Research summary

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Model theory is a branch of mathematical logic that studies classes of models, for example models of first-order theories. Using set theory, two main pieces of information can be obtained: the number of non-isomorphic models and the number of types of elements. To obtain results beyond the first-order context, we follow Shelah's approach in studying abstract elementary classes (AECs). Rather than working in a specific logic, it consists of a list of axioms that describe the properties of the models. The techniques are usually semantic and involve diagram chasing.

Given the hundreds of pages of literature in recent years, we extended known results in two directions: adding extra assumptions to better approximate the results in first-order theories; relaxing hypotheses to generalize results to a wider context. In the following, we overview four main themes of AECs and our achievements.

Stability spectrum: Vasey assumed the amalgamation property to bound the first stability cardinal below the Hanf number. There he blackboxed Shelah's results, which we gave details. We constructed AECs with first stability cardinals up to the Hanf number, but they do not satisfy amalgamation. It is still open whether amalgamation can allow such examples. Boney conjectured that the joint-embedding property was necessary in a stability result. We gave examples to show that it actually depends on non-ZFC axioms.

Axiomatization: Shelah showed that each AEC is a PC-class with complexity parameters λ and 2^{λ}, where λ is the Löwenheim-Skolem number. The original proof did not tell whether 2^{λ} could be lowered. We provided a more general proof and replaced both parameters by $I_2(\lambda,K)$, which is between λ and 2^{λ}. Under extra assumptions, we could deduce $I_2(\lambda,K)=\lambda$. As an application we lowered the threshold cardinal of the existence of models (assuming successive categoricities). Using the same strategy, we axiomatized AECs in an infinitary logic with game quantification. Our proofs generalize to μ-AECs.

Superstability: in first-order theories, superstability has various equivalent criteria. Grossberg and Vasey showed that under nice assumptions, there is also a list of equivalent criteria of superstable AECs. We first lowered the high cardinal threshold and the high cardinal jump between different criteria. Then we added two new criteria to the list: boundedness of U-rank and a weaker form of solvability. Next we provided another list of equivalent criteria in the strictly stable context, assuming continuity of non-splitting.

Categoricity spectrum: Vasey showed that tame AECs with primes and amalgamation can transfer categoricity upwards. Later Vasey and Shelah used multidimensional diagrams to transfer categoricity at the cost of non-ZFC axioms. We adapted their results to prove in ZFC that tame AECs with a stronger version of amalgamation can transfer categoricity. Our result recovers the classical categoricity theorem by Morley and Shelah, but using purely semantic techniques.