a], building on numerous papers, defined many more properties of a frame which cater for his coheir construction, which will not be considered here.

Definition 6.2.6. Let $\mathbf{K}$ be an AEC with a monster model $\mathfrak{C}$, $\mu \geq \mathrm{LS}(\mathbf{K})$ be a cardinal and $\alpha \geq 2$ be an ordinal or ∞. A $(<\alpha, \geq \mu)$-good frame is a ternary relation $\perp$ such that:

1. If $(a, M_0, M_1) \in \perp$, then $a \in |M_1|^{<\alpha}$, $M_0 \leq_{\mathbf{K}} M_1$ and $M_0, M_1 \in K_{\geq \mu}$. We write $a \underset{M_0}{\perp} M_1$ and say $\mathrm{gtp}(a/M_1)$ *does not fork over* M_0 (well-defined by invariance below).
2. (Invariance) If $f \in \mathrm{Aut}(\mathfrak{C})$ and $a \underset{M_0}{\perp} M_1$, then $f(a) \underset{f(M_0)}{\perp} f(M_1)$.
3. (Monotonicity) If $a \underset{M_0}{\perp} M_1$, $M_0 \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N_1 \leq_{\mathbf{K}} M_1$, $a' \subseteq a$ and $a' \in |N'|$, then $a' \underset{N_0}{\perp} N_1$.
4. (Stability) For $M \in K_{\geq \mu}$, $|\mathrm{gS}(M)| \leq \|M\|$.
5. (Existence) For $M \in K_{\geq \mu}$ and $a \in |M|^{<\alpha}$, $a \underset{M}{\perp} M$.
6. (Extension) If $p \in \mathrm{gS}^{<\alpha}(M_1)$ does not fork over M_0, $M_1 \leq_{\mathbf{K}} M_2$ and $l(p) \leq \beta < \alpha$, then there is $q \in \mathrm{gS}^{\beta}(M_2)$ such that $q^{\beta} \restriction M = p$ and q does not fork over M_0.
7. (Uniqueness) If $p, q \in \mathrm{gS}^{<\alpha}(M_1)$ do not fork over M_0 and $p \restriction M_0 = q \restriction M_0$, then $p = q$.
8. (Transitivity) If $a \underset{M_0}{\perp} M_1$ and $a \underset{M_1}{\perp} M_2$, then $a \underset{M_0}{\perp} M_2$.
9. (Local character) If δ is regular, $\langle M_i \in K_{\geq \mu} : i \leq \delta \rangle$ is increasing and continuous, $p \in \mathrm{gS}^{<\delta}(M_\delta)$, then there is $i < \delta$ such that p does not fork over M_i.
10. (Continuity) If δ is a limit ordinal, both $\langle M_i \in K_{\geq \mu} : i \leq \delta \rangle$ and $\langle \alpha_i < \alpha : i \leq \delta \rangle$ both increasing and continuous, $p_i \in \mathrm{gS}^{\alpha_i}(M_i)$ increasing in $i < \delta$, then there is some $p \in \mathrm{gS}^{\alpha_\delta}(M_\delta)$ such that for all $i < \delta$, $p^{\alpha_i} \restriction M_i = p_i$. If each p_i does not fork over M_0, then neither does p.

11. (Symmetry) If $a_2 \underset{M_0}{\downarrow} M_1$ and $a_1 \in |M_1|^{<\alpha}$, then there is M_2 containing a_2 such that $a_1 \underset{M_0}{\downarrow} M_2$.

We define $(<\alpha,\mu)$-frame similarly when the models must have size μ. We omit "$(<\alpha)$" when $\alpha = 2$. We call $\downarrow$ an independence relation if it only has invariance and monotonicity.

Remark 6.2.7. There are weaker versions of a good frame which still have nice properties (for example [JS13, MA19] and Chapter 3), which will not be discussed here because we will focus on the full strength of a good frame (and more under categoricity).

6.3 A $(<\infty, \geq (2^{\mathrm{LS}(\mathbf{K})})^+)$-NONFORKING RELATION

Assuming superstability and shortness, we will build a $(<\infty, \geq (2^{\mathrm{LS}(\mathbf{K})})^+)$-nonforking relation with nice properties. This will allow us to use [Vas16a, Section 11] to conclude that the underlying good frame is weakly successful. The result was sketched in [Vas17e, Lemma A.14] but it drew technical results from [Vas16a, Sections 1-10]. In this section, we will construct the nonforking relation and derive its properties directly. Readers can blackbox this section and skip to Section 6.4.

Definition 6.3.1. Let $\mathbf{K}$ be an AEC with a monster model, $\lambda \geq \mathrm{LS}(\mathbf{K})$.

1. Let $M \leq_{\mathbf{K}} N$, we say that N is an *universal extension of* M if for any $N' \in K_{\|M\|}$ with $M \leq_{\mathbf{K}} N'$, there is $f : N' \underset{M}{\longrightarrow} N$. We say a chain $\langle M_i \in K_\lambda : i \leq \delta\rangle$ is *universally increasing* if for each $i < \delta$, M_{i+1} is a universal extension of M_i.
2. Let $N \in K$ and $p \in \mathrm{gS}(N)$, we say that p λ-splits over M if there exists $N_1, N_2 \in K_\lambda$ such that $M \leq_{\mathbf{K}} N_1, N_2 \leq_{\mathbf{K}} N$, $f : N_1 \underset{M}{\longrightarrow} N_2$ with $f(p) \restriction N_2 \neq p \restriction N_2$.
3. $\mathbf{K}$ is *superstable in* λ if $\mathbf{K}$ is stable in λ and the following holds: for any limit ordinal $\delta < \lambda^+$, any universally increasing and continuous $\langle M_i \in K_\lambda : i \leq \delta\rangle$, $p \in \mathrm{gS}(M_\delta)$, there is $i < \delta$ such that p does not λ-split over M_i.

Remark 6.3.2. In item (1), if $M \leq_{\mathbf{K}} N$ and N is $\|M\|^+$-saturated, then N is a universal extension over M. In addition, N realizes all $(<\|M\|^+)$-types over M. In item (3), by [Vas16a, Proposition 10.10] or Corollary 5.6.11, λ-tameness and λ-superstability imply λ'-superstability for $\lambda' \geq \lambda$.

Under tameness and superstability, we can build a good frame in the successor cardinal. We remark that the original item (2) did not show whether the μ^+-saturated models form an AEC (in particular whether they are closed under unions). It was only later in item (1) that the question was fully settled.

Fact 6.3.3. Let $\mathbf{K}$ be an AEC with a monster model and $\mu \geq \mathrm{LS}(\mathbf{K})$. Suppose $\mathbf{K}$ is μ-tame and superstable in μ.

1. [VV17, Corollary 6.10] For $\lambda > \mu$, $\mathbf{K}^{\lambda\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{\lambda\text{-sat}}) = \lambda$.

2. [Vas16b, Theorem 7.1] The relation defined by: $p \in \mathrm{gS}(N)$ does not fork over $M \leq N$ if there is $M_0 \in K_\mu$ such that M is a universal extension over M_0 and p does not μ-split over M_0, induces a good $(\geq \mu^+)$-frame for $\mathbf{K}^{\mu^+\text{-sat}}$ (by (1) the μ^+-saturated models form a sub-AEC of $\mathbf{K}$).

Remark 6.3.4.

- Coheir in [BG17] is another candidate for a good frame, but one has to assume in addition the no weak order property and the extension property of coheir. To remove these assumptions, one has to raise the starting cardinal very high, so the threshold cardinal of categoricity transfer is way above μ^+. See also item 2(a) after this remark.

- One might wonder if it is possible to define the frame for K_μ. [Vas18a, Corollary 13.16] gave a weaker version where the underlying models are limit models while local character and continuity are for universally increasing chains (this argument was generalized to the strictly stable context in 3). Alternatively, [Vas19, Section 6] built a good μ-frame by assuming WGCH and drawing heavily from [JS13] (WGCH is used to establish that the frame is weakly successful, see Definition 6.4.4).

Now we have a good $(\geq \mu)$-frame and would like to extend it to longer types. However, there are difficulties in terms of proving extension and local character. Besides the use of WGCH as in the above remark, we list three main approaches in literature:

1. Using independent sequences and tameness, [BV17b] developed on [She09a, Exercise III.9.4.1] to extend the frame to longer types. But such frame is not necessarily type-full, which is assumed in other results.

2. Extend the good ($\geq \mu$)-frame to a ($< \infty, \geq \mu$)-nonforking relation, which might not be a good frame itself. [Vas16a, Section 11] gave sufficient conditions of the nonforking relation in order for the original frame to be weakly successful. Then one can quote [JS13] to extend the original frame by NF, which is a good frame. To build the nonforking relation, there are two ways:

   (a) [Vas16a, Sections 1-10] built an axiomatic framework that allows one to use coheir to produce a good ($\geq \mu$)-frame (instead of using nonsplitting). To obtain the sufficient conditions above, he went on with a highly convoluted construction, which also uses canonicity to obtain properties from nonsplitting. Moreover, the threshold cardinal μ is very high (fixed points of the beth function) in order to use the no-order property.

   (b) Using nonsplitting (Fact 6.3.3), [Vas17e, Lemma A.14] sketched that it can be extended to a nonforking relation that satisfies the sufficient conditions. However, the details were sparse (about two paragraphs) and he invoked technical results from [Vas16a, Sections 1-10], which have numerous definitions and go back and forth between coheir and nonsplitting.

We will adopt approach 2(b), but give an alternative proof that such nonforking relation satisfies the desired properties. In particular we do not need [Vas16a] in this section but refer to the simple construction in Fact 6.3.3(2). Our starting cardinal is μ^+ for the same reason as the successor cardinal in Fact 6.3.3(2). Meanwhile [Vas17e, Lemma A.14] starts at μ, but we cannot verify the claims there. At the end it does not affect the categoricity transfer by virtue of Fact 6.6.12(2).

Definition 6.3.5. Let $\mathbf{K}$ be an AEC with a monster model, $\mu = 2^{\mathrm{LS}(\mathbf{K})}$ and assume $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-short and superstable in $\mathrm{LS}(\mathbf{K})$.

1. Since shortness implies tameness (Remark 6.2.5), we can define the nonforking relation as in Fact 6.3.3(2) but for $< (\mathrm{LS}(\mathbf{K})^+)$-types (instead of 1-types). This is a $(< \mathrm{LS}(\mathbf{K})^+, \geq \mu^+)$-nonforking relation $\mathop{\smile}\limits^{|}$ over the μ^+-saturated models.

2. Extend $\downarrow$ to a $(<\infty, \geq \mu^+)$-nonforking relation $\bar{\downarrow}$ by coheir : $a \underset{M_0}{\bar{\downarrow}} M_1$ iff for any subsequence $a' \subseteq a$ of length $< \mathrm{LS}(\mathbf{K})^+$, we have $a' \underset{M_0}{\downarrow} M_1$.

The following collection of facts helps us establish local character properties. The second item below is from [Bon17, Theorem 3.5], which was usually cited as [Bon17, Theorem 3.1] (the issue was clarified in Theorem 3.2.2). The statement of the third item can be found in [Vas17e, Lemma A.12] and is essentially [GV06b, Fact 4.6].

Fact 6.3.6. Let $\mathbf{K}$ be an AEC with a monster model, $\mu \geq \mathrm{LS}(\mathbf{K})$ and $\alpha \geq 1$.

1. [She99, Lemma 3.3] If $\mathbf{K}$ is stable in μ, $M \in K_{\geq \mu}$ and $p \in \mathrm{gS}(M)$, then there is $M_0 \leq_{\mathbf{K}} M$, $\|M\| = \mu$ such that p does not μ-split over M_0.
2. If $\mathbf{K}$ is stable in μ and $\mu = \mu^\alpha$, then it is α-stable in μ.
3. If κ satisfies $\mu = \mu^{<\kappa}$, then item (1) is still true for $p \in \mathrm{gS}^{<\kappa}(M)$.

Proof. We sketch (3): by stability and (2), $\mathbf{K}$ is $(<\kappa)$-stable in μ. The proof of (1) shows that if the conclusion of (1) fails, one can build a tree of types and models to contradict 1-stability in μ, where "1" comes from $l(p)$. The same proof goes through for (3) because we now have $(<\kappa)$-stability in μ. $\square$

We now state the nice properties of $\bar{\downarrow}$ we constructed. Items (c) and (d) can be strengthened but they are sufficient for the next section. Notice that shortness is the key to obtain uniqueness in item (e) below.

Proposition 6.3.7. *Let* $\mathbf{K}$ *be an AEC with a monster model,* $\mu = 2^{\mathrm{LS}(\mathbf{K})}$ *and assume* $\mathbf{K}$ *is* $\mathrm{LS}(\mathbf{K})$*-short and* $\mathrm{LS}(\mathbf{K})$*-superstable. The relation* $\bar{\downarrow}$ *defined in Definition 6.3.5 satisfies the following:*

a. $\bar{\downarrow}$ *is a* $(<\infty, \geq \mu^+)$*-nonforking relation over the* μ^+*-saturated models.*

b. When restricted to 1-types, $\bar{\downarrow}$ *is a good* $(\geq \mu^+)$*-frame.*

c. For $n \geq 2$, $\mathbf{K}^{\mu^{+n}\text{-}sat}$ *is an AEC with* $\mathrm{LS}(\mathbf{K}^{\mu^{+n}\text{-}sat}) = \mu^{+n}$.

d. For $n \geq 2$, $\bar{\downarrow}$ restricted to $(\leq \mu^{+n})$-types has local character for chains of length $\geq \mu^{+(n+1)}$. Namely, for any a of length $(\leq \mu^{+n})$, any regular $\delta \geq \mu^{+(n+1)}$, any increasing and continuous chain $\langle M_i : i \leq \delta \rangle \subseteq K^{\mu^+\text{-sat}}$, there is $i < \delta$ such that $a \mathop{\bar{\downarrow}}_{M_i} M_\delta$.

e. $\bar{\downarrow}$ has uniqueness.

f. $\bar{\downarrow}$ has the left $(\leq \mu^+)$-witness property: $a \mathop{\bar{\downarrow}}_{M_0} M_1$ iff for any $a' \subseteq a$ of length $\leq \mu^+$, we have $a' \mathop{\bar{\downarrow}}_{M_0} M_1$.

g. $\bar{\downarrow}$ has the right $(\leq \mu^+)$-model witness property: $a \mathop{\bar{\downarrow}}_{M_0} M$ iff for any $M_1 \in K^{\mu^+\text{-sat}}$ with $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$, $\|M_1\| \leq \mu^+$, we have $a \mathop{\bar{\downarrow}}_{M_0} M_1$.

Proof. Items (a) and (b) follow from the construction of $\bar{\downarrow}$ which extends the original frame. Item (c) is by Fact 6.3.3(1).

For item (d), we first assume that a has length $< \mathrm{LS}(\mathbf{K})^+$. Since $\mu = \mu^{<\mathrm{LS}(\mathbf{K})^+}$, by Fact 6.3.6(3) there is $M^* \leq_{\mathbf{K}} M_\delta$, $\|M^*\| = \mu$ such that $\mathrm{gtp}(a/M_\delta)$ does not μ-split over M^*. Since $\delta \geq \mu^{+n} > \mu$, there is $i < \delta$ such that $M^* \leq_{\mathbf{K}} M_i$. Since M_i is $\|M^*\|^+$-saturated, by Remark 6.3.2 M_i is universal over M^*. By definition, $a \mathop{\downarrow}_{M_i} M_\delta$ as desired. Now for general a of length $(\leq \mu^{+n})$, there are at most $(\mu^{+n})^{\mathrm{LS}(\mathbf{K})}$, which is μ^{+n} many subsequences of length $< \mathrm{LS}(\mathbf{K})^+$, therefore we can take the maximum i from the previous case, which is still less than δ by a cofinality argument.

For item (e), let $M \leq_{\mathbf{K}} N \in K^{\mu^+\text{-sat}}$, $p, q \in \mathrm{gS}^{<\infty}(N)$ both do not fork over M and $p \restriction M = q \restriction M$. By shortness we may assume that $p, q \in \mathrm{gS}^{<\mathrm{LS}(\mathbf{K})^+}(N)$. Then the uniqueness proof for the case of 1-types in Fact 6.3.3(2) goes through, because it uses universal extensions only and our types p, q have length $< \mathrm{LS}(\mathbf{K})^+$ less than the sizes of the models.

Item (f) is true by coheir in the construction, in particular we have $(\leq \mathrm{LS}(\mathbf{K}))$-witness property which is stronger. We show the backward direction of item (g): by coheir and monotonicity, it suffices to consider the case $l(a) < \mathrm{LS}(\mathbf{K})^+$. By Fact 6.3.6(3), there is $M^* \leq_{\mathbf{K}} M$, $\|M^*\| = \mu$ such that $\mathrm{gtp}(a/M)$ does not μ-split over M^*. Pick $N_0 \in K^{\mu^+\text{-sat}}$

such that $M_0 \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} M$ and N_0 is a universal extension over M^*. By definition, $\operatorname{gtp}(a/M)$ does not fork over N_0. Since $\|N_0\| = \mu^+$, by assumption $\operatorname{gtp}(a/N_0)$ does not fork over M_0. Now we can quote the transitivity proof for the case of 1-types in Fact 6.3.3(2), which generalizes to $< \mathrm{LS}(\mathbf{K})^+$-types for the same reason as in the previous paragraph. Thus we have $\operatorname{gtp}(a/M)$ does not fork over M_0 as desired. $\square$

6.4 A WEAKLY SUCCESSFUL FRAME

By Proposition 6.3.7, we will show that the nonforking relation in Definition 6.3.5 satisfies [Vas16a, Hypothesis 11.1]. This allows us to quote results from [Vas16a, Sections 11, 12] and conclude that the underlying good $(\geq (2^{\mathrm{LS}(\mathbf{K})})^+)$-frame is weakly successful, can be extended by NF, is ω-successful and has full model continuity (in the third successor cardinal). This will allow us to do categoricity transfer in Section 6.6. On the other hand, we compare our extended frame with the results in [Vas16a, Section 15], which was constructed from coheir (instead of nonsplitting).

Proposition 6.4.1. *Let* $\mathbf{K}$ *be an AEC with a monster model,* $\mu = 2^{\mathrm{LS}(\mathbf{K})}$ *and assume* $\mathbf{K}$ *is* $\mathrm{LS}(\mathbf{K})$*-short and* $\mathrm{LS}(\mathbf{K})$*-superstable. The relation* $\bar{\downarrow}$ *defined in Definition 6.3.5 satisfies [Vas16a, Hypothesis 11.1].*

Proof. The hypothesis is a list of requirements on the nonforking relation $\bar{\downarrow}$. By substituting "λ" and "μ" there by μ^{++} and μ^+ respectively. We check the items in the same numbering as in the hypothesis.

1. This is exactly Proposition 6.3.7(a). There they use the term "independence relation" to allow the right hand side of $\bar{\downarrow}$ to be sets (instead of models), which is just a generalization and does not affect the rest of the proof.

2. This is Proposition 6.3.7(b).

3. By the substitution above, clearly $\mu^{++} > \mu^+$.

4. This is Proposition 6.3.7(c)(d).

5. Base monotonicity is built in our definition of nonforking relation. Uniqueness is by Proposition 6.3.7(e).

6. This is Proposition 6.3.7(f)(g).

$\square$

Under [Vas16a, Hypothesis 11.1], Vasey imitated the proofs in [MS90] and showed that the underlying good ($\geq \mu^{++}$)-frame has domination triples (see Definition 6.4.2). Then he connected domination triples with uniqueness triples, which allowed him to conclude that the frame is weakly successful. In the following we state the relevant definitions and results.

The term "domination triples" came from the later [Vas17e, Definition A.17] and [Vas17a, Definition 2.9] even though [Vas16a, Definition 11.5] had already investigated the idea of domination.

Definition 6.4.2. Let $\lambda > \mathrm{LS}(\mathbf{K})$ and $\downarrow$ be a $(<\infty, \geq \lambda)$-nonforking relation over the λ-saturated models.

1. A triple (a, M, N) is a *domination triple* if $M \leq_{\mathbf{K}} N$ both λ-saturated, $a \in |N| \backslash |M|$ and for any λ-saturated N', $a \underset{M}{\downarrow} N'$ implies $N \underset{M}{\downarrow} N'$.
2. $\downarrow$ has the λ-*existence property for domination triples* if for any M saturated in K_λ, any nonalgebraic $p \in \mathrm{gS}(M)$, there exists a domination triple (a, M, N) such that $p = \mathrm{gtp}(a/M; N)$.

The following fact [Vas16a, Lemma 11.12] shows the existence property for domination triples. It will be applied to Corollary 6.5.3 to show that the sufficiently saturated models have primes.

Fact 6.4.3. In Proposition 6.4.1, for $\lambda > \mu^+$, $\bar{\downarrow}$ has the λ-existence property for domination triples.

Now we look at uniqueness triples and weak successfulness.

Definition 6.4.4. [Vas16a, Definition 11.4]. Let $\lambda > \mathrm{LS}(\mathbf{K})$ and $\downarrow$ be a good λ-frame over the saturated models in K_λ. Let $M_0 \leq_{\mathbf{K}} M_1$ and $M_0 \leq_{\mathbf{K}} M_2$ all λ-saturated.

1. An *amalgam* of M_1 and M_2 over M_0 is a triple (f_1, f_2, N) such that N is λ-saturated, $f_i : M_i \xrightarrow[M_0]{} N$ for $i = 1, 2$.

2. Two amalgams (f_1^a, f_2^a, N^a), (f_1^b, f_2^b, N^b) of M_1 and M_2 over M_0 are *equivalent* if there are $N \in K_\lambda^{\lambda\text{-sat}}$, $f^a : N^a \to N$ and $f^a : N^a \to N$ such that the following diagram commutes:

$$\begin{tikzcd}
 & N^b \arrow[r, dotted, "f^b"] & N \\
M_1 \arrow[ur, "f_1^b"] \arrow[rr, "f_1^a"'] & & N^a \arrow[u, dotted, "f^a"] \\
M_0 \arrow[u] \arrow[r] & M_2 \arrow[uu, "f_2^b"] \arrow[ur, "f_2^a"'] &
\end{tikzcd}$$

3. A triple (a, M, N) is a *uniqueness triple* if M, N are saturated models in K_λ, $a \in |N| \backslash |M|$ and for any M_1 saturated in K_λ, there exists an amalgam (f_1, f_2, N_1) of N and M_1 over M such that $\mathrm{gtp}(f_1(a)/f_2[M_1]; N_1)$ does not fork over M and the amalgam is unique up to equivalence (see item (2)).

4. $\mathop{\smile}\limits^{\vert}$ is *weakly successful* if it has the *existence property* for uniqueness triples: for any M saturated in K_λ, any nonalgebraic $p \in \mathrm{gS}(M)$, we can find a uniqueness triple (a, M, N) such that $p = \mathrm{gtp}(a/M; N)$.

The following fact translates [Vas16a, Theorem 11.13] into our context.

Fact 6.4.5. Under [Vas16a, Hypothesis 11.1], the relation $\mathop{\smile}\limits^{\bar{\vert}}$ defined in Definition 6.3.5 (when restricted to 1-types and μ^{++}-saturated models) induces a weakly successful good μ^{++}-frame over the μ^{++}-saturated models.

Corollary 6.4.6. *Let* $\mathbf{K}$ *be an AEC with a monster model and* $\mu = 2^{\mathrm{LS}(\mathbf{K})}$. *Suppose* $\mathbf{K}$ *is* $\mathrm{LS}(\mathbf{K})$*-short and superstable in* $\mathrm{LS}(\mathbf{K})$. *Then the good* $(\geq \mu^+)$*-frame defined in Fact 6.3.3(2) induces a weakly successful good* μ^{++}*-frame over the* μ^{++}*-saturated models.*

Proof. Since $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-short and superstable in $\mathrm{LS}(\mathbf{K})$, it is also μ-short and superstable in μ and we can use Fact 6.3.3(2) to build a good $(\geq \mu^+)$-frame $\mathop{\smile}\limits^{\vert}$. By Definition 6.3.5, Proposition 6.3.7 and Proposition 6.4.1, we can extend $\mathop{\smile}\limits^{\vert}$ to a nonforking relation $\mathop{\smile}\limits^{\bar{\vert}}$ that satisfies [Vas16a, Hypothesis 11.1]. By Fact 6.4.5, $\mathop{\smile}\limits^{\bar{\vert}}$ induces a weakly successful good μ^{++}-frame over the μ^{++}-saturated models. But this frame is just $\mathop{\smile}\limits^{\vert}$ restricted to μ^{++}-saturated models. $\square$

One more ingredient for categoricity transfer is the property of full model continuity. Vasey drew results from [She09a, JS13, Jar16] and showed that the weakly successful frame we obtained is ω-successful. And if we move up by three successors (so we consider μ^{+5}-saturated models), then it can be extended to a good frame with full model continuity.

Definition 6.4.7. Let $\mathbf{K}$ be an AEC with a monster model, $\lambda \geq \mathrm{LS}(\mathbf{K})$ and $\downarrow$ be a $(<\infty, \geq \lambda)$-nonforking relation on $K_{\geq\lambda}$. $\downarrow$ has *full model continuity* if the following holds: for any limit ordinal δ, any $\langle M_i^k : i \leq \delta\rangle$ increasing and continuous in $K_{\geq\lambda}$ where $k = 0, 1, 2$, if $M_i^1 \underset{M_i^0}{\downarrow} M_i^2$ for each $i < \delta$, then $M_\delta^1 \underset{M_\delta^0}{\downarrow} M_\delta^2$.

We sum up the previous paragraph in the following fact. The original results were from [Vas16a, Sections 11, 12] but applied them to our context (in the same spirit as Corollary 6.4.6). In particular item (1) is from [Vas16a, Theorem 11.21]; item (2) is from [Vas16a, Theorem 12.16]. We will not define *ω-successfulness* because under amalgamation and tameness, it coincides with weak successfulness [Vas16a, Facts 11.15, 11.19]. Also, *good+* will be automatically satisfied by the new frame [Vas16a, Fact 11.17] so we skip its definition.

Fact 6.4.8. Let $\mathbf{K}$ be an AEC with a monster model and $\mu = 2^{\mathrm{LS}(\mathbf{K})}$. Suppose $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-short and superstable in $\mathrm{LS}(\mathbf{K})$.

1. The weakly successful good μ^{++}-frame from Corollary 6.4.6 is also ω-successful.

2. Let $\lambda = (\mu^{++})^{+3} = \mu^{+5}$. The frame can be extended by NF (defined for quadruples of models) and then closed to a good $(\leq \lambda, \geq \lambda)$-frame over the λ-saturated models. Moreover, the new frame is good+ and has full model continuity.

The rest of this section discusses what happens if we combine our results with [Vas16a, Sections 13-15]. Readers only interested in categoricity transfer can skip to Fact 6.5.5 which will be used in Section 6.6.

After obtaining a good $(\leq \lambda, \geq \lambda)$-frame with full model continuity, Vasey [Vas16a, Sections 13,14] went on extending the right hand side of $\downarrow$ to arbitrary sets, and then the left hand side to arbitrary lengths. Such results still apply to our construction because we have shortness and amalgamation in our background assumptions (see also [Vas16a, Hypotheses 13.1, 14.1]). We first state what Vasey had obtain in [Vas16a, Theorem 15.6].

Fact 6.4.9. Let $\mathbf{K}$ be a $(<\kappa)$-short AEC with a monster model. Suppose there are λ, θ such that

1. $\mathrm{LS}(\mathbf{K}) < \kappa = \beth_\kappa < \lambda = \beth_\lambda \leq \theta$;
2. $\mathrm{cf}(\lambda) \geq \kappa$;
3. $\mathbf{K}$ is categorical in θ;

then there is a $(<\infty, \geq \lambda^{+4})$-good frame over the λ^{+4}-saturated models except that extension holds over saturated models only. Moreover it has full model continuity.

We state one more fact from [She99] about categoricity. A complete proof can be found in [BGVV17].

Fact 6.4.10. Let $\mathbf{K}$ be an AEC with a monster model. Suppose $\mathbf{K}$ is categorical in some $\lambda > \mathrm{LS}(\mathbf{K})$, then $\mathbf{K}$ is superstable in $\mathrm{LS}(\mathbf{K})$.

To compare Fact 6.4.9 with our results, we replace our assumptions of $\mathrm{LS}(\mathbf{K})$-shortness by κ-shortness, and superstability in $\mathrm{LS}(\mathbf{K})$ by superstability in κ.

Corollary 6.4.11. *Let $\mathbf{K}$ be a κ-short AEC with a monster model where $\kappa \geq \mathrm{LS}(\mathbf{K})$. Suppose $\mathbf{K}$ is categorical in some $\theta > \kappa$ (superstability in κ is sufficient), then there is a $(<\infty, \geq (2^\kappa)^{+5})$-good frame over the $(2^\kappa)^{+5}$-saturated models models except that extension holds over saturated models only. Moreover it has full model continuity.*

Proof sketch. By categoricity and Fact 6.4.10, $\mathbf{K}$ is superstable in κ. By Fact 6.4.8 (replacing $\mathrm{LS}(\mathbf{K})$ there by κ), there is a $(<(2^\kappa)^{+5}, \geq (2^\kappa)^{+5})$-good frame over the $(2^\kappa)^{+5}$-saturated models. Extend the frame to arbitrarily long types as in [Vas16a, Sections 13,14]. □

As we can see, using nonsplitting to build a good frame has a much lower threshold than using coheir in obtaining Fact 6.4.9. The fixed points of beth function are to guarantee no order property (see [Vas16a, Fact 2.21]), which currently lacks a good upper bound (under amalgamation and stability). [Vas17e, Corollary A.16] claimed a result similar to our corollary and we highlight the differences here:

1. The threshold he obtained is $(\mathrm{LS}(\mathbf{K})^{<\kappa})^{+5}$ while ours is $(2^\kappa)^{+5}$.

2. He used $(<\kappa)$-shortness directly but we weakened it to κ-shortness. We did so both for convenience and to readily apply Fact 6.4.8.

3. In verifying [Vas16a, Hypothesis 11.1], he drew heavy machinery from [Vas16a, Sections 1-10] but we proved them directly in Proposition 6.4.1.

6.5 PRIMES FOR SATURATED MODELS

We will combine the results from the previous section and Fact 6.5.2 below to conclude that $\mathbf{K}$ has primes for saturated models. However, it is not clear whether this implies primes for models in general, so we cannot invoke categoricity transfer of AECs with primes and amalgamation. Readers only interested in categoricity transfer can skip to Fact 6.5.5 which will be used in the next section.

Definition 6.5.1. [Vas17a, Definition 2.13] Let $\mathbf{K}$ be an AEC.

1. A triple (a, M, N) is a *prime triple* if $M \leq_{\mathbf{K}} N$, $a \in |N|\backslash|M|$, and the following holds: for any $N' \in K$ with $a' \in |N'|$ and $\mathrm{gtp}(a/M; N) = \mathrm{gtp}(a'/M; N')$ then there exists $f : N \underset{M}{\longrightarrow} N'$ such that $f(a) = a'$.

2. $\mathbf{K}$ *has primes* if for each $M \in K$ and each nonalgebraic $p \in \mathrm{gS}(M)$, there exists a prime triple (a, M, N) such that $p = \mathrm{gtp}(a/M; N)$.

The original statement of the following fact is about K^* only but we strengthen the monster model assumption to K. Vasey allowed the right hand side of $\downarrow$ to be sets (and had extra axioms) but we stick to models (see also the proof of Proposition 6.4.1(1)).

Fact 6.5.2. [Vas17a, Theorem 3.6] Let $\mathbf{K}$ be an AEC with a monster model. Suppose there is $\lambda_0 \geq \mathrm{LS}(\mathbf{K})$ and $\mathbf{K}^*$ such that:

1. $\mathbf{K}^* \subseteq \mathbf{K}$ is a sub-AEC of $\mathbf{K}$;

2. $\mathbf{K}^*$ is categorical in λ_0;

3. There is a good $(<\infty, \geq \lambda_0)$-frame with full model continuity over $\mathbf{K}^*$;

4. $\mathbf{K}^*_{\lambda_0}$ has the λ_0-existence property for domination triples (see Definition 6.4.2);

Then for any $\lambda > \lambda_0$, the saturated models of $\mathbf{K}^*_\lambda$ has primes.

Corollary 6.5.3. *Let* $\mathbf{K}$ *be an AEC with a monster model and* $\lambda_0 = (2^{\mathrm{LS}(\mathbf{K})})^{+5}$*. Suppose* $\mathbf{K}$ *is* $\mathrm{LS}(\mathbf{K})$*-short and superstable in* $\mathrm{LS}(\mathbf{K})$*, then for* $\lambda > \lambda_0$*,* $\mathbf{K}^{\lambda\text{-}sat}_\lambda$ *has primes.*

Proof. Let $\mathbf{K}^* = K^{\lambda_0\text{-sat}}$. $\mathbf{K}^*$ is a sub-AEC of $\mathbf{K}$ by Fact 6.3.3(1) and is categorical in λ_0 by a back-and-forth argument. Substituting $\kappa = \mathrm{LS}(\mathbf{K})$ in Corollary 6.4.11, there is a good $(<\infty, \geq \lambda_0)$-frame with full model continuity over $\mathbf{K}^*$. We would like to invoke Fact 6.4.3 (substituting λ there by λ_0) and say that the good frame has λ_0-existence property for domination triples. While the good frame might not agree with the nonforking relation in Fact 6.4.3 for longer types, they both extend the good $(<2, \geq \lambda_0)$-frame from Fact 6.3.3(2). Since domination triples are about 1-types only, we can conclude that the nonforking relation from Fact 6.4.3 and hence the good frame from Corollary 6.4.11 has the λ_0-existence property for domination triples. By Fact 6.5.2, for $\lambda > \lambda_0$, $(\mathbf{K}^*)^{\lambda\text{-sat}}_\lambda = \mathbf{K}^{\lambda\text{-sat}}_\lambda$ has primes. $\square$

Remark 6.5.4. The above proof went back to the notion of domination triples (instead of uniqueness triples) to quote Fact 6.4.3 because it was used in the assumptions of [Vas17a]. We suspect that one can derive a version of Fact 6.5.2(4) with uniqueness triples, which can simplify the proof because we have the existence property of the latter (see Fact 6.4.5). In the original construction, [Vas16a, Section 11] built domination triples and showed that they are also uniqueness triples. [Vas16a, Remark 11.8] claimed that if the nonforking relation has extension (to longer types), then uniqueness triples are domination triples. [Vas17e, Fact A.18] cited [Vas16a, Lemma 11.7] without proof that it is true in general (without assuming extension). We cannot verify those claims so we follow the longer route to obtain the existence property for domination triples.

It would be ideal if Corollary 6.5.3 concluded that $\mathbf{K}_\lambda$, instead of $\mathbf{K}^{\lambda\text{-sat}}_\lambda$, has primes, because we have the following fact:

Fact 6.5.5. [Vas17b, Corollary 10.9] Let $\mathbf{K}^*$ be an LS($\mathbf{K}^*$)-tame AEC with primes and arbitrarily large models. If $\mathbf{K}^*$ is categorical in some $\lambda > \mathrm{LS}(\mathbf{K}^*)$, then it categorical in all $\lambda' \geq \min(\lambda, h(\mathrm{LS}(\mathbf{K}^*)))$.

The main component of the proof came from [Vas17e] (or see [Vas18a] for a written-up version). The idea is that $\mathbf{K}$ to show categoricity $\lambda' > \lambda$, one can pick a bigger categorical cardinal λ'' (guaranteed by [Vas17b, Theorem 9.8]). Suppose $K^*_{\lambda'}$ is not categorical, then one can use primes to transfer non-saturation from λ' to λ''. Since we cannot assume $K^*_{\lambda'}$ is categorical in the first place, we need primes for $K^*_{\lambda'}$ rather than the saturated models of $K^*_{\lambda'}$.

Question 6.5.6. Using the assumptions in Corollary 6.5.3 (or more), is it possible to obtain primeness for sufficiently saturated models? A positive answer will simplify the rest of the proof and remove the assumption of amalgamation over sets to obtain categoricity transfer.

6.6 *AP* OVER SETS AND MULTIDIMENSIONAL DIAGRAMS

In this section, we will add the extra assumption of amalgamation over sets (Definition 6.2.1) to obtain excellence (Definition 6.6.8) over sufficiently saturated models. This allows us to use [SV18b] and show that those models have primes. Then we can invoke Fact 6.5.5 to do categoricity transfer.

In [SV18b, Section 7], given a categorical good λ-frame (for example a good frame over the λ-saturated models), they defined when a frame *reflects down*, is *extendible*, *very good* etc. We do not need the precise definitions but only the following fact:

Fact 6.6.1. Let $\mathbf{K}$ be a LS($\mathbf{K}$)-short AEC with a monster model. Suppose $\mathbf{K}$ is superstable in LS($\mathbf{K}$) and let $\lambda = (2^\kappa)^{+5}$, then there is a $(< \omega)$-extendible categorical good $(\geq \lambda)$-frame over the λ-saturated models.

Proof sketch. Readers familiar with [SV18b] and Vasey's papers can consult [SV18b, Fact 7.21], which applied the same idea on compact AECs. Notice that "LS($\mathbf{K}$)$^{+6}$" there should be κ^{+6}.

Alternatively, we use the frame from Corollary 6.4.11 and verify directly the extra conditions (see [SV18b, Section 7] for relevant definitions):

1. There is a two-dimensional nonforking relation that extends our frame: this is witnessed by NF in Fact 6.4.8(2).

2. The two-dimensional nonforking relation is *good*: namely the frame it extends is a good frame; the nonforking relation has long transitivity and local character. Our NF satisfies these by [Vas16a, Facts 12.2, 12.10].

3. The two-dimensional nonforking relation *reflects down*: by [SV18b, Remark 7.8] it suffices to check that it is good and extends to λ^+. This is true again by Fact 6.4.8(2).

4. The two-dimensional nonforking relation has full model continuity (which makes the relation *very good*). This is true by Fact 6.4.8(2).

5. The frame is $(<\omega)$-extendible: by [SV18b, Fact 7.20], it suffices to show that it is ω-successful and good+, which is true by Fact 6.4.8(1)(2).

$\square$

Given a $(<\omega)$-extendible good frame, [SV18b, Sections 8-11] went on to build multidimensional independence relations from the two-dimensional nonforking relation (which extends the good frame). Basically a multidimensional independence relation takes in models indexed by a general partial order instead of $\mathcal{P}(2)$ as in a two-dimensional nonforking relation (see [SV18b, Definition 8.11] for a precise definition). We state some relevant definitions:

Definition 6.6.2. Let $\mathbf{K}$ be an abstract class and $(I, \leq)$ be a partial order.

1. [SV18b, Definition 8.1] An $(I, \mathbf{K})$-*system* is a sequence $\mathbf{m} = \langle M_u : u \in I \rangle$ such that $u \leq v \Rightarrow M_u \leq_{\mathbf{K}} M_v$. We omit $\mathbf{K}$ if the context is clear. Usually $I = \mathcal{P}(n)$ or $I = \mathcal{P}(n) \backslash \{n\}$ for some $n < \omega$.

2. [SV18b, Definition 8.8] The language of $(I, \mathbf{K})$-systems is $\tau^I := L(\mathbf{K}) \cup \{P_i : i \in I\}$ where each P_i is a unary predicate. The *abstract class of* $(I, \mathbf{K})$-*systems* is $\mathbf{K}^I =$

$(K^I, \leq_{K^I})$ where for each $\mathbf{m} \in K^I$, $\langle (P_i)^M : i \in I \rangle$ forms a disjoint system and the models in K^I are ordered by disjoint extensions (see [SV18b, Definition 8.6]).

Remark 6.6.3. For our purpose, we only need to know that if $\mathbf{m} \in K^{\mathcal{P}(n)}$, then $\langle (P_i)^{\mathbf{m}} : i \in \mathcal{P}(n) \rangle$ is an $\mathcal{P}(n)$-system whose models are at least ordered by $\leq_{\mathbf{K}}$.

We now define a generalized version of amalgamation as well as higher-dimensional uniqueness properties. These were key to establish excellence and to build primes.

Definition 6.6.4. 1. [SV18b, Definition 5.6] Let $\mathbf{K}$ be an abstract class in τ and let ϕ be a first-order quantifier-free formula in τ.

   (a) $M, N \in K$ are *ϕ-equal* if $\phi(M) = \phi(N)$ and the induced partial τ-structures by ϕ on M, N are equal: for each relation and function symbol $R \in \tau$, $R^M \restriction \phi(M) = R^N \restriction \phi(N)$.

   (b) A *ϕ-span* is a triple (M_0, M_1, M_2) such that $M_0 \leq_{\mathbf{K}} M_1$, $M_0 \leq_{\mathbf{K}} M_2$; and M_1, M_2 are ϕ-equal.

   (c) A *ϕ-amalgam* of a ϕ-span (M_0, M_1, M_2) is a triple (N, f_1, f_2) such that $N \in K$, $f_i : M_i \underset{M_0}{\longrightarrow} N$ for $i = 1, 2$ and $f_1 \restriction \phi(M_1) = f_2 \restriction \phi(M_2)$.

   (d) $M \in K$ is a *ϕ-amalgamation base* if every ϕ-span of the form (M, M_1, M_2) has a ϕ-amalgam.

2. [SV18b, Definition 10.14] For $n < \omega$, let ϕ_n be the formula in the language of $(n, \mathbf{K})$-systems such that for any $\mathbf{m} = \langle M_u : u \in \mathcal{P}(n) \rangle$, $a \in |\mathbf{m}|$, we have $\mathbf{m} \vDash \phi_n[a]$ iff $a \in \bigcup_{u \in \mathcal{P}(n) \setminus \{n\}} M_u$.

3. [SV18b, Definition 10.2]

   (a) For $n < \omega$, let $\mathcal{I}_n$ be the class of all partial orders isomorphic to an initial segment of $\mathcal{P}(n)$ and let $\mathcal{I}_{<\omega} = \bigcup_{n<\omega} \mathcal{I}_n$.

   (b) Let $\mathbf{i}$ be a multidimensional independence relation and P be either existence, extension or uniqueness (see [SV18b, Definitions 8.11, 8.16]; we do not need the precise descriptions here). Let $I \subseteq \mathcal{I}_{<\omega}$ be a partial order and $\lambda \geq \mathrm{LS}(\mathbf{K})$.

i. $\mathfrak{i}$ has n-P if I is defined on $\mathcal{P}(n)$-systems and $\mathfrak{i} \restriction \mathcal{I}_n$ has P.

ii. $\mathfrak{i}$ has (λ, n)-P if $\mathfrak{i} \restriction K_\lambda$ has n-P.

We will adapt the proof of item (2) below to transfer uniqueness to higher dimensions. They used WGCH and we will replace it by amalgamation over sets. The construction of $\mathbf{K}^{\text{proper},*}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}$ is very complicated and spans several sections. We only need to know that it is a sub-abstract class of $\mathbf{K}^{\mathcal{P}(n)}$.

Fact 6.6.5. Let $n < \omega$, $\mathfrak{i}$ be a very good (see [SV18b, Definition 11.2]) multidimensional independence relation defined on $\mathcal{P}(n+1)$-systems, $\mathfrak{i}^*$ be its restriction to limit models ordered by universal extensions. Write $\mathbf{K}^* = \mathbf{K}^{\text{proper},*}_{\mathfrak{i},\mathfrak{i}^*,\mathcal{P}(n)}$.

1. [SV18b, Lemma 10.15(5)] Let $(\mathbf{m}^0, \mathbf{m}^1, \mathbf{m}^2)$ be a ϕ_n-span in $\mathbf{K}^*$ and write $\mathbf{m}^i = \langle M^i_u : u \in \mathcal{P}(n) \rangle$ for $i = 0, 1, 2$. Then $(\mathbf{m}^0, \mathbf{m}^1, \mathbf{m}^2)$ has a ϕ_n-amalgam in $\mathbf{K}^*$ iff there exists $N \in K$, $f_i : M^i_{\mathcal{P}(n)} \underset{\mathbf{m}^0}{\longrightarrow} N$ for $i = 1, 2$ such that $f_1 \restriction M^1_u = f_2 \restriction M^2_u$ for $u \in \mathcal{P}(n) \backslash \{n\}$.
2. [SV18b, Lemma 11.16(2)] Let λ, λ^+ be in the domain of $\mathfrak{i}$. Suppose $2^\lambda < 2^{\lambda^+}$ and for $\mu = \lambda, \lambda^+$, $\mathfrak{i}^*$ has (μ, n)-existence and (μ, n)-uniqueness. Then $\mathfrak{i}^*$ also has $(\lambda, n+1)$-uniqueness.

Corollary 6.6.6. *Let $n < \omega$, $\mathfrak{i}$ be a very good multidimensional independence relation defined on $\mathcal{P}(n+1)$-systems, $\mathfrak{i}^*$ be its restriction to limit models ordered by universal extensions. Let λ, λ^+ be in the domain of $\mathfrak{i}$. Suppose $\mathbf{K}$ has amalgamation over sets and for $\mu = \lambda, \lambda^+$, $\mathfrak{i}^*$ has (μ, n)-existence and (μ, n)-uniqueness. Then $\mathfrak{i}^*$ also has $(\lambda, n+1)$-uniqueness.*

Proof. WGCH was used in the proof of Fact 6.6.5(2) to show that there is a ϕ_n-amalgamation base K^*_λ. It suffices to show that the second part of Fact 6.6.5(1) is always true under amalgamation over sets, which will imply that *any* $\mathbf{m}^0$ is a ϕ_n-amalgamation base.

Let $(\mathbf{m}^0, \mathbf{m}^1, \mathbf{m}^2)$ as in Fact 6.6.5(1). We observe the following:

1. The models in $\mathbf{m}^0$ are $\mathbf{K}$-substructures of $M^0_{\mathcal{P}(n)} \leq M^i_{\mathcal{P}(n)}$ for $i = 1, 2$. In particular $\mathbf{m}^0$ is a common subset of the latter two.

2. Since $(\mathbf{m}^0, \mathbf{m}^1, \mathbf{m}^2)$ is a ϕ_n-span, $\mathbf{m}^1$ and $\mathbf{m}^2$ agree on ϕ_n, which means that for $u \in \mathcal{P}(n) \backslash \{n\}$, $M^1_u = M^2_u$.

Now take A be the union of the models in $\mathbf{m}^0$ as well as M^i_u for $u \in \mathcal{P}(n)\backslash\{n\}$, $i = 1, 2$. Then we can invoke amalgamation over sets to obtain $f_i : M^i_{\mathcal{P}(n)} \underset{A}{\rightarrow} N$ for some $N \in K$. By (2), $f_1 \restriction M^1_u = \mathrm{id} = f_2 \restriction M^2_u$. $\square$

Remark 6.6.7. 1. In the above proof, we can relax amalgamation over sets to amalgamation over multiple models. Namely, let $\langle M_u : u \in I\rangle$ be a finite set of models in $\mathbf{K}$. Suppose each M_u is a $\mathbf{K}$-substructure of N_1 and N_2, then there are $N \in K$ and $f_i : N_i \xrightarrow[\bigcup_{u\in I} M_u]{} N$ for $i = 1, 2$. The point in the original proof of Fact 6.6.5(2) is to restrict the class to a nice enough $\mathbf{K}^*$ so that WGCH is sufficient.

2. A natural question is whether we can simply work in a usual monster model to dispense with amalgamation over sets. One difficulty is in the proof of [SV18b, Lemma 12.4] where they claimed to be "similar" to that of [Vas17a, Theorem 3.6]. The latter makes use of saturated models being model-homogeneous. If we generalize this to higher-dimensional systems, we need to justify the notion of saturation over sets or set-homogeneity (the set comes from a system of models). While we are not able to infer how they close this gap, a strong enough monster model (with amalgamation over sets) is sufficient for the proof to proceed.

We will show that the frame in Fact 6.6.1 guarantees that the AEC of sufficiently saturated models is excellent, has primes and hence allows categoricity transfer.

Definition 6.6.8. [SV18b, Definition 13.1]

1. Let $\mathfrak{i}$ be a multidimensional independence relation. $\mathfrak{i}$ is *excellent* if
   (a) $\mathfrak{i}$ is defined on an AEC $\mathbf{K}^*$;
   (b) $\mathfrak{i}$ is very good [SV18b, Definition 11.2];
   (c) $\mathfrak{i}$ has extension and uniqueness [SV18b, Definitions 8.11, 8.16].

2. An AEC $\mathbf{K}^*$ is *excellent* if there is an excellent multidimensional independence relation defined on $\mathbf{K}^*$.

Fact 6.6.9. 1. [SV18b, Theorem 13.6] Let $\mathbf{K}$ be an AEC. Suppose there is a $(<\omega)$-extendible categorical very good λ-frame $\mathfrak{s}$ defined on some $\mathbf{K}_{\mathfrak{s}}$. Let $\mathbf{K}^*$ be the AEC generated by $\mathbf{K}_{\mathfrak{s}}$. If WGCH holds, then $\mathbf{K}^*$ is excellent.

2. [SV18b, Theorem 13.9] Let $\mathbf{K}^*$ be an AEC. If $\mathbf{K}^*$ is excellent, then $\mathbf{K}^{\mathrm{LS}(\mathbf{K}^*)^+\text{-sat}}$ has primes.

Corollary 6.6.10. *Let* $\mathbf{K}$ *be a* $\mathrm{LS}(\mathbf{K})$*-short AEC with amalgamation over sets and arbitrarily large models. Suppose* $\mathbf{K}$ *is superstable in* $\mathrm{LS}(\mathbf{K})$ *and let* $\lambda = (2^{\mathrm{LS}(\mathbf{K})})^{+6}$*, then* $\mathbf{K}^{\lambda\text{-sat}}$ *is excellent and has primes.*

Proof. Let λ^- be the predecessor cardinal of λ. By Fact 6.6.1, there is a $(<\omega)$-extendible categorical very good $(\geq \lambda^-)$-frame $\mathfrak{s}$ defined on $\mathbf{K}^* := K^{\lambda^-\text{-sat}}$ (which is also the AEC generated by $\mathbf{K}_{\lambda^-}^{\lambda^-\text{-sat}}$; see Fact 6.3.3(1)). In the proof of Fact 6.6.9(1), the only usage of WGCH is to show Fact 6.6.5(2), which can be replaced by amalgamation over sets due to Corollary 6.6.6. Hence $\mathbf{K}^*$ is excellent. By Fact 6.6.9(2), $\mathbf{K}^{\lambda\text{-sat}}$ has primes. Restart the whole proof with λ^- replaced by λ to obtain excellence for $\mathbf{K}^{\lambda\text{-sat}}$. □

Remark 6.6.11. Excellence (a nice enough multidimensional independence relation) is an important tool to generalize the main gap theorem to uncountable theories. [SV18b, Section 1.3] already hinted that their result (with non-ZFC assumptions) satisfies (part of) the axioms of [GL05]. Here we obtain a ZFC version of excellence by assuming amalgamation over sets. This is perhaps not a strong assumption because we still do not have a proof of the main gap theorem for uncountable first-order theories. Future work in this direction could be verifying [GL05, Axioms 8-10] on regular types. Relevant results can be found in [She09a, III] but the definitions are different from those in [GL05].

We state three last facts before proving the categoricity transfer in the abstract. The proof of the first fact uses orthogonality calculus while the proof of the second fact uses Shelah's omitting type theorem in [MS90] (see also [Bon20]).

Fact 6.6.12. 1. [Vas17b, Theorem 0.1] Let $\mathbf{K}$ be an AEC and $\mathrm{LS}(\mathbf{K}) \leq \lambda < \theta$. Suppose $\mathbf{K}$ has a (type-full) good $[\lambda, \theta]$-frame and is categorical in λ, θ^+, then it is categorical in all $\mu \in [\lambda, \theta]$.

2. [Vas17b, Theorem 9.8] Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation and arbitrarily large models. If it is categorical in some $\lambda > \mathrm{LS}(\mathbf{K})$, then the categoricity spectrum contains $h(\mathrm{LS}(\mathbf{K}))$ and is unbounded.

3. If an AEC $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-tame and has amalgamation over sets, then it is $\mathrm{LS}(\mathbf{K})$-short.

Proof sketch of (3). Let $\bar{a} = \langle a_i : i < \alpha \rangle$ and $\bar{b} = \langle b_i : i < \alpha \rangle$ such that $\mathrm{gtp}(\bar{a}'/\emptyset) = \mathrm{gtp}(\bar{b}'/\emptyset)$ for small $\bar{a}' \subseteq \bar{a}$ and small $\bar{b}' \subseteq \bar{b}$. It suffices to define $\langle f_j : j < \alpha \rangle$ increasing and continuous such that $f_j(a_i) = b_i$ for $i \leq j < \alpha$. We handle the successor case: Suppose f_j is defined. Extend it to an automorphism of $\mathfrak{C}$. Observe that $\mathrm{gtp}(f_j(a_{j+1})/f_j[\{a_i : i \leq j\}]) = \mathrm{gtp}(f_j(a_{j+1})/\{b_i : i \leq j\})$ so it remains to check that $\mathrm{gtp}(f_j(a_{j+1})/\{b_i : i \leq j\}) = \mathrm{gtp}(b_{j+1}/\{b_i : i \leq j\})$. By tameness, we can replace $\{b_i : i \leq j\}$ by a small subsequence $\bar{c}$. Apply $a' = \langle a_{j+1} \rangle ^\frown f_j^{-1}[\bar{c}]$ and $b' = \langle b_{j+1} \rangle ^\frown \bar{c}$ in the assumption. □

Theorem 6.6.13. *Let $\mathbf{K}$ be an AEC which is $\mathrm{LS}(\mathbf{K})$-tame, has amalgamation over sets and arbitrarily large models. Suppose $\mathbf{K}$ is categorical in some $\xi > \mathrm{LS}(\mathbf{K})$, then it is categorical in all $\xi' \geq \min(\xi, h(\mathrm{LS}(\mathbf{K})))$.*

Proof. By Fact 6.6.12(3), we have $\mathrm{LS}(\mathbf{K})$-shortness. We follow the same idea in [Vas17b, SV18b], where we obtain primes for sufficiently saturated models by the results in this section, then transfer categoricity by Section 6.5 and the above fact. Categoricity also bootstraps the original AEC to be eventually categorical.

1. By Fact 6.4.10, $\mathbf{K}$ is superstable in $\mathrm{LS}(\mathbf{K})$. Let $\lambda = (2^{\mathrm{LS}(\mathbf{K})})^{+6}$. By Corollary 6.6.10, $\mathbf{K}^* := \mathbf{K}^{\lambda\text{-sat}}$ has primes.

2. By Fact 6.6.12(2), we may assume that $\mathbf{K}$ (hence $\mathbf{K}^*$) is categorical in some $\theta > \lambda = \mathrm{LS}(\mathbf{K}^*)$. By Fact 6.5.5, $\mathbf{K}^*$ is categorical in all $\lambda' \geq \min(\theta, h(\mathrm{LS}(\mathbf{K}^*))) = \min(\theta, h(\mathrm{LS}(\mathbf{K})))$. In particular it is categroical in θ^+.

3. Since $\mathbf{K}^*$ is categorical in λ (by saturation) and θ^+, by Fact 6.6.12(1) it is categorical in all $\lambda' \in [\lambda, \theta]$. Combining with (2), it is categorical in all $\lambda' \geq \lambda$.

4. By Fact 6.6.12(2), we may assume $\xi \leq h(\mathrm{LS}(\mathbf{K}))$. We consider two cases:

(a) $\xi \geq \lambda$: the models in $\mathbf{K}_\xi$ are saturated, in particular λ-saturated. Hence $\mathbf{K}_{\geq\xi} = \mathbf{K}^*_{\geq\xi}$ is totally categorical as desired.

(b) $\xi < \lambda$: by Fact 6.3.3(2), there is a good $(\geq \xi)$-frame over $\mathbf{K}^{**} := \mathbf{K}^{\xi\text{-sat}}$. $\mathbf{K}^{**}$ is categorical in ξ by saturation. By substituting ξ by $h(\mathrm{LS}(\mathbf{K}))$ in (a), we have $\mathbf{K}$, and hence $\mathbf{K}^{**}$ is categorical in all $\xi' \geq h(\mathrm{LS}(\mathbf{K}))$. In particular $\mathbf{K}^{**}$ is categorical in $h(\mathrm{LS}(\mathbf{K}))^+$. By the same argument as (3), $\mathbf{K}^{**}$ is categorical in all $\xi' \geq \xi$. Now we end up in the scenario of (a) with the new "λ" being ξ so $\mathbf{K}_{\geq\xi} = \mathbf{K}^{**}_{\geq\xi}$ which is totally categorical.

□

We apply our theorem to prove known results:

Example 6.6.14. 1. Complete first-order theories: by compactness the models of a complete first-order theory T satisfy amalgamation over sets, joint-embedding and no maximal models. It has Löwenheim-Skolem number $|T|$ and is $(< \aleph_0)$-short. Therefore, we can use Theorem 6.6.13 transfer categoricity in any $\mu > |T|$ to all $\mu' \geq \mu$. However, we cannot conclude categoricity down to all $\mu' > |T|$ as in [Mor65a, She74] which used syntactic proofs.

2. Homogeneous diagrams with a monster model: let T be a first-order theory and D be a subset of syntactic T-types over the empty set. Let $\mathbf{K}_D$ be the class of models of T such that the only types over the empty set they realize are from D, where the models are ordered by elementary substructures. Assuming the existence of a monster model (see the precise statements in [GL02, Hypothesis 2.5] or [Vas18b, Definition 4.2]), we have the same properties as those in (1). Hence we can transfer categoricity in any $\mu > |T|$ to all $\mu' \geq \mu$. [Vas18b, Theorem 4.22] proved the same result using Fact 6.5.5 but also syntactic results from [She71]. Our approach is purely semantic.

3. Classes with intersections, assuming tameness, amalgamation and arbitrarily large models: we need to justify amalgamation over sets. Work in a monster model $\mathfrak{C}$, let $M, N \leq_{\mathbf{K}} \mathfrak{C}$ and $A \subseteq |M| \cap |N|$. By [Vas17e, Proposition 2.14(4)], the closure of A is

the same among M, N and $\mathfrak{C}$. Hence amalgamation over A amounts to amalgamation over the closure of A, which is a model. Although this approach is more convoluted than [Vas17e, Remark 5.3] (classes with intersections immediately have primes), we can show the extra property of excellence which [Vas17e] could not. In the special case of universal classes, tameness is for free [Vas17e, Theorem 3.7]. Hence we can conclude that universal classes with amalgamation and arbitrarily large models can transfer categoricity upwards, recovering [Vas17b, Corollary 10.11] ([Vas17f] removed the assumption of amalgamation and arbitrarily large models, but at the expense of a high categoricity threshold).

Our theorem does not exclude the possibility that the first categoricity cardinal to be arbitrarily close to $h(\mathrm{LS}(\mathbf{K}))$. The following example shows such categoricity behavior but unfortunately it fails amalgamation and joint-embedding.

Example 6.6.15. Let $\lambda \geq \aleph_0$ and $\lambda \leq \alpha < (2^\lambda)^+$. By the construction of $\mathbf{K}_0$ and $\mathbf{K}_1$ in Proposition 3.4.1, there is $\mathbf{K}^\alpha$, an AEC that encodes the cumulative hierarchy $V_\alpha(\alpha)$. $\mathbf{K}^\alpha$ is ordered by $\mathrm{L}(\mathbf{K}^\alpha)$-substructures, $\mathrm{LS}(\mathbf{K}^\alpha) = \lambda$ and the models have sizes up to $\beth_\alpha(\lambda)$. Also, $\mathbf{K}^\alpha$ has joint-embedding but not amalgamation. Taking the disjoint union of $\mathbf{K}^\alpha$ with a totally categorical AEC, we obtain an AEC $\mathbf{K}$ whose first categoricity cardinal is $\beth_\alpha(\lambda)$, but it fails amalgamation and joint-embedding.

Remark 6.6.16. [Vas19, Example 9.10(2)] claimed that by encoding the cumulative hierarchy, one could get such an example with amalgamation (which would provide a complete list of examples for his categoricity spectra). However, he did not provide the exact encoding or the ordering (which amalgamation is sensitive to), so we cannot verify his claim. A similar problem occurs in [Vas19, Example 9.10(3)] when he encoded an AEC $\mathbf{K}$ categorical only in $[\mathrm{LS}(\mathbf{K})^{+m}, \mathrm{LS}(\mathbf{K})^{+n}]$ where $m, n < \omega$. If we use $\mathrm{L}(\mathbf{K})$-substructures as the ordering, amalgamation again fails because the functions (see F in [Vas19, Fact 9.8]) might be computed differently.

Question 6.6.17. Let $\kappa \geq \aleph_0$. For $\mu < h(\kappa)$, is there an AEC $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \kappa$ which is κ-short, has amalgamation over sets and arbitrarily large models such that the first categoricity cardinal (exists and) is greater than μ? What if we replace amalgamation over sets by the usual amalgamation property (see also table(10)(11))?

CHAPTER 7

ADDITIONAL RESULTS ON CATEGORICITY TRANSFER

Using topos theory, Espíndola proved:

Fact 7.0.1. [Esp22, Theorems 8.3, 9.1, 10.1] Let $\mathbf{K}$ be an AEC.

1. Suppose $\mathbf{K}$ has amalgamation everywhere and is categorical in some $\mu > \kappa \geq \mathrm{LS}(\mathbf{K})$. Then it is categorical in all cardinals between κ and μ.

2. Suppose $\mu > 2^\kappa$ and $\mathbf{K}$ is categorical in both κ and μ. Then $\mathbf{K}_{\geq\mu}$ has amalgamation.

In [Esp22, Corollary 9.6], he obtained the threshold of categoricity transfer to be the maximum of the Hanf numbers for categoricity and for non-categoricity. Amalgamation and tameness were not assumed. Here we give a variation on his result by assuming amalgamation and tameness.

Recall the following result by Vasey (it was cited in Chapter 6 but in a weaker form):

Fact 7.0.2. [Vas17b, Theorem 9.8] Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation and arbitrarily large models. If it is categorical in some $\lambda > \mathrm{LS}(\mathbf{K})$, then it is categorical in all $\beth_\delta$ where δ is divisible by $(2^{\mathrm{LS}(\mathbf{K})})^+$.

Combining Espíndola's and Vasey's results, we obtain:

Corollary 7.0.3. *Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation and arbitrarily large models. Suppose $\mathbf{K}$ is categorical in some $\mu > \mathrm{LS}(\mathbf{K})$, then it is categorical in all $\mu' \geq \min(\mu, h(\mathrm{LS}(\mathbf{K})))$.*

Proof. By Fact 7.0.2, $\mathbf{K}$ is categorical in all $\beth_\delta$ where δ is divisible by $(2^{\mathrm{LS}(\mathbf{K})})^+$, including $h(\mathrm{LS}(\mathbf{K}))$. Apply Fact 7.0.1 to any two categoricity cardinals. We obtain categoricity for cardinals greater than or equal to $\min(\mu, h(\mathrm{LS}(\mathbf{K})))$. $\square$

This corollary is stronger than Theorem 6.6.13 because it assumes amalgamation over models but not amalgamation over sets. The price is to borrow powerful results from topos theory and tell little about the good frames and primes of the AECs.

We look at an application of the above corollary. Mazari-Armida [MA22] used algebraic techniques to obtain categoricity transfer for several classes of modules:

Fact 7.0.4. Let R be an associative ring with unity.

1. [MA22, Theorem 3.4] Let $\mathbf{K}$ be the class of locally pure-injective modules ordered by pure submodule. The following are equivalent:

   (a) $\mathbf{K}$ is categorical in all $\lambda > |R| + \aleph_0$;

   (b) $\mathbf{K}$ is categorical in some $\lambda > 2^{|R|+\aleph_0}$.

2. [MA22, Theorem 3.11] Let $\mathbf{K}$ be the class of absolutely pure modules ordered by pure submodule. The following are equivalent:

   (a) $\mathbf{K}$ is categorical in all $\lambda > |R| + \aleph_0$;

   (b) $\mathbf{K}$ is categorical in some $\lambda > |R| + \aleph_0$.

3. [MA22, Theorem 3.19] Let $\mathbf{K}$ be the class of locally injective modules ordered by pure submodule. The following are equivalent:

   (a) $\mathbf{K}$ is categorical in all $\lambda > |R| + \aleph_0$;

   (b) $\mathbf{K}$ is categorical in some $\lambda > 2^{|R|+\aleph_0}$.

He asked [MA22, Question 4.3] whether the above results could be achieved by model-theoretic techniques only. We give a positive answer to the upward transfers if we are also allowed to use Fact 7.0.1 which is topos-theoretic.

Corollary 7.0.5. *Let $\mathbf{K}$ be one of the three classes in Fact 7.0.4. If $\mathbf{K}$ is categorical in some $\lambda > 2^{|R|+\aleph_0}$, then it is categorical in all $\lambda' \geq \lambda$.*

Proof. Using [MA22, Facts 3.3, 3.10] and the proof of [MA22, Theorem 3.19], each of the class $\mathbf{K}$ in Fact 7.0.4 is an $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation and arbitrarily large models. Also, $|R| + \aleph_0 \leq \mathrm{LS}(\mathbf{K}) \leq 2^{|R|+\aleph_0}$. Apply Corollary 7.0.3. $\square$

References

[Bal09] John T. Baldwin. *Categoricity.* University lecture series; v. 50. American Mathematical Society, Providence, R.I, 2009.

[Bal18] John T. Baldwin. Categoricity: Errata, October 2018. `http://www.math.uic.edu/~jbaldwin/pub/bookerrata.pdf`.

[BB16] Will Boney and John Baldwin. Hanf numbers and presentation theorems in AECs. 2016. Preprint: `https://arxiv.org/abs/1511.02935`.

[BG17] Will Boney and Rami Grossberg. Forking in short and tame abstract elementary classes. *Annals of Pure and Applied Logic*, 168(8):1517–1551, 2017.

[BGKV16] Will Boney, Rami Grossberg, Alexei Kolesnikov, and Sebastien Vasey. Canonical forking in AECs. *Annals of Pure and Applied Logic*, 167(7):590–613, 2016.

[BGL$^+$16] Will Boney, Rami Grossberg, Michael Lieberman, Jiří Rosický, and Sebastien Vasey. μ-abstract elementary classes and other generalizations. *Journal of Pure and Applied Algebra*, 220(9):3048–3066, 2016.

[BGVV17] Will Boney, Rami Grossberg, Monica VanDieren, and Sebastien Vasey. Superstability from categoricity in abstract elementary classes. *Annals of Pure and Applied Logic*, 168(7):1383–1395, 2017.

[BKV06] John Baldwin, David Kueker, and Monica VanDieren. Upward stability transfer for tame abstract elementary classes. *Notre Dame Journal of Formal Logic*, 47(2):291–298, 2006.

[Bon14a] Will Boney. Tameness and extending frames. *Journal of Mathematical Logic*, 14(2):1450007–, 2014.

[Bon14b] Will Boney. Tameness from large cardinal axioms. *The Journal of Symbolic Logic*, 79(4):1092–1119, 2014.

[Bon17] Will Boney. Computing the number of types of infinite length. *Notre Dame Journal of Formal Logic*, 58(1), 2017.

[Bon20] Will Boney. Shelah's omitting types theorem in excruciating detail,. `https://cpb-us-e1.wpmucdn.com/wp.txstate.edu/dist/0/1970/`

`files/2020/07/SOTTproofv2.pdf`, July 2020.

[BV15a] Will Boney and Monica M VanDieren. Limit models in strictly stable abstract elementary classes. 2015. Preprint: `https://arxiv.org/abs/1508.04717`.

[BV15b] Will Boney and Sebastien Vasey. A survey on tame abstract elementary classes. 2015.

[BV17a] Will Boney and Sebastien Vasey. Chains of saturated models in AECs. *Archive for Mathematical Logic*, 56(3-4):187–213, 2017.

[BV17b] Will Boney and Sebastien Vasey. Tameness and frames revisited. *The Journal of Symbolic Logic*, 82(3):995–1021, 2017.

[Cha68] Chen Chung Chang. Some remarks on the model theory of infinitary languages. In Jon Barwise, editor, *The Syntax and Semantics of Infinitary Languages*, pages 36–63, Berlin, Heidelberg, 1968. Springer.

[Che21] Hanif Cheung. Notions of amalgamation for AECs and categoricity, 2021. Preprint: `https://arxiv.org/abs/2104.13867`.

[Dic75] M. A. Dickmann. *Large Infinitary Languages*, volume 83 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1975.

[Dru13] Fred Drueck. *Limit Models, Superlimit Models, and Two Cardinal Problems in Abstract Elementary Classes*. PhD thesis, 2013.

[Esp22] Christian Espíndola. A proof of Shelah's eventual categoricity conjecture and an extension to accessible categories with directed colimits. 2022.

[GL02] Rami Grossberg and Olivier Lessmann. Shelah's stability spectrum and homogeneity spectrum in finite diagrams. *Archive for Mathematical Logic*, 41(1):1–31, 2002.

[GL05] Rami Grossberg and Olivier Lessmann. Abstract decomposition theorem and applications. *Contemporary Mathematics*, Vol 380, AMS:73–108, 2005.

[GMA21] Rami Grossberg and Marcos Mazari-Armida. Simple-like independence relations in abstract elementary classes. *Annals of Pure and Applied Logic*, 172(7):102971–, 2021.

[Gro02] Rami Grossberg. Classification theory for abstract elementary classes. *Logic* *and Algebra,* ed. Yi Zhang*, Contemporary Mathematics*, Vol 302, AMS:165–204, 2002.

[Gro21] Rami Grossberg. *A Course in Model Theory I: Introduction*. 2021. A book in preparation.

[GS86] Rami. Grossberg and Saharon Shelah. On the number of nonisomorphic models of an infinitary theory which has the infinitary order property. I. *J. Symbolic Logic*, 51(2):302–322, 1986.

[GS98] Rami Grossberg and Saharon Shelah. On Hanf numbers of the infinitary order property. 1998. Preprint.

[GV06a] Rami Grossberg and Monica VanDieren. Categoricity from one successor cardinal in tame abstract elementary classes. *Journal of Mathematical Logic*, 6(2):181–201, 2006.

[GV06b] Rami Grossberg and Monica VanDieren. Galois-stability for tame abstract elementary classes. *Journal of Mathematical Logic*, 6(1):25–48, 2006.

[GV06c] Rami Grossberg and Monica VanDieren. Shelah's categoricity conjecture from a successor for tame abstract elementary classes. *The Journal of Symbolic Logic*, 70(2):165–660, 2006.

[GV17] Rami Grossberg and Sebastien Vasey. Equivalent definitions of superstability in tame abstract elementary classes. *The Journal of Symbolic Logic*, 82(4):1387–1408, 2017.

[GVV16] Rami Grossberg, Monica VanDieren, and Andrés Villaveces. Uniqueness of limit models in classes with amalgamation. *Mathematical Logic Quarterly*, 62(4-5):367–382, 2016.

[Hen19] Simon Henry. An abstract elementary class nonaxiomatizable in $L_{\infty,\kappa}$. *The Journal of Symbolic Logic*, 84(3):1240–1251, 2019.

[HK11] Tapani Hyttinen and Meeri Kesälä. Categoricity transfer in simple finitary abstract elementary classes. *The Journal of Symbolic Logic*, 67(3):231–660, 2011.

[HL02] Tapani Hyttinen and Olivier Lessmann. A rank for the class of elementary submodels of a superstable homogeneous model. *The Journal of Symbolic Logic*, 66(4):77–1482, 2002.

[HS00] Tapani Hyttinen and Saharon Shelah. Strong splitting in stable homogeneous models. *Annals of Pure and Applied Logic*, 103(1):201–228, 2000.

[Jar16] Adi Jarden. Tameness, uniqueness triples and amalgamation. *Annals of Pure and Applied Logic*, 167(2):155–188, 2016.

[Jec03] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer, Berlin, Germany, 3rd edition, 2003.

[Joh10] Gregory Johnson. Abstract elementary classes with Löwenheim-Skolem number cofinal with ω. *Notre Dame Journal of Formal Logic*, 51(3):361–371, 2010.

[JS13] Adi Jarden and Saharon Shelah. Non-forking frames in abstract elementary classes. *Ann. Pure Appl. Logic*, 164(3):135–191, 2013. arXiv: 0901.0852.

[Kol85] Ph. G. Kolaitis. Game quantification. In J. Barwise and S. Feferman, editors, *Model-Theoretic Logics*, New York, 1985. Springer-Verlag.

[Kue77] David Kueker. Countable approximations and Löwenheim-Skolem theorems. *Annals of Mathematical Logic*, 11(1):57–103, 1977.

[Kue08] David Kueker. Abstract elementary classes and infinitary logics. *Annals of Pure and Applied Logic*, 156(2):274–286, 2008.

[Les00] Olivier Lessmann. Ranks and pregeometries in finite diagrams. *Annals of Pure and Applied Logic*, 106(1):49–83, 2000.

[MA19] Marcos Mazari-Armida. Non-forking w-good frames. *Archive for Mathematical Logic*, 59(1-2):31–56, 2019.

[MA20] Marcos Mazari-Armida. Algebraic description of limit models in classes of abelian groups. *Annals of Pure and Applied Logic*, 171(1):102723–, 2020.

[MA22] Marcos Mazari-Armida. Characterizing categoricity in several classes of modules, 2022. Preprint: `https://arxiv.org/abs/2202.07900`.

[Mar02] David Marker. *Model Theory : An Introduction*. Graduate Texts in Mathematics, 217. Springer New York, New York, NY, 1st edition, 2002.

[MAV18] Marcos Mazari-Armida and Sebastien Vasey. Universal classes near $\aleph_1$. *The Journal of Symbolic Logic*, 83(4):1633–1643, 2018.

[Mor65a] Michael Morley. Categoricity in power. *Transactions of the American Mathe-*

matical Society, 114(2):514–538, 1965.

[Mor65b] Michael Morley. Omitting classes of elements. In Addison, Henkin, and Tarski, editors, *The Theory of Models*, pages 265–273. North-Holland, Amsterdam, 1965.

[MS90] Michael Makkai and Saharon Shelah. Categoricity of theories in $L_{\kappa\omega}$, with κ a compact cardinal. *Ann. Pure Appl. Logic*, 47(1):41–97, 1990.

[She71] Saharon Shelah. Finite diagrams stable in power. *Ann. Math. Logic*, 2(1):69–118, 1970/1971.

[She72] Saharon Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific J. Math.*, 41:247–261, 1972.

[She74] Saharon Shelah. Categoricity of uncountable theories. In *Proceedings of the Tarski Symposium*, volume XXV of *Proc. Sympos. Pure Math.*, pages 187–203. Amer. Math. Soc., Providence, R.I., 1974.

[She83a] Saharon Shelah. Classification theory for nonelementary classes. I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A. *Israel J. Math.*, 46(3):212–240, 1983.

[She83b] Saharon Shelah. Classification theory for nonelementary classes. I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B. *Israel J. Math.*, 46(4):241–273, 1983.

[She87] Saharon Shelah. Classification of nonelementary classes. II. Abstract elementary classes. In *Classification Theory (Chicago, IL, 1985)*, volume 1292 of *Lecture Notes in Math.*, pages 419–497. Springer, Berlin, 1987.

[She90] Saharon Shelah. *Classification Theory and the Number of Non-isomorphic Models*. Studies in logic and the foundations of mathematics; v. 92. North-Holland, Amsterdam, rev. edition, 1990.

[She99] Saharon Shelah. Categoricity for abstract classes with amalgamation. *Ann. Pure Appl. Logic*, 98(1-3):261–294, 1999.

[She00] Saharon Shelah. On what I do not understand (and have something to say), model theory. *Math. Japon.*, 51(2):329–377, 2000.

[She01] Saharon Shelah. Categoricity of an abstract elementary class in two successive cardinals. *Israel J. Math.*, 126:29–128, 2001.

[She09a] Saharon Shelah. *Classification Theory for Abstract Elementary Classes*, volume 18 of *Studies in Logic (London)*. College Publications, London, 2009.

[She09b] Saharon Shelah. *Classification Theory for Abstract Elementary Classes. Vol. 2*, volume 20 of *Studies in Logic (London)*. College Publications, London, 2009.

[SV99] Saharon Shelah and Andrés Villaveces. Toward categoricity for classes with no maximal models. *Ann. Pure Appl. Logic*, 97(1-3):1–25, 1999.

[SV18a] Saharon Shelah and Sebastien Vasey. Abstract elementary classes stable in $\aleph_0$. *Annals of Pure and Applied Logic*, 169(7):565–587, 2018.

[SV18b] Saharon Shelah and Sebastien Vasey. Categoricity and multidimensional diagrams. 2018.

[SV21] Saharon Shelah and Andrés Villaveces. Infinitary logics and abstract elementary class. *Proceedings of the American Mathematical Society*, pages 1–, 2021.

[Van06] Monica VanDieren. Categoricity in abstract elementary classes with no maximal models. *Annals of Pure and Applied Logic*, 141(1):108–147, 2006.

[Van16a] Monica VanDieren. Superstability and symmetry. *Annals of Pure and Applied Logic*, 167(12):1171–1183, 2016.

[Van16b] Monica VanDieren. Symmetry and the union of saturated models in superstable abstract elementary classes. *Annals of Pure and Applied Logic*, 167(4):395–407, 2016.

[Vas16a] Sebastien Vasey. Building independence relations in abstract elementary classes. *Annals of Pure and Applied Logic*, 167(11):1029–1092, 2016.

[Vas16b] Sebastien Vasey. Forking and superstability in tame AECs. *The Journal of Symbolic Logic*, 81(1):357–383, 2016.

[Vas16c] Sebastien Vasey. Infinitary stability theory. *Archive for Mathematical Logic*, 55(3):567–592, 2016.

[Vas17a] Sebastien Vasey. Building prime models in fully good abstract elementary classes. *Mathematical Logic Quarterly*, 63(3-4):193–201, 2017.

[Vas17b] Sebastien Vasey. Downward categoricity from a successor inside a good frame. *Annals of Pure and Applied Logic*, 168(3):651–692, 2017.

[Vas17c] Sebastien Vasey. On the uniqueness property of forking in abstract elementary classes. *Mathematical Logic Quarterly*, 63(6):598–604, 2017.

[Vas17d] Sebastien Vasey. Saturation and solvability in abstract elementary classes with amalgamation. *Archive for Mathematical Logic*, 56(5):671–690, 2017.

[Vas17e] Sebastien Vasey. Shelah's eventual categoricity conjecture in universal classes: Part i. *Annals of Pure and Applied Logic*, 168(9):1609–1642, 2017.

[Vas17f] Sebastien Vasey. Shelah's eventual categoricity conjecture in universal classes: part ii. *Selecta Mathematica (Basel, Switzerland)*, 23(2):1469–1506, 2017.

[Vas18a] Sebastien Vasey. Math 269X - model theory for abstract elementary classes, spring 2018 lecture notes. `https://svasey.github.io/academic-homepage-may-2020/aec-spring-2018/aec-lecture-notes_04_26_2018.pdf`, April 2018.

[Vas18b] Sebastien Vasey. Shelah's eventual categoricity conjecture in tame AECs with primes. *Mathematical Logic Quarterly*, 64(1-2):25–36, 2018.

[Vas18c] Sebastien Vasey. Toward a stability theory of tame abstract elementary classes. *Journal of Mathematical Logic*, 18(2):1850009–, 2018.

[Vas19] Sebastien Vasey. The categoricity spectrum of large abstract elementary classes. *Selecta Mathematica (Basel, Switzerland)*, 25(5):1–51, 2019.

[Vas20] Sebastien Vasey. On categoricity in successive cardinals. *The Journal of Symbolic Logic*, pages 1–19, 2020.

[VV17] Monica M VanDieren and Sebastien Vasey. Symmetry in abstract elementary classes with amalgamation. *Archive for Mathematical Logic*, 56(3-4):423–452, 2017.

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