RELAXED ENERGIES FOR $H^{1/2}$–MAPS WITH VALUES INTO THE CIRCLE AND MEASURABLE WEIGHTS

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Abstract. We consider, for maps $f \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{S}^1)$, an energy $\mathcal{E}(f)$ related to a seminorm equivalent to the standard one. This seminorm is associated to a measurable matrix field in the half space. Under structure assumptions on it, we show that the infimum of $\mathcal{E}$ over a class of maps two prescribed singularities induces a natural geodesic distance on the plane. In case of a continuous matrix field, we determine the asymptotic behavior of minimizing sequences. We prove that, for such minimizing sequences, the energy concentrates near a geodesic curve on the plane. We describe this concentration in terms of bubbling-off of circles along this curve. Then we explicitly compute the relaxation with respect to the weak $\dot{H}^{1/2}$-convergence of the functional $f \mapsto \mathcal{E}(f)$ if f is smooth and $+\infty$ otherwise. The formula involves the length of a minimal connection between the singularities of f computed in terms of the distance previously obtained.

Keywords: fractional Sobolev space; trace space; topological singularity; minimal connection; Cartesian current; relaxed energy.

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1. Introduction

In the recent years, several papers were devoted to the study of the fractional Sobolev space $H^{1/2}$ with values into the unit circle $\mathbb{S}^1$, in particular in the framework of the Ginzburg-Landau model (see [7], [8], [9], [14], [22], [25], [27], [33] and [34]), but also into more general target manifolds (see [23], [24]). In this paper, we are interested in one of the simplest case of such spaces, namely in

$$X := \left\{ f \in L^2_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2) \ ; \ |f| = 1 \text{ a.e. and } |f|_{1/2} < +\infty \right\}, \tag{1.1}$$

where $|\cdot|_{1/2}$ denotes the standard (Gagliardo) $H^{1/2}$–seminorm

$$|f|_{1/2} = \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|f(x) - f(y)|^2}{|x-y|^3} \, dxdy \right)^{1/2} \tag{1.2}$$

which makes X modulo constants a complete metric space. In this way, X naturally appears as a closed subset of the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$ (see Section 2 for the definitions and the basic properties of the homogeneous Sobolev spaces we are interested in). As it is well known, up to a multiplicative constant,

$$|f|_{1/2} = \|\nabla u_f\|_{L^2(\mathbb{R}^3_+)}\,, \tag{1.3}$$

where u_f is the (unique, finite energy) harmonic extension of f to the half space $\mathbb{R}^3_+ := \mathbb{R}^2 \times (0,+\infty)$. Alternatively, $|f|^2_{1/2} = \int_{\mathbb{R}^2} |\xi||\hat{f}(\xi)|^2 d\xi$, where $\hat{f}$ denotes the Fourier transform of f. Throughout the paper, we might identify $\mathbb{R}^2$ with $\partial\mathbb{R}^3_+ = \mathbb{R}^2 \times \{0\}$ and for $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+$, $(x_1, x_2, 0) = \hat{x} \in \mathbb{R}^2$.

On the space X, we will consider a family of seminorms including (1.2) and equivalent to it. These seminorms arise naturally from Riemannian metrics on $\mathbb{R}^3_+$. The main goal of this paper is to study variational problems for energy functionals corresponding to Dirichlet integrals as in (1.3) with respect to Riemannian metrics with measurable coefficients. The lack of regularity will force us to introduce suitable length structures in the sense of [26] and corresponding geodesic distances related to Finsler metrics that will be the proper substitutes for the Euclidean metric. For $\mathbb{S}^1$-valued maps, the present study recovers some recent results proved in [9] and [22,25] in the setting of Cartesian currents. In contrast with the aforementioned papers, our analysis is performed in the entire space which motivates the use of homogeneous Sobolev spaces. In the case of the Euclidean metric, most of the present results could be derived from the ones in [22,25] once adapted to the unbounded domain situation.

In order to describe the variational problems in details, we recall some properties of maps in X related to the nontrivial topology of the target. These properties are well known in the bounded domain case (see [9], [34] and see also [25] for a different approach) and their proofs can be found in the next sections. In particular, the strong density of the subspace of smooth maps $X \cap C^\infty(\mathbb{R}^2)$ is known to fail (see e.g. [14]) and the sequential weak density to hold (see [34]). However strong density holds for maps with finitely many singualrities (see [34] and Section 2). For any $f \in X$, a characterization of the *topological singularities*, i.e., the topologically relevant part of the singular set of f, can be obtained in terms of a distribution $T(f)$ as in [9], [27] and [34] (see [22] and [25] for an alternative approach in terms of currents and [10], [27] and [23] for higher dimensional extensions). Roughly speaking, this distribution measures how much f fails to preserve closed forms under pull-back.

For $f = f(x_1, x_2) \in X$ and $\varphi = \varphi(x_1, x_2) \in \mathrm{Lip}(\mathbb{R}^2;\mathbb{R})$, we consider $u = u(x_1, x_2, x_3) \in \dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$ and $\Phi = \Phi(x_1, x_2, x_3) \in \mathrm{Lip}(\mathbb{R}^3_+;\mathbb{R})$ with $u_{|\mathbb{R}^2} = f$ and $\Phi_{|\mathbb{R}^2} = \varphi$. Setting

$$H(u) = 2\big(\partial_2 u \wedge \partial_3 u, \partial_3 u \wedge \partial_1 u, \partial_1 u \wedge \partial_2 u\big)\,,$$

the distribution $T(f)$ is defined through its action on φ by

$$< T(f), \varphi >= \int_{\mathbb{R}^3_+} H(u) \cdot \nabla\Phi \, dx\,. \tag{1.4}$$

It is not hard to check (see [9] for details) that such a definition makes sense, i.e., it is independent of the extensions u and Φ, and $T(f) \in \big(\mathrm{Lip}(\mathbb{R}^2)\big)'$. As shown in [34] (see also [9] and Section 2), $T(f) = 0$ if and only if f can be approximated strongly by smooth functions. For maps which are slightly more regular, namely if $f \in X \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$, an integration by parts in (1.4) yields

$$< T(f), \varphi >= -\int_{\mathbb{R}^2} (f \wedge \partial_1 f)\partial_2\varphi - (f \wedge \partial_2 f)\partial_1\varphi \quad \forall \varphi \in C^1_0(\mathbb{R}^2;\mathbb{R})\,, \tag{1.5}$$

and the same formula holds for $f \in \dot{H}^{1/2}$ whenever $\varphi \in C^2_0(\mathbb{R}^2;\mathbb{R})$, interpreting (1.5) in terms of $\dot{H}^{1/2} - \dot{H}^{-1/2}$ duality (see Remark 2.2 in Section 2). In addition, if f is smooth except at a finite number of points $\{a_j\}_{j=1}^k$ and u is taken to be smooth in the open half space, then

$$H(u)\cdot\nabla\Phi\,dx = u^{\#}d\omega\wedge d\Phi = d(u^{\#}\omega\wedge d\Phi)\,,$$

where $\omega(y_1,y_2) = y_1dy_2 - y_2dy_1$ induces the standard volume form on $\mathbb{S}^1 = \{y_1^2+y_2^2=1\}$. In this way,

$$< T(f),\varphi >= \int_{\mathbb{R}^3_+} u^{\#}d\omega\wedge d\Phi = -\int_{\mathbb{R}^2} f^{\#}\omega\wedge d\varphi = -2\pi\sum_{j=1}^k d_j\,\varphi(a_j)\,,$$

where the integer $d_j = \deg(f,a_j)$ is the topological degree of f around a_j and $\sum_{j=1}^k d_j = 0$ because $f\in\dot H^{1/2}$ (see [11], [17] and Lemma 2.2 in Section 2). The same finite sum representation holds if $T(f)$ is a finite measure (see [9]), this result being the $H^{1/2}$–counterpart of the same statement for $W^{1,1}$–maps proved in [22],[29] and [16].

As a consequence of the strong $\dot H^{1/2}$–continuity of $T(f)$, we easily see that, no matter which seminorm $\langle\cdot\rangle$ equivalent to the standard one is used, given $f_0\in X$ such that $T_0 := T(f_0)\neq 0$, we have

$$m_{\langle\cdot\rangle}(T_0) := \mathrm{Inf}\left\{\langle f\rangle^2\,;\ f\in X\,,\ T(f)=T_0\right\} > 0\,. \tag{1.6}$$

A slightly different quantity will play the decisive role in the sequel. It can be introduced as follows:

$$\bar m_{\langle\cdot\rangle}(T_0) := \mathrm{Inf}\left\{\liminf_{n\to+\infty}\langle f_n\rangle^2\,;\ \{f_n\}_{n\in\mathbb{N}}\subset X\,,\ T(f_n)=T_0\,,\right. \tag{1.7}$$
$$\left. f_n\rightharpoonup\alpha \text{ weakly in } \dot H^{1/2} \text{ for some constant } \alpha\in\mathbb{S}^1\right\}.$$

It is a nontrivial fact that $\bar m_{\langle\cdot\rangle}(T_0)$ is well defined. This issue will be discussed in Sections 2 and 7 (see also [9]). In any case, we obviously have $\bar m_{\langle\cdot\rangle}(T_0)\geq m_{\langle\cdot\rangle}(T_0)$ since the sequences converging weakly to a constant are the only ones allowed in the definition of $\bar m_{\langle\cdot\rangle}(T_0)$.

In the particular case $T_0 = 2\pi(\delta_P-\delta_Q)$ with $P,Q\in\mathbb{R}^2$, it is tempting to show that the numbers

$$\rho(P,Q) := m_{\langle\cdot\rangle}\big(2\pi(\delta_P-\delta_Q)\big) \quad\text{and}\quad \bar\rho(P,Q) := \bar m_{\langle\cdot\rangle}\big(2\pi(\delta_P-\delta_Q)\big)\,, \tag{1.8}$$

as functions of P and Q are distances on the plane. At least for suitable seminorms, this will be the case, these functions giving heuristically the minimal $\dot H^{1/2}$–energy necessary to move the singularity P up to the singularity Q.

In this paper, we discuss two natural questions concerning (1.8), namely

(Q1) Can we compute (1.8) in terms of $\langle\cdot\rangle$?

(Q2) What is the behavior of a minimizing sequence in (1.8) ?

Both the questions are very delicate in nature and intimately related to the specific choice of the seminorm. Since smooth maps are dense in the weak topology and $T(f)=0$ for any such map, it is obvious that the constraint $T(f)=T_0$ is not sequentially weakly closed. Hence each of the minimization problems above is highly nontrivial.

In analogy with the minimization problem studied in [33], we confine ourselves to a class of seminorms which come from second order linear elliptic operators in the half space. As we shall see, no matter which regularity we assume on the coefficients of the operators, concentration phenomena occur near the boundary of the half space. These phenomena can be regarded as the boundary analogues of the concentration phenomena in the Ginzburg-Landau theories and they will be explained in terms of concentration and quantization effects for Jacobians.

The class of seminorms we are interested in is defined as follows. Let $\mathcal{S}^+$ be the set of all positive definite symmetric 3×3 matrices and consider $A : \mathbb{R}^3_+ \to \mathcal{S}^+$ satisfying the ellipticity assumption

$$\lambda|\xi|^2 \leq A(x)\xi\cdot\xi \leq \Lambda|\xi|^2, \quad \text{for a.e. } x\in\mathbb{R}^3_+\,, \forall \xi\in\mathbb{R}^3, \tag{1.9}$$

for some constants $\lambda = \lambda(A) > 0$ and $\Lambda = \Lambda(A) > 0$ independent of x. We denote by $\mathcal{A}$ the set of all measurable matrix fields satisfying (1.9). Thus, $\mathcal{A}\subset L^\infty(\mathbb{R}^3_+;\mathcal{S}^+)$. We shall also consider $\mathcal{A}_0\subset\mathcal{A}$ the subset of those $A\in\mathcal{A}$ of product-type, i.e. such that

$$A = \begin{pmatrix} B & 0 \\ 0 & b\end{pmatrix}\,, \tag{1.10}$$

for some 2×2 matrix field B and scalar function b such that $B(x) = B(x_1,x_2)$ and $b(x) = b(x_1,x_2)$. Given $A\in\mathcal{A}$, we introduce a functional on $\dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)$ as follows

$$E_A(u) = \frac{1}{2}\int_{\mathbb{R}^3_+} \operatorname{tr}\big(\nabla u\, A(\nabla u)^t\big)dx\,. \tag{1.11}$$

We define an energy $\mathcal{E}_A$ and a seminorm $\langle\cdot\rangle_A$ on $\dot H^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$ by setting

$$\mathcal{E}_A(f) = \langle f\rangle^2_A := \operatorname{Inf}\left\{E_A(u)\,;\ u\in\dot H^1_f(\mathbb{R}^3_+;\mathbb{R}^2)\right\} \quad \forall f\in\dot H^{1/2}(\mathbb{R}^2;\mathbb{R}^2)\,, \tag{1.12}$$

where $\dot H^1_f = \big\{u\in\dot H^1\ ;\ u_{|\mathbb{R}^2} = f\big\}$. Due to the uniform ellipticity assumption (1.9), this seminorm is equivalent to (1.2). Moreover, as we recall in Section 2, the infimum in (1.12) is precisely attained by the (unique) map $u = u_f\in\dot H^1_f(\mathbb{R}^3_+;\mathbb{R}^2)$ satisfying

$$\operatorname{div}(A\nabla u) = 0 \quad \text{in} \ \ \dot H^{-1}\,. \tag{1.13}$$

In the sequel, we will refer to u_f as the A–harmonic extension of f.

Assuming the choice $\langle\cdot\rangle = \langle\cdot\rangle_A$ in (1.6) (respectively (1.7)) for the rest of the paper, we will denote by $m_A(T_0)$ (respectively $\bar m_A(T_0)$) the corresponding quantities and similarly $\rho_A(P,Q)$ (respectively $\bar\rho_A(P,Q)$) the functions as defined in (1.8).

To a given a matrix field $A\in\mathcal{A}$ which is continuous, one may associate natural geometric distances on the half space and its boundary (see Section 3). We will introduce the integrand $\mathcal{L}_A(x,\tau) = \sqrt{(\operatorname{Cof}A(x))\tau\cdot\tau}$ in order to define the length functional $\mathbb{L}_A : \operatorname{Lip}\big([0,1];\overline{\mathbb{R}^3_+}\big)\to\mathbb{R}_+$ by setting

$$\mathbb{L}_A(\gamma) = \int_0^1 \mathcal{L}_A\big(\gamma(t),\dot\gamma(t)\big)dt\,. \tag{1.14}$$

In the case of a measurable field A, the previous formula is meaningless since A has no trace on sets of null Lebesgue measure. However, it is possible to construct a generalized length functional on $\operatorname{Lip}\big([0,1];\overline{\mathbb{R}^3_+}\big)$ associated to A, still denoted by $\mathbb{L}_A$, such that (1.14) holds whenever A is continuous. This issue has been partially pursued in [31], [32] and will be presented in Section 3. To the functional $\mathbb{L}_A$, we associate the geodesic distance $d_A : \overline{\mathbb{R}^3_+}\times\overline{\mathbb{R}^3_+}\to\mathbb{R}_+$ defined for $P,Q\in\overline{\mathbb{R}^3_+}$ by

$$d_A(P,Q) = \operatorname{Inf}\left\{\mathbb{L}_A(\gamma)\,;\ \gamma\in\operatorname{Lip}\big([0,1];\overline{\mathbb{R}^3_+}\big)\,,\ \gamma(0)=P\,,\ \gamma(1)=Q\right\}.$$

In the same way, $\mathbb{L}_A$ induces a distance $\bar d_A$ on $\mathbb{R}^2$ by taking the previous infimum over curves lying on $\partial\mathbb{R}^3_+\simeq\mathbb{R}^2$, i.e., for $P,Q\in\mathbb{R}^2$,

$$\bar d_A(P,Q) = \operatorname{Inf}\left\{\mathbb{L}_A(\gamma)\,;\ \gamma\in\operatorname{Lip}([0,1];\partial\mathbb{R}^3_+)\,,\ \gamma(0)=P\,,\ \gamma(1)=Q\right\}.$$

Both these distances are equivalent to the Euclidean ones and reduce to the respective Riemannian distances whenever A is continuous. In addition, formula (1.14) still holds for a suitable Finsler metric $\mathcal{L}_A^\infty$ (see Proposition 3.3).

The first result of this paper compares the functions ρ_A, $\bar\rho_A$, d_A and $\bar d_A$.

Theorem 1.1. *Let $A \in \mathcal{A}$, ρ_A, $\bar\rho_A$, d_A and $\bar d_A$ as defined above.*

(i) The function $\bar\rho_A$ is a distance on $\mathbb{R}^2$. Moreover,

$$\pi\lambda|P-Q| \le \bar\rho_A(P,Q) \le \pi\Lambda|P-Q| \quad \forall P,Q \in \mathbb{R}^2\,,$$

where $\lambda = \lambda(A)$ and $\Lambda = \Lambda(A)$ are the ellipticity bounds of A.

(ii) We have

$$\pi d_A(P,Q) \le \rho_A(P,Q) \quad \text{and} \quad \pi\bar d_A(P,Q) \le \bar\rho_A(P,Q) \quad \forall P,Q \in \mathbb{R}^2\,.$$

(iii) If $A \in \mathcal{A}_0$, then $\rho_A = \bar\rho_A = \pi\bar d_A = \pi d_A$. In particular, ρ_A is a distance.

(iv) If $\pi d_A(P,Q) = \rho_A(P,Q)$ for some distinct points $P,Q \in \mathbb{R}^2$, then, up to subsequences, any minimizing sequence $\{f_n\}_{n\in\mathbb{N}}$ for $\rho_A(P,Q)$ tends weakly to some constant $\alpha \in \mathbb{S}^1$. As a consequence, $\pi d_A(P,Q) = \rho_A(P,Q) = \bar\rho_A(P,Q) = \pi\bar d_A(P,Q)$ and $\rho_A(P,Q)$ is not attained.

Except for the upper bound in *(i)*, both *(i)* and *(ii)* come from a duality argument involving the characterization of 1−Lipschitz functions with respect to the distance d_A as subsolutions of suitable Hamilton-Jacobi type equations with measurable matrices in the spirit of [18]. This argument was originally introduced in [13] for the Dirichlet integral in the context of $\mathbb{S}^2$−valued maps from domains in $\mathbb{R}^3$. Here we follow the same strategy of [31,32], where, still in the $\mathbb{S}^2$−valued case, this approach was extended to the conformally flat case $A(x) = w(x)\,\mathrm{Id}$. Essentially the same characterization, combined with a differentiation argument yields the equality $\bar\rho_A = \pi\bar d_A$ in *(iii)*. Another basic ingredient providing the upper bound in *(i)*, is the construction of an explicit optimal *dipole* $\{f_n\}_{n\in\mathbb{N}}$ with respect to a constant matrix. As in [33], the crucial role is played by Möbius transformations. Under the structure assumption (1.10), d_A and $\bar d_A$ coincide as distances on the plane (see Corollary 3.2) and this fact leads to the full equality in *(iii)*. About claim *(iv)*, we will show that the energy has to stay in a bounded set, therefore concentration follows from the strong maximum principle.

Remark 1.1. We point out that the first inequality in Theorem 1.1, claim *(ii)*, may be strict. More precisely, we may construct a matrix field $A \in \mathcal{A}$ of the form $A(x) = a(x_3)\mathrm{Id}$ with $a \in C^0([0,+\infty))$ such that $\pi d_A(P,Q) < \rho_A(P,Q) < \bar\rho_A(P,Q) = \pi\bar d_A(P,Q)$ whenever $P \ne Q$ (see Example 4.1).

In the case of a general matrix field A depending also on the x_3-variable, we would need, and this is indeed the missing ingredient, a useful boundary version of the eikonal equation on $\partial\mathbb{R}^3_+$ to characterize functions which are 1-Lipschitz with respect to $\bar d_A$. Though not yet completely satisfactory, the use of measurable eikonal equations seems of interest for this problem. Indeed, due to the lack of regularity of the matrix A, both the usual coarea type argument for the lower bounds (see e.g.[1]) and the direct construction of optimal dipoles for the equality $\overline{\rho}_A = \pi\overline{d}_A$ seem to be impossible.

In the case of a continuous matrix field A, the situation simplifies a lot and much more can be said. In particular, the asymptotic behavior of optimal sequences for $\bar\rho_A$ can be described. We have the following.

Theorem 1.2. *With the notation of Theorem 1.1, assume that $A \in \mathcal{A}$ is continuous in $\overline{\mathbb{R}^3_+}$ and $\bar{A} \in \mathcal{A}_0$ where $\bar{A} = \bar{A}(x_1, x_2) = A(x_1, x_2, 0)$. Let $P, Q \in \mathbb{R}^2$ be two distinct points, $\{f_n\}_{n\in\mathbb{N}} \subset X$ an optimal sequence for $\bar{\rho}_A(P,Q)$ and $\{u_n\}_{n\in\mathbb{N}} \subset \dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$ the corresponding A–harmonic extensions.*

(i) *We have $\bar{\rho}_A(P,Q) = \pi \bar{d}_A(P,Q)$.*

(ii) *Up to subsequences, there exists an injective curve $\gamma \in \mathrm{Lip}([0,1];\partial\mathbb{R}^3_+)$ satisfying $\gamma(0) = Q$, $\gamma(1) = P$ and $\mathbb{L}_A(\gamma) = \bar{d}_A(P,Q)$ such that*

$$\frac{1}{2}\,\mathrm{tr}(\nabla u_n\, A(\nabla u_n)^t)dx \overset{*}{\rightharpoonup} \pi \mathcal{L}_A(x,\tau_x)\mathcal{H}^1 \llcorner \Gamma \quad \text{as } n \to +\infty, \tag{1.15}$$

weakly-$\star$ in the sense of measures, where $\Gamma = \gamma([0,1])$ and τ_x denotes a unit tangent vector to Γ at the point x.

(iii) *Up to subsequences, the graph current G_n associated to f_n satisfies*

$$< G_n, \beta > \underset{n\to+\infty}{\longrightarrow} < G_\alpha, \beta > + < \vec{\Gamma} \times [\mathbb{S}^1], \beta > \quad \forall \beta \in \mathcal{D}^2(\mathbb{R}^2 \times \mathbb{S}^1)\,, \tag{1.16}$$

where $\vec{\Gamma}$ is the 1–rectifiable current relative to the oriented curve γ.

(iv) *The energy is carried by the vorticity sets, i.e., for any $0 < R < 1$, we have*

$$\frac{1}{2}\int_{\{|u_n|\leq R\}} \mathrm{tr}\big(\nabla u_n\, A(\nabla u_n)^t\big)dx \underset{n\to+\infty}{\longrightarrow} \pi R^2 \bar{d}_A(P,Q)\,. \tag{1.17}$$

In the light of Theorem 1.1, claim *(i)* is not surprising and actually holds even if $\bar{A}$ does not satisfy the structure assumption (1.10). The upper bound could be obtained from a direct dipole construction but we prefer to deduce it from claim *(iii)* of the previous theorem. Claim *(ii)* describes lack of compactness of optimal sequences and the structure of the limiting defect measure. The analysis of this quantization phenomena is based on a study of pre-Jacobians of $\dot{H}^{1/2}$-maps and their limits. Claim *(iii)* interprets the topological counterpart of the energy concentration in terms of bubbling-off of a vertical current in the framework of Cartesian currents (see [21]) as already pursued in the $H^{1/2}$–setting in [22,25] and [33] in the $\mathbb{S}^1$-valued case and [23,24] for more general target manifolds. Our approach to graph currents is quite direct and does not rely heavily on Geometric Measure Theory. Instead, it essentially relies on a representation formula for the pre-Jacobian current $J(f)$ in terms of a suitable lifting of the map f. Our lifting construction is based on a deep result in [9] (see Section 6 for details). Finally, claim *(iv)* asserts that the energy is carried by the vorticity sets of the extensions, much in the spirit of the Ginzburg-Landau theories. This statement is the higher dimensional analogue of [33], Remark 7, formula (3.54), and it is proved using the oriented coarea formula of [1].

The previous results are very useful in order to deal with approximation and relaxation type problems. We recall that smooth maps are dense in X only for the $\dot{H}^{1/2}$–weak topology (see Section 2 and [34]). Then, a natural question is to know, for a given $f \in X$, how far from f remains a smooth approximating sequence. Given the energy functional $\mathcal{E}_A$ on X, we study the *smooth approximation defect* via the relaxed functional $\overline{\mathcal{E}}_A : X \to \mathbb{R}$ defined by

$$\overline{\mathcal{E}}_A(f) := \mathrm{Inf}\bigg\{ \liminf_{n\to+\infty} \mathcal{E}_A(f_n)\,;\, \{f_n\}_{n\in\mathbb{N}} \subset X \cap C^\infty(\mathbb{R}^2)\,,\, f_n \rightharpoonup f \text{ weakly in } \dot{H}^{1/2} \bigg\}\,.$$

Obviously $\overline{\mathcal{E}}_A \geq \mathcal{E}_A$ and the determination of the gap between $\overline{\mathcal{E}}_A(f)$ and $\mathcal{E}_A(f)$ for a given $f \in X$, gives an answer to the previous question. We point out that in the definition of $\overline{\mathcal{E}}_A$ (as well as for $\bar{m}_A$), we could take the convergence a.e. instead of the weak convergence. Actually, in view of Theorem 2.5, this alternative choice gives the same quantity.

In the context of $\mathbb{S}^2$–valued maps from three dimensional domains, it has been proved in [6] and [32], that the gap occurring in the approximation process is proportional to the *length of a minimal connection* between the topological singularities of positive and negative degree. In our setting, the length of a minimal connection relative to the distance $\bar{d}_A$, corresponds to the functional $L_A : X \to \mathbb{R}_+$ defined by

$$L_A(f) = \frac{1}{2\pi}\,\mathrm{Sup}\bigg\{ < T(f), \varphi > \,;\, \varphi \in \mathrm{Lip}(\mathbb{R}^2, \mathbb{R})\,, \qquad |\varphi(P) - \varphi(Q)| \leq \bar{d}_A(P,Q)\ \forall P, Q \in \mathbb{R}^2 \bigg\}. \tag{1.18}$$

For $f \in X$, $L_A(f)$ can be viewed as the dual norm of $T(f) \in \big(\mathrm{Lip}(\mathbb{R}^2;\mathbb{R})\big)'$ with $\mathbb{R}^2$ endowed with the metric $\bar{d}_A$, and in the particular case $T(f) = 2\pi(\delta_P - \delta_Q)$, $L_A(f)$ is equal to $\bar{d}_A(P,Q)$. Here we can not assert that $L_A(f)$ is the right quantity to consider for computing $\overline{\mathcal{E}}_A(f) - \mathcal{E}_A(f)$. In view of Theorem 1.1 (and as suggested in [16] for $W^{1,1}$–energies), a natural candidate is the length of a minimal connection relative to the distance $\bar{\rho}_A$, i.e., the functional $\tilde{L}_A : X \to \mathbb{R}_+$ defined by

$$\tilde{L}_A(f) = \frac{1}{2\pi}\,\mathrm{Sup}\bigg\{ < T(f), \varphi > \,;\, \varphi \in \mathrm{Lip}(\mathbb{R}^2, \mathbb{R})\,, \qquad |\varphi(P) - \varphi(Q)| \leq \frac{1}{\pi}\,\bar{\rho}_A(P,Q)\ \forall P, Q \in \mathbb{R}^2 \bigg\}. \tag{1.19}$$

For a general measurable matrix field $A \in \mathcal{A}$, we have the following result.

Theorem 1.3. *Let $A \in \mathcal{A}$ and $\bar{d}_A$, $\bar{\rho}_A$, $\bar{m}_A$, $\mathcal{E}_A$, $\overline{\mathcal{E}}_A$, L_A, $\tilde{L}_A$ as above. Then for every $f \in X$,*

$$\mathcal{E}_A(f) + \pi L_A(f) \leq \overline{\mathcal{E}}_A(f) \leq \mathcal{E}_A(f) + \pi\tilde{L}_A(f)\,, \tag{1.20}$$

$$\pi L_A(f) \leq \bar{m}_A\big(T(f)\big) \leq \pi\tilde{L}_A(f)\,. \tag{1.21}$$

If in addition $\bar{\rho}_A = \pi\bar{d}_A$ (e.g. if $A \in C^0(\overline{\mathbb{R}^3_+})$ or $A \in \mathcal{A}_0$) then equality holds in (1.20) *and* (1.21)*. Conversely, if*

$$\overline{\mathcal{E}}_A(f) = \mathcal{E}_A(f) + \pi L_A(f) \tag{1.22}$$

or $\bar{m}_A(T(f)) = \pi L_A(f)$ for every $f \in X$, then $\bar{\rho}_A = \pi\bar{d}_A$.

For the upper bounds, the heart of the matter is a combination of the density of maps with finitely many singularities with Theorem 1.1 through a dipole removing technique. The lower bounds are obtained again by duality arguments. As already mentioned, when $A = \mathrm{Id}$ formula (1.22) could be proved using the theory of Cartesian currents, adapted to the case of the entire space, combining the lower semicontinuity of the energy functional and the approximation in energy (see [25], Prop.2.11 and Thm. 6.1). As for Theorem 1.1, it would be very interesting to know if the representation formula (1.22) fails for a certain class of matrix fields $A \in \mathcal{A}$, e.g., highly oscillating in the x_3-variable.

We point out that it seems difficult to have a representation formula for $\overline{\mathcal{E}}_A(f)$ which is local, i.e., with the term $L_A(f)$ written as an integral with respect to some measure. Indeed, in the case $T(f) = 2\pi(\delta_P - \delta_Q)$, Theorem 1.2 suggests to write $L_A(f)$ as an integral with respect to $\mathcal{H}^1$ on a $\bar{d}_A$-geodesic running from P to Q. However there would be no canonical choice for such geodesic whenever P and Q are conjugate points. Moreover, for measurable matrix fields, we may even believe that different recovery sequences could concentrate energy on different geodesics.

The plan of the paper is as follows. In Section 2, we recall and prove basic properties of homogeneous Sobolev spaces, $\mathbb{S}^1$-valued maps and A–harmonic extensions. In Section 3, we introduce the notions of length structures and geodesic distances related to a measurable field of matrices and we prove a characterization of the corresponding 1–Lipschitz functions in terms of Hamilton-Jacobi equations. In Section 4, we combine the previous notions with duality arguments and dipole-type constructions to prove Theorem 1.1. In Section 5, we develop a theory for pre-Jacobians and graph currents associated to $\dot{H}^{1/2}$–maps. In Section 6, we describe the quantization properties of Jacobians of harmonic extensions and we prove Theorem 1.2. In Section 7, we apply the previous techniques to the relaxation problem and we prove Theorem 1.3. We collect in a separate appendix the proof of the density of maps in X with finitely many singularities.

2. Function spaces and A–harmonic extensions

In this section, we collect the definitions and the basic properties of the homogeneous function spaces we use throughout the paper and the related results concerning A–harmonic extensions. Many proofs will be omitted since they are similar to the ones for bounded domains. The main reference is [20], Chap. II.

2.1 Homogeneous Sobolev spaces.

The main function spaces we are interested in are the homogeneous Sobolev spaces $\dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$ and $\dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$. In Section 5, we will also consider the homogeneous BV-space $\dot{BV}(\mathbb{R}^2;\mathbb{R})$. We start with $\dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$ which is defined by

$$\dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2) = \Big\{ f \in L^2_{\rm loc}(\mathbb{R}^2;\mathbb{R}^2)\ ;\ |f|_{1/2} < +\infty \Big\}\,,$$

where $|\cdot|_{1/2}$, given by (1.2), is just a seminorm since it vanishes on constant functions. Clearly $C^\infty_0(\mathbb{R}^2;\mathbb{R}^2) \subset \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$ and the same holds for the Schwartz class $\mathcal{S}(\mathbb{R}^2;\mathbb{R}^2)$. In addition, as already recalled in the introduction, a direct computation yields $|f|^2_{1/2} = \int_{\mathbb{R}^2}|\xi||\hat{f}(\xi)|^2 d\xi$, where $\hat{f}$ denotes the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^2;\mathbb{R}^2)$. The following proposition clarifies the behavior of $\dot{H}^{1/2}$–functions at infinity.

Proposition 2.1. *For every $f \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$, there exists a uniquely determined $f^\infty \in \mathbb{R}^2$ such that*

$$⨍_{B_R} (f - f^\infty)d\hat{x} = O(R^{-1/2}) \quad \text{as } R \to +\infty\,. \tag{2.1}$$

Moreover, there exists a constant $C > 0$ independent of f such that

$$\|f - f^\infty\|_{L^4(\mathbb{R}^2)} \leq C|f|_{1/2} \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{|f - f^\infty|^2}{|\hat{x}|}\,d\hat{x} \leq C|f|^2_{1/2}\,. \tag{2.2}$$

In view of Proposition 2.1, we have the following result.

Theorem 2.1. *The space $\dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$ endowed with the scalar product*

$$\langle f, g\rangle_{1/2} := f^\infty \cdot g^\infty + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\big(f(\hat{x}) - f(\hat{y})\big)\cdot\big(g(\hat{x}) - g(\hat{y})\big)}{|\hat{x} - \hat{y}|^3}\,d\hat{x}d\hat{y} \tag{2.3}$$

is an Hilbert space. Moreover, setting $\|\cdot\|_{1/2}$ to be the induced norm, we have :

(i) the subspace $C^\infty_{\rm const}(\mathbb{R}^2;\mathbb{R}^2) := C^\infty_0(\mathbb{R}^2;\mathbb{R}^2)\bigoplus\mathbb{R}^2$ of smooth functions constant outside a compact set is dense. In addition, $f^\infty = 0$ if and only if $f \in \overline{C^\infty_0(\mathbb{R}^2;\mathbb{R}^2)}^{|\cdot|_{1/2}}$,

(ii) $\dot H^{1/2}(\mathbb{R}^2;\mathbb{R}^2) \hookrightarrow L^2_{\rm loc}(\mathbb{R}^2;\mathbb{R}^2)$ and the embedding is compact.

We postpone the proofs to the end of this subsection when we will derive them as a consequence of other results below.

Remark 2.1. We observe that the operator $f \in \dot H^{1/2} \mapsto f^\infty$ is linear and continuous. In particular, if $f_n \rightharpoonup f$ weakly in $\dot H^{1/2}$ then $f_n^\infty \to f^\infty$ as $n \to +\infty$.

Remark 2.2. Let us consider the subspace

$$\dot H^{1/2}_0(\mathbb{R}^2;\mathbb{R}^2) := \overline{C^\infty_0(\mathbb{R}^2;\mathbb{R}^2)}^{\|\cdot\|_{1/2}} = \Big\{ f \in L^4(\mathbb{R}^2;\mathbb{R}^2)\,,\ |f|_{1/2} < +\infty \Big\}$$

endowed with the scalar product (2.3) and let $\dot H^{-1/2}(\mathbb{R}^2;\mathbb{R}^2)$ be its dual space (as an Hilbert space). If $f \in \dot H^{1/2}_0$ (actually if $f \in \dot H^{1/2}$) then $\partial_j f \in \dot H^{-1/2}$ for $j = 1, 2$. Indeed, for $f, g \in C^\infty_0(\mathbb{R}^2;\mathbb{R}^2)$, an easy application of the Fourier transform leads to the inequality

$$\left| \int_{\mathbb{R}^2} \partial_j f \wedge g \right| \leq |f|_{1/2} |g|_{1/2}\,.$$

Hence, arguing by density, for $f \in \dot H^{1/2}_0$ (actually, for $f \in \dot H^{1/2}$) the distribution $\partial_j f$ extends to an element in $\dot H^{-1/2}$ (actually in $(\dot H^{1/2})'$). As a consequence, given $f, g \in \dot H^{1/2}$ and $\{f_n\}_{n\in\mathbb{N}}, \{g_n\}_{n\in\mathbb{N}} \subset C^\infty_{\rm const}(\mathbb{R}^2;\mathbb{R}^2)$ such that $f_n \to f$ and $g_n \to g$ in $\dot H^{1/2}$, we may set

$$\int_{\mathbb{R}^2} \partial_j f \wedge g := \lim_{n\to+\infty} \int_{\mathbb{R}^2} \partial_j f_n \wedge g_n\,.$$

This number is well defined (i.e. independent of the sequences) and

$$\int_{\mathbb{R}^2} \partial_j f \wedge g = -\int_{\mathbb{R}^2} f \wedge \partial_j g \quad , \quad \left| \int_{\mathbb{R}^2} f \wedge \partial_j g \right| \leq |f|_{1/2} |g|_{1/2}\,. \tag{2.4}$$

Now we recall that the homogeneous Sobolev space $\dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)$ is defined as

$$\dot H^1(\mathbb{R}^3_+;\mathbb{R}^2) = \Big\{ u \in L^2_{\rm loc}(\mathbb{R}^3_+;\mathbb{R}^2)\,;\ \nabla u \in \big(L^2(\mathbb{R}^3_+;\mathbb{R}^2)\big)^3 \Big\}\,,$$

where $\|\nabla u\|_{L^2(\mathbb{R}^3_+)} = \big(E_{\rm Id}(u)\big)^{1/2} =: |u|_1$ is the natural seminorm associated to the Dirichlet energy which vanishes precisely on constant functions. The following proposition parallels Proposition 2.1 and clarifies the behavior of $\dot H^1$–functions at infinity.

Proposition 2.2. *For every $u \in \dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)$, we have $u \in L^2_{\rm loc}\big(\overline{\mathbb{R}^3_+};\mathbb{R}^2\big)$ and there exists a uniquely determined $u^\infty \in \mathbb{R}^2$ such that*

$$\fint_{B_R\cap\mathbb{R}^3_+} (u - u^\infty)dx = O(R^{-1/2}) \quad \text{as } R \to +\infty. \tag{2.5}$$

Moreover, there exists a constant $C > 0$ independent of u such that for any $x_0 \in \mathbb{R}^3$,

$$\|u - u^\infty\|^2_{L^6(\mathbb{R}^3_+)} \leq C \int_{\mathbb{R}^3_+} |\nabla u|^2 dx \ \ \text{and} \ \ \int_{\mathbb{R}^3_+} \frac{|u - u^\infty|^2}{|x - x_0|^2}\,dx \leq C \int_{\mathbb{R}^3_+} |\nabla u|^2 dx\,. \tag{2.6}$$

Proof. According to [20] Chapt. II, Remark 4.1, we have $u \in L^2_{\rm loc}\big(\overline{\mathbb{R}^3_+};\mathbb{R}^2\big)$. Setting $\tilde u$ to be the even reflection of u across the plane $\{x_3=0\}$, we have $\tilde u \in \dot H^1(\mathbb{R}^3;\mathbb{R}^2)$ and $\|\nabla \tilde u\|^2_{L^2(\mathbb{R}^3)} = 2\|\nabla u\|^2_{L^2(\mathbb{R}^3_+)}$. By Lemma 5.2 and Theorem 5.1 in [20], Chapt. II, there exists

$$u^\infty = \lim_{R\to+\infty} \fint_{\partial B_R} \tilde u \, dx$$

and for any $x_0 \in \mathbb{R}^3$,

$$\|\tilde u - u^\infty\|^2_{L^6\left(\mathbb{R}^3\setminus B_1(x_0)\right)} + \int_{\mathbb{R}^3\setminus B_1(x_0)} \frac{|\tilde u - u^\infty|^2}{|x-x_0|^2}\, dx \le C \int_{\mathbb{R}^3} |\nabla \tilde u|^2 dx\,.$$

Therefore,

$$u^\infty = \lim_{R\to+\infty} \fint_{\partial B_R} \tilde u \, dx = \lim_{R\to+\infty} \fint_{B_R} \tilde u \, dx = \lim_{R\to+\infty} \fint_{B_R\cap\mathbb{R}^3_+} u \, dx\,, \tag{2.7}$$

and $u - u^\infty \in L^6\big(\mathbb{R}^3_+ \setminus B_1(x_0)\big)$. Then (2.5) follows from Hölder inequality. To complete the proof of (2.6), we consider a cut-off function $\chi \in C^\infty_0(B_2(x_0))$, $0 \le \chi \le 1$, with $\chi \equiv 1$ in $B_1(x_0)$. Clearly $(\tilde u - u^\infty)\chi^2 \in H^1_0(B_2(x_0);\mathbb{R}^2)$. Combining the standard Hardy and Sobolev inequalities in $B_2(x_0)$ with (2.7), the conclusion easily follows. Finally, the uniqueness of u^∞ is an obvious consequence of (2.6). ∎

As a consequence of Proposition 2.2, we have the following classical theorem.

Theorem 2.2. *The space $\dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)$ endowed with the scalar product*

$$\langle u, v\rangle_1 := u^\infty \cdot v^\infty + \int_{\mathbb{R}^3_+} \operatorname{tr}\big(\nabla u (\nabla v)^t\big)\, dx$$

is an Hilbert space. Moreover, setting $\|\cdot\|_1$ to be the induced norm, we have :

(i) The subspace $C^\infty_{\rm const}\big(\overline{\mathbb{R}^3_+};\mathbb{R}^2\big) := C^\infty_0\big(\overline{\mathbb{R}^3_+};\mathbb{R}^2\big) \bigoplus \mathbb{R}^2$ of smooth functions constant outside a compact set is dense. In addition, $u^\infty = 0$ if and only if $u \in \overline{C^\infty_0(\overline{\mathbb{R}^3_+};\mathbb{R}^2)}^{|\cdot|_1}$,

(ii) $\dot H^1(\mathbb{R}^3_+;\mathbb{R}^2) \hookrightarrow L^2_{\rm loc}(\overline{\mathbb{R}^3_+};\mathbb{R}^2)$ and the embedding is compact.

Proof. The proof of the Hilbert space structure and *(ii)* are easy consequences of Proposition 2.2 together with standard arguments. Claim *(i)* is given in [20], Chapt. II, Theorem 6.6. ∎

Remark 2.3. The proof of Thm. 6.6 in [20], Chapt. II, is based on standard convolution and suitable truncation arguments. From this proof, it follows that, for any $u \in \dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)\cap L^\infty$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty_{\rm const}\big(\overline{\mathbb{R}^3_+};\mathbb{R}^2\big)$ such that $u_n \to u$ in $\dot H^1$ and $\|u_n\|_\infty \le \|u\|_\infty$ for each n. In view of Theorem 2.3 below, the analogous property holds for $f \in \dot H^{1/2}(\mathbb{R}^2;\mathbb{R}^2)\cap L^\infty$ since such a map f admits a bounded extension of finite energy.

The homogeneous Sobolev spaces introduced above are naturally related through the notions of traces and extensions. Recall that for any $\Phi \in C^\infty_{\rm const}(\overline{\mathbb{R}^3_+};\mathbb{R}^2)$ the function $\varphi(\hat x) \equiv \Phi(\hat x, 0) \in C^\infty_{\rm const}(\mathbb{R}^2;\mathbb{R}^2)$ is the trace of Φ on the plane $\mathbb{R}^2$, i.e., $\varphi = \Phi_{|\mathbb{R}^2}$. In Theorem 2.3 below, we introduce the trace operator Tr acting on $\dot H^1$ but we might use in the next sections the notation $u_{|\mathbb{R}^2}$ for $u \in \dot H^1$ instead of $\operatorname{Tr} u$.

Theorem 2.3. *There exits a unique bounded linear trace operator* Tr *from* $\dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)$ *onto* $\dot H^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$ *such that* $\mathrm{Tr}\,\Phi=\Phi_{|\mathbb{R}^2}$ *for any* $\Phi\in C^\infty_{\rm const}\big(\overline{\mathbb{R}^3_+};\mathbb{R}^2\big)$. *Moreover,*

(i) $(\mathrm{Tr}\ u)^\infty=u^\infty$ *for every* $u\in\dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)$,

(ii) the kernel of the trace operator Tr *is given by* $N(\mathrm{Tr})=\overline{C^\infty_0(\mathbb{R}^3_+;\mathbb{R}^2)}^{\|\cdot\|_1}$,

(iii) there exists a bounded linear operator $\mathrm{Ext}:\dot H^{1/2}(\mathbb{R}^2;\mathbb{R}^2)\to\dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)$ *such that* $\mathrm{Tr}(\mathrm{Ext}(f))=f$ *for every* $f\in\dot H^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$. *In particular, the trace operator* Tr *is surjective.*

Remark 2.4. From the continuity and linearity of the operator Tr, we infer that, if $u_n\rightharpoonup u$ weakly in $\dot H^1$, then $\mathrm{Tr}\,u_n\rightharpoonup\mathrm{Tr}\,u$ weakly in $\dot H^{1/2}$ as $n\to+\infty$.

Proof of Proposition 2.1. We start with three auxiliary facts. Let $\varphi\in C^\infty_0(\mathbb{R}^2;\mathbb{R}^2)$ and define $g=\mathcal{F}^{-1}(|\xi|^{1/2}\mathcal{F}(\varphi))$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are respectively the Fourier and inverse Fourier transforms. Clearly $g\in L^2(\mathbb{R}^2;\mathbb{R}^2)$ with $\|g\|_{L^2(\mathbb{R}^2)}=|\varphi|_{1/2}$ and $\varphi=\mathcal{F}^{-1}(|\xi|^{-1/2}\mathcal{F}(g))=C|x|^{-3/2}*g$. By Hardy-Littlewood-Sobolev fractional integration theorem (see e.g. [37]),

$$\|\varphi\|_{L^4(\mathbb{R}^2)}\le C\|\,|x|^{-3/2}*g\|_{L^4(\mathbb{R}^2)}\le C\|g\|_{L^2(\mathbb{R}^2)}=C|\varphi|_{1/2}\,, \tag{2.8}$$

for some constant $C>0$ independent of φ. The second fact we need is the following version of the Hardy inequality (see [30], Theorem 2). For $\varphi\in C^\infty_0(\mathbb{R}^2;\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2}\frac{|\varphi|^2}{|\hat x|}\,d\hat x\le C|\varphi|^2_{1/2} \tag{2.9}$$

for some constant $C>0$ independent of φ. Finally, given $\Phi\in C^\infty_0(\overline{\mathbb{R}^3_+};\mathbb{R}^2)$, the function $\varphi(\hat x)\equiv\Phi(\hat x,0)\in C^\infty_0(\mathbb{R}^2;\mathbb{R}^2)$ satisfies

$$|\varphi|^2_{1/2}\le C\int_{\mathbb{R}^3_+}|\nabla\Phi|^2dx\,, \tag{2.10}$$

for some constant $C>0$ independent of Φ. Now let $f\in\dot H^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$ and consider $\varrho\in C^\infty_0(\mathbb{R}^2;\mathbb{R})$ such that $\varrho\ge0$ and $\int_{\mathbb{R}^2}\varrho=1$. We define

$$u(\hat x,x_3)=\int_{\mathbb{R}^2}f(\hat x+x_3z)\varrho(z)dz\,. \tag{2.11}$$

Standard calculations (see [30], proof of Theorem 2) lead to $u\in C^\infty(\mathbb{R}^3_+;\mathbb{R}^2)$ with $\nabla u\in L^2(\mathbb{R}^3_+)$ and clearly $u(\cdot,x_3)\to f(\cdot)$ a.e. in $\mathbb{R}^2$ as $x_3\to0$. Since $u\in\dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)$, we infer from Theorem 2.2 that there exist $u^\infty\in\mathbb{R}^2$ and $\{\Phi_n\}_{n\in\mathbb{N}}\subset C^\infty_0(\overline{\mathbb{R}^3_+};\mathbb{R}^2)$ such that $u-u^\infty\in L^6(\mathbb{R}^3_+;\mathbb{R}^2)$ and $\nabla\Phi_n\to\nabla u$ in $L^2(\mathbb{R}^3_+)$. Combining (2.9), (2.8) and (2.10) we obtain

$$\int_{\mathbb{R}^2}\frac{|\Phi_n(\hat x,x_3)|^2}{|\hat x|}\,d\hat x+\|\Phi_n(\cdot,x_3)\|^2_{L^4(\mathbb{R}^2)}\le C|f|^2_{1/2}\,,$$

for some constant $C>0$ independent of x_3 and f. Up to subsequences, we have $\Phi_n(\cdot,x_3)\to u(\cdot,x_3)-u^\infty$ as $n\to+\infty$ for every $x_3>0$ by the standard trace theory. Then Fatou lemma yields

$$\int_{\mathbb{R}^2}\frac{|u(\hat x,x_3)-u^\infty|^2}{|\hat x|}\,d\hat x+\|u(\cdot,x_3)-u^\infty\|^2_{L^4(\mathbb{R}^2)}\le C|f|^2_{1/2}\,,$$

for every $x_3>0$. Applying again Fatou lemma as $x_3\to0$, we conclude that $f-f^\infty\in L^4(\mathbb{R}^2)$ and $|\hat x|^{-1/2}(f-f^\infty)\in L^2(\mathbb{R}^2)$ for $f^\infty=u^\infty$. This complete the proof of (2.2) and then (2.1) follows from Hölder inequality. Finally, the uniqueness of f^∞ is an obvious consequence of (2.2). ■

Proof of Theorem 2.1. The proof of the Hilbert space structure is a standard consequence of Proposition 2.1. In addition, one also has the continuity of the restriction operator $f\to f_{|\Omega}$ from

$\dot H^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$ into $H^{1/2}(\Omega;\mathbb{R}^2)$ for any smooth bounded domain $\Omega\subset\mathbb{R}^2$. Then claim *(ii)* follows from the well-known compact embedding $H^{1/2}(\Omega)\hookrightarrow L^2(\Omega)$. In order to prove claim *(i)*, we go back to the proof of Proposition 2.1. For $\varepsilon>0$, we set $f_\varepsilon=u(\cdot,\varepsilon)$ and by the arguments in the previous proof, we have $f_\varepsilon-f^\infty\in\overline{C_0^\infty(\mathbb{R}^2;\mathbb{R}^2)}^{\|\cdot\|_{1/2}}$. A simple computation yields $|f_\varepsilon|_{1/2}\le|f|_{1/2}$. Since we already proved that $(f_\varepsilon)^\infty\equiv f^\infty$ and $f_\varepsilon\to f$ a.e. as $\varepsilon\to0$, we infer from claim *(ii)* that $f_\varepsilon\rightharpoonup f$ weakly in $\dot H^{1/2}$. Then , by Fatou lemma, we have $|f|_{1/2}\le\liminf_{\varepsilon\to0}|f_\varepsilon|_{1/2}\le|f|_{1/2}$ so that $f_\varepsilon\to f$ strongly in $\dot H^{1/2}$. Therefore $f-f^\infty\in\overline{C_0^\infty(\mathbb{R}^2;\mathbb{R}^2)}^{\|\cdot\|_{1/2}}$ and the proof is complete. ∎

Proof of Theorem 2.3. By Theorem 2.2 we can argue by density and the continuity of the trace operator follows from (2.10). Claim *(ii)* is a direct consequence of Theorem 6.5 in [20], Chapt. II. To prove claim *(iii)*, we argue exactly as in the proof of Proposition 2.1. We easily see that $f(\hat x)\mapsto u(\hat x,x_3)$ given by (2.11), is a continuous linear mapping from $\dot H^{1/2}$ to $\dot H^1$. In addition, it is a left inverse for the trace operator, as one may check on the dense subspace $C^\infty_{\rm const}(\mathbb{R}^2;\mathbb{R}^2)$. ∎

An easy consequence of the definitions of the spaces $\dot H^{1/2}$ and $\dot H^1$ and their properties recalled above is that both $\dot H^{1/2}\cap L^\infty$ and $\dot H^1\cap L^\infty$ are algebras and the trace operator well behave under pointwise multiplication. The proof of these facts are elementary and it will be omitted.

Proposition 2.3. *Let* $Y=\dot H^{1/2}(\mathbb{R}^2;\mathbb{R}^2)\cap L^\infty$ *and* $Z=\dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)\cap L^\infty$ *endowed with the respective norms* $\|f\|_Y=\|f\|_{1/2}+\|f\|_\infty$ *and* $\|u\|_Z=\|u\|_1+\|u\|_\infty$. *Then* Y *and* Z *are Banach algebras for the complex product of functions where* $\mathbb{R}^2\simeq\mathbb{C}$, *and for every* $f,g\in Y$, *resp.* $u,v\in Z$, *we have*

$$\|fg\|_Y\le|f^\infty||g^\infty|+|f|_{1/2}\|g\|_\infty+|g|_{1/2}\|f\|_\infty+\|f\|_\infty\|g\|_\infty\le\|f\|_Y\|g\|_Y\,,$$
$$\|uv\|_Z\le|u^\infty||v^\infty|+\|\nabla u\|_2\|v\|_\infty+\|\nabla v\|_2\|u\|_\infty+\|u\|_\infty\|v\|_\infty\le\|u\|_Z\|v\|_Z\,.$$

Moreover, for any $u,v\in Z$, *we have* $\mathrm{Tr}(uv)\in Y$ *and* $\mathrm{Tr}(uv)=\mathrm{Tr}(u)\mathrm{Tr}(v)$.

The last fact we need about the space $\dot H^1$ is the lattice property for scalar valued functions. Since the proof is exactly as in the case of bounded domains, it will be omitted.

Proposition 2.4. *Let* $u\in\dot H^1(\mathbb{R}^3_+;\mathbb{R})$ *and denote by* u^+ *and* u^- *the positive and negative parts of* u. *Then,*

(i) $u^+,u^-\in\dot H^1(\mathbb{R}^3_+;\mathbb{R})$ *and* $\nabla u^+=\chi_{\{u>0\}}\nabla u\,,\ \nabla u^-=-\chi_{\{u<0\}}\nabla u$ *a.e. in* $\mathbb{R}^3_+$,

(ii) the map $u\mapsto(u^+,u^-)$ *is continuous,*

(iii) $\mathrm{Tr}\,u^+=0$ *(resp.* $\mathrm{Tr}\,u^-=0$*) iff* $\mathrm{Tr}\,u\le0$ *(resp.* $\mathrm{Tr}\,u\ge0$*) a.e. in* $\mathbb{R}^2$.

We conclude this subsection with the homogeneous BV-space $\dot{BV}(\mathbb{R}^2;\mathbb{R})$ defined by

$$\dot{BV}(\mathbb{R}^2;\mathbb{R})=\Big\{\psi\in L^1_{\rm loc}(\mathbb{R}^2;\mathbb{R})\,;\ D\psi=(D_1\psi,D_2\psi)\in\mathcal{M}(\mathbb{R}^2;\mathbb{R}^2)\Big\}$$

where $\mathcal{M}(\mathbb{R}^2;\mathbb{R}^2)$ denotes the set of all $\mathbb{R}^2$-valued finite Radon measures in $\mathbb{R}^2$. We shall use the notation $|\psi|_{BV}:=|D\psi|(\mathbb{R}^2)$. By classical results (see e.g. [3]), we also have

$$|\psi|_{BV}=\mathrm{Sup}\left\{\int_{\mathbb{R}^2}\psi\,\mathrm{div}\phi\,dx\ :\ \phi\in C^1_0(\mathbb{R}^2;\mathbb{R}^2)\,,\ \|\phi\|_{L^\infty(\mathbb{R}^2)}\le1\right\}.$$

It is well know (see e.g. [3]) that $\psi\mapsto|\psi|_{BV}$ is sequentially lower semicontinuous with respect to the $L^1_{\rm loc}(\mathbb{R}^2)$-topology. The following proposition parallels Proposition 2.1, Proposition 2.2 and clarifies the behavior of $\dot{BV}$–functions at infinity.

Proposition 2.5. *For every $\psi \in \dot{BV}(\mathbb{R}^2;\mathbb{R})$, there exists a uniquely determined $\psi^\infty \in \mathbb{R}$ such that*

$$\fint_{B_R} (\psi - \psi^\infty) d\hat{x} = O(R^{-1}) \quad \text{as } R \to +\infty\,. \tag{2.12}$$

Moreover, there exists a constant $C > 0$ independent of ψ such that

$$\|\psi - \psi^\infty\|_{L^2(\mathbb{R}^2)} \le C|\psi|_{BV}\,. \tag{2.13}$$

Proof. Consider a smooth mollifier $\varrho \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$, i.e., $\varrho \ge 0$, $\int_{\mathbb{R}^2} \varrho = 1$, and set for $\varepsilon > 0$, $\varrho_\varepsilon(z) = \varepsilon^{-2}\varrho(z/\varepsilon)$. Then, for $\psi \in \dot{BV}(\mathbb{R}^2;\mathbb{R})$, we define $\psi_\varepsilon = \varrho_\varepsilon * \psi$ so that $\psi_\varepsilon \in C^\infty(\mathbb{R}^2) \cap L^1_{\rm loc}(\mathbb{R}^2)$ and $\nabla\psi_\varepsilon = \varrho_\varepsilon * D\psi \in L^1(\mathbb{R}^2)$. Then the conclusion of Proposition 2.5 holds for ψ_ε by Lemma 5.2 and Theorem 5.1 in [20], Chapt. II, together with arguments similar to those used in the proof of Proposition 2.2. It is well known (see e.g. [3]) that $\psi_\varepsilon \to \psi$ a.e., in $L^1_{\rm loc}(\mathbb{R}^2)$ and $|\psi_\varepsilon|_{BV} = |D\psi_\varepsilon|(\mathbb{R}^2) \to |D\psi|(\mathbb{R}^2) = |\psi|_{BV}$ as $\varepsilon \to 0$. Setting $\bar{\psi}_\varepsilon = \fint_{B_1(0)} \psi_\varepsilon$, we observe that

$$|\psi_\varepsilon^\infty| \le \left| \fint_{B_1(0)} (\psi_\varepsilon - \psi_\varepsilon^\infty) \right| + |\bar{\psi}_\varepsilon| \le C\|\psi_\varepsilon - \psi_\varepsilon^\infty\|_{L^2(\mathbb{R}^2)} + |\bar{\psi}_\varepsilon| \le C|\psi_\varepsilon|_{BV} + |\bar{\psi}_\varepsilon|$$

so that ψ_ε^∞ remains bounded. Hence, for a subsequence $\varepsilon_n \to 0$, $\psi^\infty = \lim\limits_{n\to+\infty} \psi^\infty_{\varepsilon_n}$ exists. In particular, $\psi_{\varepsilon_n} - \psi^\infty_{\varepsilon_n} \to \psi - \psi^\infty$ a.e. as $n \to +\infty$ and we infer from Fatou lemma that

$$\|\psi - \psi^\infty\|_{L^2(\mathbb{R}^2)} \le \liminf_{n\to+\infty} \|\psi_{\varepsilon_n} - \psi^\infty_{\varepsilon_n}\|_{L^2(\mathbb{R}^2)} \le C \lim_{n\to+\infty} |\psi_{\varepsilon_n}|_{BV} = C|\psi|_{BV}\,.$$

Then (2.12) easily follows from Hölder inequality. ∎

As a consequence of Proposition 2.5, we have the following classical result.

Theorem 2.4. *The space $\dot{BV}(\mathbb{R}^2;\mathbb{R})$ endowed with the norm*

$$\|\psi\|_{BV} := |\psi^\infty| + |\psi|_{BV}$$

is a Banach space. Moreover, we have :

(i) For every $\psi \in \dot{BV}(\mathbb{R}^2;\mathbb{R})$, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset C^\infty_{\rm const}(\mathbb{R}^2;\mathbb{R})$ such that $\psi_n \to \psi$ in $L^1_{\rm loc}(\mathbb{R}^2)$, $\psi_n^\infty \to \psi^\infty$ and $|\psi_n|_{BV} \to |\psi|_{BV}$ as $n \to +\infty$.

(ii) $\dot{BV}(\mathbb{R}^2;\mathbb{R}) \hookrightarrow L^1_{\rm loc}(\mathbb{R}^2;\mathbb{R})$ and the embedding is compact.

Proof. The proof of the Banach space structure and *(ii)* are easy consequences of Proposition 2.5 together with standard arguments. Given $\psi \in \dot{BV}$ and ψ_ε as in the previous proof, claim *(i)* for ψ_ε classically follows by multiplying $\psi_\varepsilon - \psi_\varepsilon^\infty$ by a suitable sequence of cut-off functions (see [20]). Then we derive *(i)* for ψ using a standard diagonal argument. ∎

2.2. $\dot{H}^{1/2}$–maps with values into $\mathbb{S}^1$.

According to the definition of X (see (1.1)) and the properties of $\dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$ recalled in the previous subsection, we easily derive the following structure result.

Theorem 2.5. *We have*

$$X = \left\{ f \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2) \;:\; |f| = 1 \;\text{ a.e.} \right\}$$

and it defines a complete metric space under the distance induced by $\dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$. *In addition,* $f^\infty \in \mathbb{S}^1$ *for every* $f \in X$. *Moreover, for any bounded sequence* $\{f_n\}_{n\in\mathbb{N}} \subset X$, *there exist a subsequence* $\{f_{n_k}\}$ *and a map* $f \in X$ *such that* $f_{n_k} \rightharpoonup f$ *weakly in* $\dot{H}^{1/2}$, $f_{n_k} \to f$ *a.e. and in* $L^2_{\rm loc}(\mathbb{R}^2)$.

Remark 2.5. In the next sections, we shall need a diagonalization procedure for sequences in X with multi-indices. The generic situation we will have to deal with can be described as follows. Consider a sequence $\{f_{n,m}\}_{(n,m)\in\mathbb{N}^2} \subset X$ such that $f_{n,m} \to f_n \in X$ a.e. as $m \to +\infty$ and $f_n \to f \in X$ a.e. as $n \to +\infty$ with $\limsup_n \limsup_m \mathcal{E}_A(f_{n,m}) < +\infty$. We claim that we can find a diagonal sequence $f_k = f_{n_k,m_k}$ such that $\limsup_k \mathcal{E}_A(f_k) \leq \limsup_n \limsup_m \mathcal{E}_A(f_{n,m})$, $f_k \rightharpoonup f$ weakly in $\dot{H}^{1/2}$ and $f_k \to f$ a.e. as $k \to +\infty$. Indeed, we have $|f_{n,m}(x) - f(x)|(1+|x|^2)^{-2} \leq 2(1+|x|^2)^{-2}$ for a.e. $x \in \mathbb{R}^2$ and $2(1+|x|^2)^{-2} \in L^1(\mathbb{R}^2)$. Hence, by dominated convergence,

$$\lim_{n\to+\infty}\lim_{m\to+\infty}\int_{\mathbb{R}^2} |f_{n,m}(x) - f(x)|(1+|x|^2)^{-2}dx = 0\,.$$

Consequently, we may find a diagonal sequence $f_k = f_{n_k,m_k}$ such that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^2} |f_k(x) - f(x)|(1+|x|^2)^{-2}dx = 0$$

and $\limsup_k \mathcal{E}_A(f_k) \leq \limsup_n \limsup_m \mathcal{E}_A(f_{n,m})$. Then, extracting a subsequence if necessary, we have also $f_k \to f$ a.e. and $f_k \rightharpoonup f$ as $k \to +\infty$.

As already recalled in the Introduction, density of smooth maps in X is a delicate issue. In general the best one can hope is the density of maps smooth except at finitely many points. The following result shows that this is indeed the case. It is the analogue in the entire space of the one in [9], [34] and [25]. The proof will be given in the Appendix.

Theorem 2.6. *For any* $f \in X$, *there exists a sequence* $\{f_n\}_{n\in\mathbb{N}} \subset X \cap W^{1,1}_{\rm loc}(\mathbb{R}^2)$ *such that for every* $n \in \mathbb{N}$, f_n *is smooth outside a finite set,* f_n *is constant outside a compact set,* $f_n^\infty = f^\infty$, $f_n \to f$ *a.e in* $\mathbb{R}^2$ *and* $|f_n - f|_{1/2} \to 0$ *as* $n \to +\infty$.

For a generic map $f \in X$, the approximability by smooth maps depends on the topological singular set as described rigorously in terms of the distribution $T(f)$ defined by (1.4). The equivalent representation formula (1.5) is given in the following lemma. The proof is a straightforward consequence of the distributional curl structure of $H(u)$, namely,

$$H(u) = \mathrm{curl}\,(u \wedge \nabla u) \quad \text{in } \mathcal{D}'(\mathbb{R}^3_+;\mathbb{R}^3)\,, \tag{2.14}$$

combined with an integration by parts which can be justified by smooth approximation.

Lemma 2.1. *If* $f \in X \cap W^{1,1}_{\rm loc}(\mathbb{R}^2)$ *then for any compactly supported function* $\varphi \in C^1_0(\mathbb{R}^2;\mathbb{R})$,

$$< T(f), \varphi > = -\int_{\mathbb{R}^2} \big((f\wedge\partial_1 f)\partial_2\varphi - (f\wedge\partial_2 f)\partial_1\varphi\big)\,.$$

For a map f with finitely many topological singularities, $T(f)$ takes a particularly simple form.

Lemma 2.2. *If* $f \in X \cap W^{1,1}_{\rm loc}(\mathbb{R}^2)$ *is smooth except at finitely many points* $a_1, \ldots, a_k$ *then*

$$T(f) = -2\pi\sum_{i=1}^k d_i\,\delta_{a_i} \tag{2.15}$$

where the integers $d_i = \deg(f, a_i)$ *satisfy* $\sum_{i=1}^k d_i = 0$.

Proof. Formula (2.15) is classical and can obtained from the previous lemma exactly as in [9]. Then, without loss of generality, we may assume that $\sum_{i=1}^k d_i \geq 0$. Then consider for $R \geq 1$, $\Phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\Phi = -1$ in B_R, $\Phi = 0$ in $\mathbb{R}^2 \setminus B_{2R}$ and $\|\nabla\Phi\|_{L^\infty(\mathbb{R}^3)} \leq 1$. Let u be a finite energy extension of f to the half space. Taking R large enough so that $a_1, \dots, a_k \in B_R$, we estimate

$$2\pi \sum_{i=1}^k d_i =< T(f), \Phi_{|\mathbb{R}^2} >= \int_{(B_{2R}\setminus B_R)\cap\mathbb{R}^3_+} H(u)\cdot\nabla\Phi \leq \int_{(B_{2R}\setminus B_R)\cap\mathbb{R}^3_+} |\nabla u|^2 \underset{R\to+\infty}{\longrightarrow} 0$$

because u has finite energy. Hence $\sum_{i=1}^k d_i = 0$. ■

The space X naturally appears as a group under the pointwise product of functions. Moreover, an elementary dominated convergence argument shows that this product is jointly continuous under strong convergence. The following proposition relates the operator T and the group law on X. The proof is obtained by a density argument as in [9] Lemma 9 and Remark 2.3, and relies in our case on Theorem 2.6 and Lemma 2.2.

Proposition 2.6. *Let* $f_1, f_2 \in X$. *We have* $f_1 f_2 \in X$ *with* $(f_1f_2)^\infty = f_1^\infty f_2^\infty$ *and* $|f_1f_2|_{1/2} \leq |f_1|_{1/2} + |f_2|_{1/2}$. *Moreover, the product is jointly continuous with respect to* f_1 *and* f_2. *In addition*

$$T(\bar{f}_1) = -T(f_1) \quad \text{and} \quad T(f_1f_2) = T(f_1) + T(f_2)$$

where $\bar{f}_1$ *is the pointwise complex conjugate of* f_1.

For a given $f \in X$, it is possible to characterize the approximability by smooth maps both in the strong and in the weak $\dot{H}^{1/2}$–convergence in terms of the distribution $T(f)$ according to the following result (see also [9], [34] and [25]).

Theorem 2.7. *Let* $f \in X$. *Then* $T(f) = 0$ *if and only if* $f \in \overline{C^\infty_{\rm const}(\mathbb{R}^2;\mathbb{S}^1)}^{\|\cdot\|_{1/2}}$. *As a consequence, for any* $f \in X$, *there exists* $\{f_n\}_{n\in\mathbb{N}} \subset C^\infty_{\rm const}(\mathbb{R}^2;\mathbb{S}^1)$ *such that* $f_n \rightharpoonup f$ *weakly in* $\dot{H}^{1/2}$ *and* $f_n \to f$ *a.e. as* $n \to +\infty$.

Proof. Clearly if $f \in \overline{C^\infty_{\rm const}(\mathbb{R}^2;\mathbb{S}^1)}^{\|\cdot\|_{1/2}}$ then $T(f) = 0$ by continuity of T under the strong $\dot{H}^{1/2}$–convergence. The converse statement holds by Proposition 7.3. Weak density of smooth maps is given by Theorem 7.2 and the approximating sequences can be assumed to be constants outside compact sets again by Proposition 7.3. ■

The density results stated above, combined with the properties of the operator T allows to give a structure theorem and, at the same time, an existence result of for sequences of maps with prescribed singularities converging weakly to a constant. The representation formulat(2.16) is essentially contained in [9], Theorem 1 and Lemma 16, for $H^{1/2}$–maps from a compact surface of spherical type into $\mathbb{S}^1$ and it is somehow implicit in the proof of Theorem 5.1 (see Section 5), since there the current t is an integration over a countable union of oriented segments. Since we will not use this result in the paper, the proof will be omitted.

Theorem 2.8. *For each* $f \in X$, *there exists two sequences* $\{P_j\}_{j\in\mathbb{N}}, \{N_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^2$ *such that* $\sum_j |P_j - N_j| < +\infty$ *and*

$$< T(f), \varphi >= 2\pi \sum_{j\in\mathbb{N}} \big(\varphi(P_j) - \varphi(N_j)\big) \quad \forall\varphi \in \mathrm{Lip}(\mathbb{R}^2;\mathbb{R})\,. \tag{2.16}$$

Conversely, for any sequences $\{P_j\}_{j\in\mathbb{N}}, \{N_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^2$ such that $\sum_j |P_j - N_j| < +\infty$ there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset X$ such that (2.16) holds for every $n \in \mathbb{N}$ and $f_n \rightharpoonup \alpha \in \mathbb{S}^1$ weakly in $\dot{H}^{1/2}$ as $n \to +\infty$.

2.3. A-harmonic extensions.

Let us define the subspace $\dot{H}^1_0 \subset \dot{H}^1$ as $\dot{H}^1_0(\mathbb{R}^3_+;\mathbb{R}^2) := \overline{C^\infty_0(\mathbb{R}^3_+;\mathbb{R}^2)}^{\|\cdot\|_1}$ and denote by $\dot{H}^{-1}(\mathbb{R}^3_+)$ its dual space. The following result is a standard consequence of the properties of the homogeneous Sobolev spaces recalled in the Subsection 2.1, the Dirichlet principle and the Lax-Milgram lemma.

Proposition 2.7. *Let $A \in \mathcal{A}$ and $f \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$. There exists a unique minimum u_f of E_A in $\dot{H}^1_f := \big\{u \in \dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)\,;\, \mathrm{Tr}\, u = f\big\}$. Moreover, u_f is the unique weak solution of*

$$\begin{cases} \mathrm{div}\,(A\nabla u) = 0 & \text{in } \dot{H}^{-1}(\mathbb{R}^3_+;\mathbb{R}^2) \\ \mathrm{Tr}\; u = f & \text{on } \mathbb{R}^2 \end{cases}, \tag{2.17}$$

and

$$C_1^{-1}|f|_{1/2} \leq \|\nabla u_f\|_{L^2} \leq C_1|f|_{1/2}\,, \tag{2.18}$$

for a constant $C_1 = C_1(A)$ which only depends on the ellipticity constants of A.

In addition, if f_1, $f_2 \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$ and u_i denotes the A_i–harmonic extension of f_i relative to some $A_i \in \mathcal{A}$, $i = 1, 2$, then

$$\|\nabla(u_1 - u_2)\|_{L^2(\mathbb{R}^3_+)} \leq C_2\bigg(|f_1 - f_2|_{1/2} + \big(|f_1|_{1/2} + |f_2|_{1/2}\big)\|A_1 - A_2\|_\infty\bigg)\,, \tag{2.19}$$

for a constant $C_2 = C_2(A_1, A_2) > 0$ which only depends on the ellipticity constants of A_1 and A_2.

We collect some standard regularity and compactness properties of solutions of (2.17). The proof is exactly the same of the corresponding result in [33], using the lattice property of Proposition 2.4, therefore it will be omitted.

Proposition 2.8. *Let $A \in \mathcal{A}$, $f \in X$ and u_f as in Proposition (2.7). Then,*

(i) if $f \in X$, then $u \in C^0(\mathbb{R}^3_+;\mathbb{R}^2)$ and $|u| \leq 1$ in $\mathbb{R}^3_+$ with strict inequality whenever f is nonconstant,

(ii) if $A \in C^\infty(\mathbb{R}^3_+;\mathcal{S}^+)$ then $u \in C^\infty(\mathbb{R}^3_+;\mathbb{R}^2)$,

(iii) let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in X and $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ a sequence of matrix fields with uniform ellipticity constants λ and Λ. Denote by $\{u_n\}_{n\in\mathbb{N}}$ the corresponding sequence of solutions of (2.17). Then $\{u_n\}_{n\in\mathbb{N}}$ is compact in $C^0_{\mathrm{loc}}(\mathbb{R}^3_+;\mathbb{R}^2)$,

(iv) if $\{A_n\}_{n\in\mathbb{N}}$ is compact in $L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}^3_+};\mathcal{S}^+)$, then $\{u_n\}_{n\in\mathbb{N}}$ is compact in $H^1_{\mathrm{loc}}(\overline{\mathbb{R}^3_+};\mathbb{R}^2)$,

(v) if $\{A_n\}_{n\in\mathbb{N}}$ is compact in $L^\infty(\mathbb{R}^3_+;\mathcal{S}^+)$ and $\{f_n\}_{n\in\mathbb{N}}$ is compact in X then $\{u_n\}_{n\in\mathbb{N}}$ is compact in $\dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$.

3. Length structures and Hamilton-Jacobi equations

In this section, our main purpose is to introduce a distance d_A associated in a canonical way to the matrix field $\mathrm{Cof}\, A$ which coincides with the standard distance δ_A below whenever A has continuous entries,

$$\delta_A(P,Q) = \operatorname*{Inf}_{\gamma} \int_0^1 \sqrt{\mathrm{Cof}\, A(\gamma(t))\dot{\gamma}(t)\cdot\dot{\gamma}(t)}\; dt, \tag{3.1}$$

where the infimum is taken over all $\gamma \in \mathrm{Lip}_{P,Q}\big([0,1];\overline{\mathbb{R}^3_+}\big)$. Here, $\mathrm{Lip}_{P,Q}\big([0,1];\overline{\mathbb{R}^3_+}\big)$ denotes the set of all Lipschitz curves γ from $[0,1]$ with values into $\overline{\mathbb{R}^3_+}$ such that $\gamma(0) = P$ and $\gamma(1) = Q$. As it will be transparent in the next sections, the construction of a generalized distance d_A will be the crucial tool in giving the lower bound for the energy functional $\mathcal{E}_A(f)$. We start by presenting both the energy and the length functional in a natural unifying way when the involved matrices are continuous. The passage to the measurable case, though harmless for the energy functional, is much more delicate for the length functional and will occupy us for the whole section.

3.1. Remark on the induced Riemannian structure.

Consider a Riemannian metric $g = \big(g_{ij}\big)$ on $\overline{\mathbb{R}^3_+}$ with continuous entries which possibly satisfies the natural product-type assumption on the boundary,

$$g(x_1,x_2,0) = g_{11}(x_1,x_2,0)dx_1^2 + 2g_{12}(x_1,x_2,0)dx_1dx_2 + \\ + g_{22}(x_1,x_2,0)dx_2^2 + g_{33}(x_1,x_2,0)dx_3^2 \qquad \forall (x_1,x_2) \in \mathbb{R}^2, \tag{3.2}$$

and let $\hat{g} = \big(g^{ij}\big)$ be the dual metric (which obviously inherits the same structure assumption on the boundary). For curves $\gamma : [0,1] \to \overline{\mathbb{R}^3_+}$, the squared length of the tangent vector $\dot\gamma(t)$ at the point $\gamma(t)$ is given by $|\dot\gamma(t)|^2_{g_{\gamma(t)}} = g_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))$. For maps $u : (\mathbb{R}^3_+, g) \to (\mathbb{R}^2, \mathrm{Id})$ the squared length of the differential $du = \partial_1 u\, dx_1 + \partial_2 u\, dx_2 + \partial_3 u\, dx_3$ at a point $x = (x_1,x_2,x_3)$ is given by $|du|^2_{\hat{g}_x} = \hat{g}_x(du^1,du^1) + \hat{g}_x(du^2,du^2)$. Taking this definitions into account, one easily get that

$$E_A(u) = \frac{1}{2}\int_{\mathbb{R}^3_+} |du|^2_{\hat g}\, d\,\mathrm{Vol}_g\,, \quad \mathbb{L}_{d_A}(\gamma) = \int_0^1 |\gamma'(t)|_{g_{\gamma(t)}}\, dt \qquad \Longleftrightarrow \qquad g = \mathrm{Cof}\, A\,,$$

so both the functional we are interested in have a natural geometric interpretation and they come from the same metric tensor $g = \mathrm{Cof}\, A$.

3.2. Construction of the distance d_A

First of all, we call the attention of the reader to the fact that, in the case of a measurable matrix field A (or in other words, a measurable metric tensor g), there is no way to define a distance via the usual formula (3.1) since A is not well defined on curves which are sets of null Lebesgue measure. This difficulty has been overcome in [31] and the main idea was to thicken the curves in order to construct a generalized length structure in the sense of [26]. Let us explain this procedure in details.

For two points P and Q in $\mathbb{R}^3_+$, we define the class $\mathcal{P}(P,Q)$ to be the set of all finite collections of segments $\mathcal{F} = ([\alpha_k,\beta_k])_{k=1}^{n(\mathcal{F})}$ such that $\alpha_k, \beta_k \in \mathbb{R}^3_+$, $\beta_k = \alpha_{k+1}$, $\beta_k \neq \alpha_k$, $\alpha_1 = P$ and $\beta_{n(\mathcal{F})} = Q$. Next we define the length $\ell_A\big(\mathcal{F}\big)$ of an element $\mathcal{F} \in \mathcal{P}(P,Q)$ by

$$\ell_A\big(\mathcal{F}\big) = \sum_{k=1}^{n(\mathcal{F})} \liminf_{\varepsilon \to 0^+} \frac{1}{\pi\varepsilon^2} \int_{\Xi\big([\alpha_k,\beta_k],\varepsilon\big)\cap\mathbb{R}^3_+} \sqrt{\mathrm{Cof}\, A(x)\tau_k \cdot \tau_k}\, dx.$$

where $\tau_k = \frac{\beta_k - \alpha_k}{|\beta_k - \alpha_k|}$ and $\Xi\big([\alpha_k,\beta_k],\varepsilon\big) = \big\{\, x \in \mathbb{R}^3\,;\, \mathrm{dist}\big(x,[\alpha_k,\beta_k]\big) \le \varepsilon \big\}$ and then we consider the function $d_A : \mathbb{R}^3_+ \times \mathbb{R}^3_+ \to \mathbb{R}$ defined by

$$d_A(P,Q) = \operatorname*{Inf}_{\mathcal{F}\in\mathcal{P}(P,Q)} \ell_A(\mathcal{F}).$$

In the next sections, we shall also consider for any $r>0$, the distance induced by $\operatorname{Cof} A$ on the strip domain $\Omega_r := \mathbb{R}^2 \times (0,r)$ corresponding to the distance

$$\delta_A^r(P,Q) = \operatorname{Inf}\left\{ \int_0^1 \sqrt{\operatorname{Cof} A(\gamma(t))\dot\gamma(t)\cdot\dot\gamma(t)}\,dt\,;\ \gamma \in \operatorname{Lip}_{P,Q}\big([0,1];\overline{\Omega}_r\big)\right\} \tag{3.3}$$

in the case where A is smooth. For $r>0$, we define $d_A^r : \Omega_r \times \Omega_r \to \mathbb{R}$ by

$$d_A^r(P,Q) = \operatorname*{Inf}_{\mathcal{F}\in\mathcal{P}_r(P,Q)} \ell_A(\mathcal{F})$$

where $\mathcal{P}_r(P,Q) = \big\{\mathcal{F} = ([\alpha_k,\beta_k])_{k=1}^n \in \mathcal{P}(P,Q)\,;\ [\alpha_k,\beta_k]\subset\Omega_r\ \forall k\big\}$.

In the following proposition, we extend d_A (resp. d_A^r) to $\overline{\mathbb{R}^3_+}\times\overline{\mathbb{R}^3_+}$ (resp. $\overline{\Omega}_r\times\overline{\Omega}_r$) and we establish the metric character of d_A (resp. d_A^r) as well as the identity between d_A and δ_A (resp. between d_A^r and δ_A^r) for a smooth matrix field A. We shall use the notations $\Omega_\infty := \mathbb{R}^3_+$, $d_A^\infty := d_A$, $\delta_A^\infty := \delta_A$ and for $P,Q\in\mathbb{R}^3_+$, $\mathcal{P}_\infty(P,Q) := \mathcal{P}(P,Q)$. The proof follows as in [31] but we reproduce it here for the convenience of the reader.

Proposition 3.1. *For any $0<r\leq\infty$, d_A^r induces a distance on $\overline{\Omega}_r$ which is equivalent to the Euclidean distance. Moreover d_A^r agrees with δ_A^r whenever A is continuous.*

Proof. *Step 1.* Let $P,Q\in\Omega_r$ and let $\mathcal{F} = ([\alpha_1,\beta_1],\dots,[\alpha_n,\beta_n])$ be any element of $\mathcal{P}_r(P,Q)$. From assumption (1.9), we get that

$$\lambda\sum_{k=1}^n|\alpha_k-\beta_k| = \sum_{k=1}^n \lim_{\varepsilon\to0^+}\frac{\lambda}{\pi\varepsilon^2}\big|\Xi\big([\alpha_k,\beta_k],\varepsilon\big)\cap\mathbb{R}^3_+\big| \leq \ell_A(\mathcal{F}) \leq$$
$$\leq \Lambda\sum_{k=1}^n \lim_{\varepsilon\to0^+}\frac{1}{\pi\varepsilon^2}\big|\Xi\big([\alpha_k,\beta_k],\varepsilon\big)\cap\mathbb{R}^3_+\big| = \Lambda\sum_{k=1}^n|\alpha_k-\beta_k|\,.$$

Taking the infimum over all $\mathcal{F}\in\mathcal{P}_r(P,Q)$, we infer that

$$\lambda|P-Q| \leq d_A^r(P,Q) \leq \Lambda|P-Q| \quad \forall P,Q\in\Omega_r\,. \tag{3.4}$$

From (3.4) we deduce that $d_A^r(P,Q)=0$ if and only if $P=Q$. Let us now prove that d_A^r is symmetric. Let $P,Q\in\Omega_r$ and $\delta>0$ arbitrary small. Obviously, we can find an element $\mathcal{F}_\delta = \big([\alpha_1,\beta_2],\dots,[\alpha_n,\beta_n]\big)$ in $\mathcal{P}_r(P,Q)$ satisfying $\ell_A\big(\mathcal{F}_\delta\big)\leq d_A^r(P,Q)+\delta$. Then for $\mathcal{F}'_\delta = \big([\beta_n,\alpha_n],\dots,[\beta_1,\alpha_1]\big)\in\mathcal{P}_r(Q,P)$, we have $d_A^r(Q,P)\leq\ell_A\big(\mathcal{F}'_\delta\big) = \ell_A\big(\mathcal{F}_\delta\big)\leq d_A^r(P,Q)+\delta$. Since δ is arbitrary, we obtain $d_A^r(Q,P)\leq d_A^r(P,Q)$ and we conclude that $d_A^r(P,Q)=d_A^r(Q,P)$ inverting the roles of P and Q. The triangle inequality is immediate since the juxtaposition of $\mathcal{F}_1\in\mathcal{P}_r(P,R)$ with $\mathcal{F}_2\in\mathcal{P}_r(R,Q)$ is an element of $\mathcal{P}_r(P,Q)$. Hence d_A^r defines a distance on Ω_r satisfying (3.4). Therefore the distance d_A^r extends uniquely to $\overline{\Omega}_r\times\overline{\Omega}_r$ into a distance function that we still denote by d_A^r. By continuity, d_A^r satisfies (3.4) in $\overline{\Omega}_r$.

Step 2. If A has continuous entries, it is easy to see that for a segment $[\alpha,\beta]\subset\Omega_r$ with $\alpha\neq\beta$, we have

$$\lim_{\varepsilon\to0^+}\frac{1}{\pi\varepsilon^2}\int_{\Xi\left([\alpha,\beta],\varepsilon\right)\cap\mathbb{R}^3_+}\sqrt{\operatorname{Cof}A(x)\tau\cdot\tau}\,dx = \int_{[\alpha,\beta]}\sqrt{\operatorname{Cof}A(s)\tau\cdot\tau}\,ds,$$

where $\tau = \frac{\beta-\alpha}{|\beta-\alpha|}$. Hence we obtain for any $\mathcal{F} = ([\alpha_1,\beta_1],\dots,[\alpha_n,\beta_n])$ in $\mathcal{P}_r(P,Q)$ with $P,Q\in\Omega_r$,

$$\ell_A\big(\mathcal{F}\big) = \int_{\cup_{k=1}^n[\alpha_k,\beta_k]}\sqrt{\operatorname{Cof}A(s)\tau_k\cdot\tau_k}\,ds. \tag{3.5}$$

Since A is continuous, the infimum in (3.3) can be taken over all piecewise affine curves $\gamma : [0,1] \to \Omega_r$ such that $\gamma(0) = P$ and $\gamma(1) = Q$ and we infer from (3.5) that $d^r_A(P,Q) = \delta^r_A(P,Q)$. Then $d^r_A \equiv \delta^r_A$ on $\Omega_r \times \Omega_r$ which implies that the equality holds in $\overline{\Omega}_r \times \overline{\Omega}_r$ by continuity. ■

3.3. Geodesic structure and the length functional

In this subsection, we study some geometric properties of the distance d^r_A. For this purpose, we introduce its associated length functional. Recall that to any metric space (M,d) is associated a length functional $\mathbb{L}_d$ defined by

$$\mathbb{L}_d(\gamma) = \mathrm{Sup}\left\{ \sum_{k=0}^{m-1} d\big(\gamma(t_k), \gamma(t_{k+1})\big)\,;\ 0 = t_0 < t_1 < \ldots < t_m = 1\,,\ m \in \mathbb{N} \right\} \tag{3.6}$$

where $\gamma : [0,1] \to M$ is any continuous curve. Note that $\mathbb{L}_d$ is always lower semicontinuous on $C^0\big([0,1], M\big)$ endowed with the topology of the uniform convergence on $[0,1]$.

Definition 3.1. *A distance d is said to be geodesic on M if for any $P,Q \in M$,*

$$d(P,Q) = \mathrm{Inf}\ \mathbb{L}_d(\gamma)$$

where the infimum is taken over all continuous curves $\gamma : [0,1] \to M$ such that $\gamma(0) = P$ and $\gamma(1) = Q$.

A geodesic distance d satisfies most of the time the usual properties of the Euclidean distance. For instance, if (M,d) is locally compact and complete, two arbitrary points in M can be linked by a curve of minimal length. Applying this concept to our distance d^r_A, we obtain the following proposition.

Proposition 3.2. *For any $0 < r \leq \infty$, the distance d^r_A is geodesic on $\overline{\Omega}_r$. In addition, for any points $P,Q \in \overline{\Omega}_r$,*

$$d^r_A(P,Q) = \mathrm{Min}\ \big\{\mathbb{L}_{d^r_A}(\gamma)\,;\ \gamma \in \mathrm{Lip}_{P,Q}\big([0,1]; \overline{\Omega}_r\big)\big\} \tag{3.7}$$

and the minimum in (3.7) *is achieved by a curve $\gamma^r_{P,Q}$ which satisfies*

$$d^r_A\left(\gamma^r_{PQ}(t), \gamma^r_{PQ}(t')\right) = \mathbb{L}_{d^r_A}(\gamma^r_{PQ})|t - t'| \quad \forall t, t' \in [0,1]. \tag{3.8}$$

Proof. The geodesic character of d^r_A follows as in the proof of Proposition 2.1 in [31] and we shall omit it. Since d^r_A is equivalent to Euclidean distance, $\overline{\Omega}_r$ endowed with d^r_A defines a complete and locally compact metric space. By the Hopf-Rinow Theorem (see [26], Chap. I), for any $P,Q \in \overline{\Omega}_r$, there exists $\tilde{\gamma} : I = [0,C] \to \overline{\Omega}_r$ such that $\tilde{\gamma}(0) = P$, $\tilde{\gamma}(C) = Q$ and

$$d^r_A\big(\tilde{\gamma}(t), \tilde{\gamma}(t')\big) = |t - t'| \quad \text{for all } t, t' \in I$$

with $C = d^r_A(P,Q)$. Setting for $t \in [0,1]$, $\gamma^r_{PQ}(t) = \tilde{\gamma}(Ct)$, we easily check that the curve γ^r_{PQ} satisfies (3.8) so that γ^r_{PQ} is Lipschitz continuous by (3.4). Then (3.7) follows from (3.8). ■

Remark 3.1. We point out that, for any $r \in (0,\infty]$ and any $P,Q \in \overline{\Omega}_r$, the curve γ^r_{PQ} given by Proposition 3.2 is a $\frac{\Lambda}{\lambda}\,|P - Q|$–Lipschitz curve for the Euclidean distance.

Remark 3.2. Since d^r_A is equivalent to the Euclidean distance on $\overline{\Omega}_r$, for any $0 < r \leq \infty$, the functional $\mathbb{L}_{d^r_A}$ is lower semicontinuous on $C^0([0,1], \overline{\Omega}_r)$ endowed with the topology of the uniform convergence on $[0,1]$ induced by the Euclidean distance on $\overline{\Omega}_r$.

Remark 3.3. It follows from the definition of $\mathbb{L}_{d^r_A}$ and (3.4) that for any $r>0$,

$$\lambda\,\mathbb{L}_{\rm Id}(\gamma)\leq \mathbb{L}_{d^r_A}(\gamma)\leq \Lambda\,\mathbb{L}_{\rm Id}(\gamma)\quad \forall\gamma\in C^0\big([0,1];\overline{\Omega}_r\big) \tag{3.9}$$

where $\mathbb{L}_{\rm Id}(\gamma)$ stands for the usual Euclidean length of γ (i.e. for $A=\mathrm{Id}$).

Through the results in [19] and [36], the geodesic property of d^r_A will allow us to obtain in Proposition 3.3 below, an integral representation of the length functional generalizing formula (3.1) to the measurable case. In general the resulting integrand has no reason to be the square root of a quadratic form (of the $\dot\gamma$–variable) as in (3.1) but it still enjoys some homogeneity and convexity properties. The proof of Proposition 3.3 follows the one in [31] and it is presented here to introduce some useful quantities.

Definition 3.2. Let $0<r\leq\infty$. A Borel measurable function $\mathcal{L}:\overline{\Omega}_r\times\mathbb{R}^3\to[0,+\infty)$ is said to be a (weak) *Finsler metric* on $\overline{\Omega}_r$ if $\mathcal{L}(x,\cdot)$ is positively 1-homogeneous for every $x\in\overline{\Omega}_r$ and convex for almost every $x\in\overline{\Omega}_r$.

Proposition 3.3. *For every $0<r\leq\infty$, there is a Finsler metric $\mathcal{L}^r_A:\overline{\Omega}_r\times\mathbb{R}^3\to[0,+\infty)$ such that for any Lipschitz curve $\gamma:[0,1]\to\overline{\Omega}_r$,*

$$\mathbb{L}_{d^r_A}(\gamma)=\int_0^1\mathcal{L}^r_A\big(\gamma(t),\dot\gamma(t)\big)dt. \tag{3.10}$$

Proof. We set $G_r=\mathbb{R}^2\times(-1,r+1)$ and Π_rP the projection of $P\in G_r$ on $\overline{\Omega}_r$. Setting $P_\perp=P-\Pi_rP$ for $P\in G_r$, we define $D^r_A:G_r\times G_r\to[0,+\infty)$ by

$$D^r_A(P,Q)=d^r_A(\Pi_rP,\Pi_rQ)+|P_\perp-Q_\perp|. \tag{3.11}$$

We easily check that D^r_A defines a distance on G_r. Then we consider for $P,Q\in G_r$,

$$\tilde D^r_A(P,Q)=\mathrm{Inf}\ \mathbb{L}_{D^r_A}(\gamma), \tag{3.12}$$

where the infimum is taken over all $\gamma\in C^0\big([0,1];G_r\big)$ satisfying $\gamma(0)=P$ and $\gamma(1)=Q$. We also easily verify that $\tilde D^r_A$ defines a distance on G_r and it follows from Proposition 1.6 in [26] that

$$\mathbb{L}_{\tilde D^r_A}=\mathbb{L}_{D^r_A}\quad\text{on } C^0\big([0,1];G_r\big). \tag{3.13}$$

Therefore $\tilde D^r_A$ is a geodesic distance on G_r. Moreover we infer from (3.4) that $\tilde D^r_A$ is equivalent to the Euclidean distance on G_r. Now we consider $\tilde{\mathcal{L}}^r_A:G_r\times\mathbb{R}^3\to[0,+\infty)$ defined by

$$\tilde{\mathcal{L}}^r_A(x,\nu)=\limsup_{t\to0^+}\frac{\tilde D^r_A(x,x+t\nu)}{t}.$$

By the results in [36], $\tilde{\mathcal{L}}^r_A$ is Borel measurable, positively 1-homogeneous in ν for every $x\in G_r$ and convex in ν for almost every $x\in G_r$. By Theorem 2.5 in [19], we have for every Lipschitz curve $\gamma:[0,1]\to G_r$,

$$\mathbb{L}_{\tilde D^r_A}(\gamma)=\int_0^1\tilde{\mathcal{L}}^r_A\left(\gamma(t),\dot\gamma(t)\right)dt. \tag{3.14}$$

Since $D^r_A=d^r_A$ in $\overline{\Omega}_r\times\overline{\Omega}_r$, we deduce from (3.6) that

$$\mathbb{L}_{\tilde D^r_A}=\mathbb{L}_{D^r_A}=\mathbb{L}_{d^r_A}\quad\text{on } C^0\left([0,1];\overline{\Omega}_r\right). \tag{3.15}$$

Setting $\mathcal{L}^r_A$ to be the restriction of $\tilde{\mathcal{L}}^r_A$ to $\overline{\Omega}_r\times\mathbb{R}^3$, we obtain (3.10) combining (3.14) and (3.15). ∎

3.4. The induced distance on the plane

With the length functional $\mathbb{L}_{d_A}$ in hands, we may now consider the induced distance $\bar{d}_A$ on $\mathbb{R}^2$ (identified with $\partial\mathbb{R}^3_+$) computing lengths of curves lying in the plane. More precisely, for $P,Q\in\mathbb{R}^2\simeq\partial\mathbb{R}^3_+$, we set

$$\bar{d}_A(P,Q)=\mathrm{Inf}\left\{\mathbb{L}_{d_A}(\gamma)\,;\ \gamma\in\mathrm{Lip}_{P,Q}\big([0,1];\partial\mathbb{R}^3_+\big)\right\} \tag{3.16}$$

The following proposition shows us that $\bar{d}_A$ inherits the geodesic property of d_A as well as its consequences.

Proposition 3.4. *The distance $\bar{d}_A$ is geodesic on $\mathbb{R}^2$ so that $\mathbb{L}_{\bar{d}_A}\equiv\mathbb{L}_{d_A}$ on $C^0([0,1];\partial\mathbb{R}^3_+)$. In addition, for any points $P,Q\in\mathbb{R}^2$, the infimum in (3.16) is achieved by a curve $\bar{\gamma}_{P,Q}$ which satisfies*

$$\bar{d}_A\big(\bar{\gamma}_{PQ}(t),\bar{\gamma}_{PQ}(t')\big)=\mathbb{L}_{d_A}(\bar{\gamma}_{PQ})|t-t'|\quad\forall t,t'\in[0,1]. \tag{3.17}$$

Proof. We just show that $\bar{d}_A$ is geodesic. Then by Proposition 1.6 in [26], $\mathbb{L}_{\bar{d}_A}=\mathbb{L}_{d_A}$ on $C^0([0,1];\partial\mathbb{R}^3_+)$ and the rest of the proof will follow as in Proposition 3.2. First we infer from (3.9) that

$$\lambda|P-Q|\le\bar{d}_A(P,Q)\le\Lambda|P-Q|\quad\forall P,Q\in\mathbb{R}^2. \tag{3.18}$$

so that $\mathbb{R}^2$ endowed with $\bar{d}_A$ defines a complete and locally compact metric space. By Theorem 1.8 in [26], to show the geodesic character of $\bar{d}_A$, it suffices to prove that for any $P,Q\in\mathbb{R}^2$ and any $\delta>0$, we can find $R\in\mathbb{R}^2$ satisfying

$$\max\big(\bar{d}_A(P,R),\bar{d}_A(R,Q)\big)\le\frac{1}{2}\,\bar{d}_A(P,Q)+\delta.$$

We proceed as follows. For $P,Q\in\mathbb{R}^2$ and $\delta>0$ given, we fix some $\gamma_\delta\in\mathrm{Lip}_{P,Q}\big([0,1];\partial\mathbb{R}^3_+\big)$ such that $\mathbb{L}_{d_A}(\gamma_\delta)\le\bar{d}_A(P,Q)+2\delta$. By Proposition 3.3, $\mathbb{L}_{d_A}(\gamma_\delta)=\int_0^1\mathcal{L}^\infty_A\big(\gamma_\delta(t),\dot{\gamma}_\delta(t)\big)dt$ so that $s\mapsto\mathcal{L}^\infty_A\big(\gamma_\delta(s),\dot{\gamma}_\delta(s)\big)$ belongs to $L^1([0,1])$. Consequently, $F(s)=\int_0^s\mathcal{L}^\infty_A\big(\gamma_\delta(t),\dot{\gamma}_\delta(t)\big)dt$ is a continuous function of $s\in[0,1]$ and it satisfies $F(0)=0$ and $F(1)=\mathbb{L}_{d_A}(\gamma_\delta)$. Hence there exists $s_\star\in(0,1)$ such that

$$\int_0^{s_\star}\mathcal{L}^\infty_A\big(\gamma_\delta(t),\dot{\gamma}_\delta(t)\big)dt=\int_{s_\star}^1\mathcal{L}^\infty_A\big(\gamma_\delta(t),\dot{\gamma}_\delta(t)\big)dt=\frac{1}{2}\,\mathbb{L}_{d_A}(\gamma_\delta)\le\frac{1}{2}\,\bar{d}_A(P,Q)+\delta.$$

We now set $R=\gamma_\delta(s_\star)\in\mathbb{R}^2$ and for $t\in[0,1]$,

$$\gamma_1(t)=\gamma_\delta(s_\star t)\quad\text{and}\quad\gamma_2(t)=\gamma_\delta(s_\star+(1-s_\star)t).$$

Then $\gamma_1\in\mathrm{Lip}_{P,R}\big([0,1];\partial\mathbb{R}^3_+\big)$ and $\gamma_2\in\mathrm{Lip}_{R,Q}\big([0,1];\partial\mathbb{R}^3_+\big)$. Using Proposition 3.3 and the homogeneity of $\mathcal{L}^\infty_A$ with respect to the second variable, we derive that

$$\bar{d}_A(P,R)\le\int_0^1\mathcal{L}^\infty_A\big(\gamma_1(t),\dot{\gamma}_1(t)\big)dt=\int_0^{s_\star}\mathcal{L}^\infty_A\big(\gamma_\delta(t),\dot{\gamma}_\delta(t)\big)dt\le\frac{1}{2}\,\bar{d}_A(P,Q)+\delta,$$
$$\bar{d}_A(R,Q)\le\int_0^1\mathcal{L}^\infty_A\big(\gamma_2(t),\dot{\gamma}_2(t)\big)dt=\int_{s_\star}^1\mathcal{L}^\infty_A\big(\gamma_\delta(t),\dot{\gamma}_\delta(t)\big)dt\le\frac{1}{2}\,\bar{d}_A(P,Q)+\delta,$$

and the proof is complete. ∎

From Proposition 3.1 and the definition of the length functional, one easily see that $\bar{d}_A$ actually coincides with the distance $\bar{\delta}_A$ defined for $P,Q\in\mathbb{R}^2$ by

$$\bar{\delta}_A(P,Q)=\mathrm{Inf}\left\{\int_0^1\sqrt{\mathrm{Cof}\,A(\gamma(t))\dot{\gamma}(t)\cdot\dot{\gamma}(t)}\;dt\,;\ \gamma\in\mathrm{Lip}_{P,Q}\big([0,1];\partial\mathbb{R}^3_+\big)\right\}$$

whenever A is continuous. Moreover $\bar{\delta}_A$ can be obtained computing lengths of curves lying in $\overline{\mathbb{R}^3_+}$ which are closer and closer to the plane. In other words, $\bar{\delta}_A$ is the limit as $r \to 0$ of the distances δ^r_A (defined in (3.3)) restricted to $\mathbb{R}^2 \times \mathbb{R}^2$. This fact, absolutely trivial in the continuous case, remains true when one deals with a measurable field A.

Proposition 3.5. *The family of distances $(d^r_A)_{r>0}$ restricted to $\mathbb{R}^2 \times \mathbb{R}^2$ converges locally uniformly to the distance $\bar{d}_A$ as $r \to 0$.*

Proof. We start the proof of Proposition 3.5 by showing that the distances d^r_A locally coincide with the distance d_A. Then we shall derive that the length functionals $\mathbb{L}_{d^r_A}$ are equal to $\mathbb{L}_{d_A}$ near the plane. This represents a crucial point of the proof.

Lemma 3.1. *Let $r > 0$ and $P_0 \in \Omega_r \cup \partial\mathbb{R}^3_+$. There exists $\eta = \eta(P_0, r) > 0$ such that $\overline{B}_\eta(P_0) \cap \overline{\mathbb{R}^3_+} \subset \Omega_r \cup \partial\mathbb{R}^3_+$ and*

$$d^r_A(P,Q) = d_A(P,Q) \quad \forall P,Q \in \overline{B}_\eta(P_0) \cap \overline{\mathbb{R}^3_+}. \tag{3.19}$$

Proof. Let $P_0 \in \Omega_r \cup \partial\mathbb{R}^3_+$ and $\delta > 0$ such that $\overline{B}_\delta(P_0) \cap \overline{\mathbb{R}^3_+} \subset \Omega_r \cup \partial\mathbb{R}^3_+$. We set $\eta = \big(1 + \frac{2\Lambda}{\lambda}\big)^{-1}\delta$. For any points $P, Q \in \overline{B}_\eta(P_0) \cap \mathbb{R}^3_+ \subset \Omega_r$, we have $d_A(P,Q) \le \Lambda|P-Q| \le 2\Lambda\eta$. Let $0 < \varepsilon < 2\Lambda\eta$ be arbitrary small. By definition, we can find $\mathcal{F}_\varepsilon = ([\alpha_1,\beta_1], \dots, [\alpha_n,\beta_n]) \in \mathcal{P}(P,Q)$ such that

$$\ell_A\big(\mathcal{F}_\varepsilon\big) \le d_A(P,Q) + \varepsilon < 4\Lambda\eta.$$

From (1.9) we infer that $\ell_A\big(\mathcal{F}_\varepsilon\big) \ge \lambda\mathcal{H}^1\big(\cup_{k=1}^n [\alpha_k,\beta_k]\big)$. Hence $\mathcal{H}^1\big(\cup_{k=1}^n [\alpha_k,\beta_k]\big) \le \frac{4\Lambda\eta}{\lambda}$ which implies that $\cup_{k=1}^n [\alpha_k,\beta_k] \subset \overline{B}_\delta(P_0) \cap \mathbb{R}^3_+$. In particular, $\cup_{k=1}^n [\alpha_k,\beta_k] \subset \Omega_r$ so that $\mathcal{F}_\varepsilon$ belongs to $\mathcal{P}_r(P,Q)$ and thus

$$d^r_A(P,Q) \le \ell_A\big(\mathcal{F}_\varepsilon\big) \le d_A(P,Q) + \varepsilon.$$

Then we derive that $d^r_A(P,Q) \le d_A(P,Q)$ from the arbitrariness of ε. On the other hand, the reverse inequality is trivial and we conclude that (3.19) holds in $\overline{B}_\eta(P_0) \cap \mathbb{R}^3_+$. Then we recover (3.19) in $\overline{B}_\eta(P_0) \cap \overline{\mathbb{R}^3_+}$ by continuity. ■

Corollary 3.1. *For any $r > 0$, we have*

$$\mathbb{L}_{d^r_A}(\gamma) = \mathbb{L}_{d_A}(\gamma) \quad \forall\, \gamma \in \mathrm{Lip}\big([0,1]; \Omega_r \cup \partial\mathbb{R}^3_+\big). \tag{3.20}$$

Proof. For $r > 0$, let P_0 be an arbitrary point in $\Omega_r \cup \partial\mathbb{R}^3_+$. By Lemma 3.1, there exists $\eta > 0$ such that $\overline{B}_\eta(P_0) \cap \overline{\mathbb{R}^3_+} \subset \Omega_r \cup \partial\mathbb{R}^3_+$ and $d^r_A \equiv d_A$ on $\overline{B}_\eta(P_0) \cap \overline{\mathbb{R}^3_+}$. Consequently, the distances D^r_A and D^∞_A defined by (3.11) coincide on $\overline{B}_\eta(P_0) \cap G_\infty \subset G_r$ (we are using the same notations as in the proof of Proposition 3.3). In particular, we have

$$\mathbb{L}_{D^r_A}(\gamma) = \mathbb{L}_{D^\infty_A}(\gamma) \quad \forall\, \gamma \in C^0\big([0,1]; \overline{B}_\eta(P_0) \cap G_\infty\big). \tag{3.21}$$

We claim that the distances $\tilde{D}^r_A$ and $\tilde{D}^\infty_A$ defined by (3.12) coincide on $\overline{B}_{\tilde{\eta}}(P_0) \cap G_\infty$ for some $0 < \tilde{\eta} < \eta$. First, we easily check that there exist two positive constants $\tilde{\lambda}$ and $\tilde{\Lambda}$ which only depend on λ and Λ, such that

$$\tilde{\lambda}|P-Q| \le D^s_A(P,Q) \le \tilde{\Lambda}|P-Q| \quad \forall\, P,Q \in G_s \text{ and } s \in \{r,\infty\}$$

(here we use (3.4) and that $\|\nabla\Pi_s\|_\infty \le C$ for a constant C independent of s). Then it follows that

$$\tilde{\lambda}\,\mathbb{L}_{\mathrm{Id}}(\gamma) \le \mathbb{L}_{D^s_A}(\gamma) \le \tilde{\Lambda}\,\mathbb{L}_{\mathrm{Id}}(\gamma) \quad \forall\, \gamma \in C^0([0,1]; G_s) \text{ and } s \in \{r,\infty\}$$

(recall that $\mathbb{L}_{\mathrm{Id}}(\gamma)$ stands for the Euclidean length of the curve γ) which leads to

$$\tilde{\lambda}|P-Q| \leq \tilde{D}^s_A(P,Q) \leq \tilde{\Lambda}|P-Q| \quad \forall\, P,Q \in G_s \text{ and } s \in \{r,\infty\}. \tag{3.22}$$

We set $\tilde{\eta} = \big(1+\frac{2\tilde{\Lambda}}{\tilde{\lambda}}\big)^{-1}\eta$. For any points $P,Q \in \overline{B}_{\tilde{\eta}}(P_0)\cap G_\infty \subset G_r$, we have $\tilde{D}^r_A(P,Q) \leq 2\tilde{\Lambda}\tilde{\eta}$ by (3.22). Let $0<\varepsilon<2\tilde{\Lambda}\tilde{\eta}$ be arbitrary small. By definition, we can find $\gamma_\varepsilon \in C^0([0,1];G_r)$ such that $\gamma_\varepsilon(0)=P$, $\gamma_\varepsilon(1)=Q$ and

$$\tilde{\lambda}\,\mathbb{L}_{\mathrm{Id}}(\gamma_\varepsilon) \leq \mathbb{L}_{D^r_A}(\gamma_\varepsilon) \leq \tilde{D}^r_A(P,Q)+\varepsilon < 4\tilde{\Lambda}\tilde{\eta}.$$

Hence the Euclidean length of γ_ε cannot exceed $\frac{4\tilde{\Lambda}\eta}{\tilde{\lambda}}$ which implies that $\gamma_\varepsilon([0,1]) \subset \overline{B}_\eta(P_0)\cap G_\infty$. By (3.21) we have

$$\tilde{D}^\infty_A(P,Q) \leq \mathbb{L}_{D^\infty_A}(\gamma_\varepsilon) = \mathbb{L}_{D^r_A}(\gamma_\varepsilon) \leq \tilde{D}^r_A(P,Q)+\varepsilon.$$

From the arbitrariness of ε we deduce that $\tilde{D}^\infty_A(P,Q) \leq \tilde{D}^r_A(P,Q)$. The reverse inequality may be obtained by the same argument inverting the roles of $\tilde{D}^\infty_A$ and $\tilde{D}^r_A$ which completes the proof of the claim. Since $\tilde{D}^r_A$ and $\tilde{D}^\infty_A$ coincide on $\overline{B}_{\tilde{\eta}}(P_0)\cap G_\infty$, we infer that

$$\limsup_{t\to 0^+} \frac{\tilde{D}^r_A\big(P_0,P_0+t\nu\big)}{t} = \limsup_{t\to 0^+} \frac{\tilde{D}^\infty_A\big(P_0,P_0+t\nu\big)}{t} \quad \forall\, \nu \in \mathbb{R}^3.$$

By definition of $\mathcal{L}^r_A$ and $\mathcal{L}^\infty_A$ (see the proof of Proposition 3.3), we conclude that $\mathcal{L}^r_A\big(P_0,\nu\big) = \mathcal{L}^\infty_A\big(P_0,\nu\big)$ for every $\nu \in \mathbb{R}^3$ and since the points P_0 is arbitrary in $\Omega_r \cup \partial\mathbb{R}^3_+$, it yields

$$\mathcal{L}^r_A \equiv \mathcal{L}^\infty_A \quad \text{on } \big(\Omega_r \cup \partial\mathbb{R}^3_+\big)\times\mathbb{R}^3. \tag{3.23}$$

Combining (3.23) with Proposition 3.3, we obtain the announced result. ∎

Proof of Proposition 3.5 completed. First, we observe that it is enough to prove the pointwise convergence of d^r_A to $\bar{d}_A$, i.e., for any $P,Q \in \mathbb{R}^2$, $d^r_A(P,Q) \to \bar{d}_A(P,Q)$ as $r \to 0$. Indeed, one may easily check (see e.g. [31], Lemma 4.1) that $d^r_{A|\mathbb{R}^2\times\mathbb{R}^2}$ is Λ–Lipschitz with respect to the Euclidean distance on $\mathbb{R}^2\times\mathbb{R}^2$. Hence the local uniform convergence follows from the Arzela-Ascoli Theorem.

By construction, for any $0<r_1\leq r_2\leq\infty$, $d^{r_1}_A \geq d^{r_2}_A$ on $\overline{\Omega}_{r_1}\times\overline{\Omega}_{r_1}$ so that, for any $r>0$, $\mathbb{L}_{d^r_A} \geq \mathbb{L}_{d_A}$ on $\mathrm{Lip}\big([0,1];\overline{\Omega}_r\big)$. By Proposition 3.2 and Remark 3.1, for any $r>0$, we can find a $\frac{\Lambda}{\lambda}|P-Q|$–Lipschitz curve $\gamma^r_{PQ}:[0,1]\to\overline{\Omega}_r$ such that $\gamma^r_{PQ}(0)=P$, $\gamma^r_{PQ}(1)=Q$ and $d^r_A(P,Q)=\mathbb{L}_{d^r_A}(\gamma^r_{PQ})$. Let $(r_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers such that $r_n\to 0$ as $n\to+\infty$. By the Arzela-Ascoli Theorem, we may extract a subsequence, still denoted (r_n), such that $\gamma^{r_n}_{PQ} \underset{n\to+\infty}{\longrightarrow} \gamma^0_{PQ}$ uniformly on $[0,1]$ for some $\gamma^0_{PQ} \in \mathrm{Lip}_{P,Q}\big([0,1];\partial\mathbb{R}^3_+\big)$. Then we infer from the lower semicontinuity of $\mathbb{L}_{d_A}$,

$$\liminf_{n\to+\infty} d^{r_n}_A(P,Q) = \liminf_{n\to+\infty} \mathbb{L}_{d^{r_n}_A}(\gamma^{r_n}_{PQ}) \geq \liminf_{n\to+\infty} \mathbb{L}_{d_A}(\gamma^{r_n}_{PQ}) \geq \mathbb{L}_{d_A}(\gamma^0_{PQ}) \geq \bar{d}_A(P,Q).$$

On the other hand, by Proposition 3.4 we can find $\bar{\gamma}_{PQ} \in \mathrm{Lip}_{P,Q}\big([0,1];\partial\mathbb{R}^3_+\big)$ such that $\bar{d}_A(P,Q) = \mathbb{L}_{d_A}(\bar{\gamma}_{PQ})$. By Corollary 3.1,

$$\bar{d}_A(P,Q) = \mathbb{L}_{d_A}(\bar{\gamma}_{PQ}) = \mathbb{L}_{d^{r_n}_A}(\bar{\gamma}_{PQ}) \geq d^{r_n}_A(P,Q) \quad \forall n \in \mathbb{N}.$$

Therefore, $\limsup_{n\to+\infty} d^{r_n}_A(P,Q) \leq \bar{d}_A(P,Q)$ and the conclusion follows. ∎

Remark 3.4. As already mentioned in the previous proof, $d^{r_1}_A \leq d^{r_2}_A \leq \bar{d}_A$ on $\mathbb{R}^2\times\mathbb{R}^2$ for any $0<r_2\leq r_1\leq\infty$. In particular, for any $P,Q\in\mathbb{R}^2$, we have $\bar{d}_A(P,Q) = \sup_{r>0} d^r_A(P,Q)$.

We close this subsection with a stability result in the case of continuous matrix fields. The proof is a standard combination of compactness properties of geodesics, dominated convergence and lower semicontinuity of Riemannian length functionals.

Proposition 3.6. *Let $A \in \mathcal{A} \cap C^0(\overline{\mathbb{R}^3_+})$ and $\{A\}_{\varepsilon>0} \subset \mathcal{A} \cap C^0(\overline{\mathbb{R}^3_+})$ with uniform ellipticity bounds such that $A_\varepsilon \to A$ locally uniformly in $\overline{\mathbb{R}^3_+}$ as $\varepsilon \to 0$. Then $\bar{d}_{A_\varepsilon} \to \bar{d}_A$ locally uniformly in $\mathbb{R}^2 \times \mathbb{R}^2$ as $\varepsilon \to 0$.*

3.5. Hamilton-Jacobi and 1–Lipschitz functions

We close this section with a characterization of 1–Lipschitz functions with respect to the distance d^r_A as subsolutions of a certain eikonal equation involving the matrix field A. As already mentioned, Proposition 3.7 below plays a central role in the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3.

Proposition 3.7. *Let $A \in \mathcal{A}$ and $r \in (0,\infty]$. Then for any function $\Phi : \overline{\Omega}_r \to \mathbb{R}$, the following properties are equivalent:*

(i) $|\Phi(P) - \Phi(Q)| \leq d^r_A(P,Q) \quad \forall P, Q \in \overline{\Omega}_r$,

(ii) Φ *is Lipschitz continuous and* $(\mathrm{Cof}\, A)^{-1}\nabla\Phi\cdot\nabla\Phi \leq 1$ *a.e. in* Ω_r.

Proof. *(i)* $\Rightarrow$ *(ii)*. Let $\Phi : \overline{\Omega}_r \to \mathbb{R}$ satisfying *(i)*. From Proposition 3.1, we infer that Φ is Lipschitz continuous. Fix $P_0 \in \Omega_r$ and $\delta > 0$ such that $B_{3\delta}(P_0) \subset \mathbb{R}^3_+$. Consider a mollifyer $\varrho \in C^\infty_0(\mathbb{R}^3;\mathbb{R})$, $\mathrm{spt}\, \varrho \subset B(0,1)$, $\varrho \geq 0$, $\int_{\mathbb{R}^3} \varrho = 1$, and set, for an integer $n > 1/\delta$, $\varrho_n(z) = n^{-3}\varrho(nz)$. We define the smooth function $\Phi_n = \varrho_n * \Phi : B_\delta(P_0) \to \mathbb{R}$. We write

$$\Phi_n(P) = \int_{B_{1/n}} \varrho_n(z)\Phi(P-z)dz$$

and therefore for any distinct points $P, Q \in B_\delta(P_0)$,

$$\begin{aligned}\big|\Phi_n(P) - \Phi_n(Q)\big| &\leq \int_{B_{1/n}} \varrho_n(z)\big|\Phi(P-z) - \Phi(Q-z)\big|dz \leq \\ &\leq \int_{B_{1/n}} \varrho_n(z)\, d^r_A(P-z, Q-z)dz \leq \int_{B_{1/n}} \varrho_n(z)\, \ell_A\left([P-z, Q-z]\right) dz\,.\end{aligned}$$

Taking an arbitrary sequence of positive numbers $\varepsilon_k \to 0$ as $k \to +\infty$ and using Fatou's lemma, we get that

$$\begin{aligned}\big|\Phi_n(P) - \Phi_n(Q)\big| &\leq \\ &\leq \int_{B_{1/n}} \varrho_n(z)\bigg(\liminf_{k\to+\infty} \frac{1}{\pi\varepsilon_k^2}\int_{\Xi([P-z,Q-z],\varepsilon_k)\cap\mathbb{R}^3_+} \sqrt{\mathrm{Cof}\, A(x)\tau\cdot\tau}\; dx\bigg)dz \\ &\leq \liminf_{k\to+\infty} \frac{1}{\pi\varepsilon_k^2}\int_{B_{1/n}}\int_{\Xi([P-z,Q-z],\varepsilon_k)\cap\mathbb{R}^3_+} \varrho_n(z)\sqrt{\mathrm{Cof} A(x)\tau\cdot\tau}\; dxdz\,.\end{aligned}$$

where $\tau = \frac{P-Q}{|P-Q|}$. For $k \in \mathbb{N}$ sufficiently large, we have $\Xi\left([P-z, Q-z], \varepsilon_k\right) \subset B_{3\delta}(P_0)$ and consequently

$$\begin{aligned}\int_{B_{1/n}}\int_{\Xi([P-z,Q-z],\varepsilon_k)\cap\mathbb{R}^3_+} &\varrho_n(z)\sqrt{\mathrm{Cof}\, A(x)\tau\cdot\tau}\; dxdz = \\ &= \int_{\Xi([P,Q],\varepsilon_k)}\int_{B_{1/n}} \varrho_n(z)\sqrt{\mathrm{Cof}\, A(x-z)\tau\cdot\tau}\; dzdx\,.\end{aligned}$$

Then Cauchy-Schwartz inequality yields

$$\int_{B_{1/n}} \varrho_n(z)\sqrt{\operatorname{Cof} A(x-z)\tau\cdot\tau}\,dz \leq \bigg(\int_{B_{1/n}} \varrho_n(z)\operatorname{Cof} A(x-z)\tau\cdot\tau\,dz\bigg)^{1/2}$$

so that, setting $M_n := \varrho_n * \big(\operatorname{Cof} A\big)$, we have

$$\big|\Phi_n(P)-\Phi_n(Q)\big| \leq \liminf_{k\to+\infty} \frac{1}{\pi\varepsilon_k^2}\int_{\Xi([P,Q],\varepsilon_k)} \sqrt{M_n(x)\tau\cdot\tau}\,dx\,.$$

Since $x\mapsto \sqrt{M_n(x)\tau\cdot\tau}$ is smooth, we derive that

$$\frac{1}{\pi\varepsilon_k^2}\int_{\Xi([P,Q],\varepsilon_k)} \sqrt{M_n(x)\tau\cdot\tau}\,dx \to \int_{[P,Q]} \sqrt{M_n(s)\tau\cdot\tau}\,ds \quad \text{as } k\to+\infty\,.$$

Thus for any distinct points $P,Q\in B_\delta(P_0)$, we have

$$|\Phi_n(P)-\Phi_n(Q)| \leq \int_{[P,Q]} \sqrt{M_n(s)\frac{P-Q}{|P-Q|}\cdot\frac{P-Q}{|P-Q|}}\,ds\,.$$

Then, for any $P\in B_\delta(P_0)$, $h\in\mathbb{S}^2$ fixed and $t>0$ small,

$$\frac{|\Phi_n(P+th)-\Phi_n(P)|}{t} \leq \frac{1}{t}\int_{[P,P+th]} \sqrt{M_n(s)h\cdot h}\,ds \underset{t\to 0}{\longrightarrow} \sqrt{M_n(P)h\cdot h}$$

and we deduce, letting $t\to 0$, that $|\nabla\Phi_n(P)\cdot h| \leq \sqrt{M_n(P)h\cdot h}$. By homogeneity and the arbitrariness of h, we infer that $|\nabla\Phi_n(P)\cdot h| \leq \sqrt{M_n(P)h\cdot h}$ for all $h\in\mathbb{R}^3$. From assumption (1.9), we easily check that $M_n(P)$ is invertible so that we can choose $h = M_n^{-1}(P)\nabla\Phi_n(P)$ and we conclude from the arbitrariness of P,

$$M_n^{-1}\nabla\Phi_n\cdot\nabla\Phi_n \leq 1 \quad \text{in } B_\delta(P_0)\,. \tag{3.24}$$

Since $M_n\to\operatorname{Cof} A$ and $\nabla\Phi_n\to\nabla\Phi$ a.e. in $B_\delta(P_0)$ as $n\to+\infty$, we conclude $(\operatorname{Cof} A)^{-1}\nabla\Phi\cdot\nabla\Phi\leq 1$ a.e. in $B_\delta(P_0)$ letting $n\to+\infty$ in (3.24). Since P_0 is arbitrary in $\mathbb{R}^3_+$, we get the result.

(ii) ⇒ (i). The reverse implication follows from Lemma 3.2 below. Its proof is similar to the one of Lemma 2.1 in [31] with minor modifications so that we shall omit it.

Lemma 3.2. *For $r\in(0,\infty]$, let $\Phi:\overline{\Omega}_r\to\mathbb{R}$ be a Lipschitz continuous function. For any distinct points $a,b\in\Omega_r$ and any $\varepsilon>0$ sufficiently small, we have*

$$|\Phi(a)-\Phi(b)| \leq \frac{1}{\pi\varepsilon^2}\int_{\Xi([a,b],\varepsilon)\cap\mathbb{R}^3_+} \left|\nabla\Phi(x)\cdot\frac{b-a}{|b-a|}\right|dx + 2\varepsilon\|\nabla\Phi\|_\infty\,.$$

Proof of Proposition 3.7 completed. Let Φ be a Lipschitz continuous function satisfying *(ii)*. We deduce from Lemma 3.2 and (1.9) that for any $\mathcal{F}=([\alpha_1,\beta_1],\ldots,[\alpha_n,\beta_n])\in\mathcal{P}_r(P,Q)$ and any parameters $\varepsilon_1,\ldots,\varepsilon_n>0$ sufficiently small, we have

$$\begin{aligned}\big|\Phi(P)-\Phi(Q)\big| &\leq \sum_{k=1}^n \big|\Phi(\beta_k)-\Phi(\alpha_k)\big|\\ &\leq \sum_{k=1}^n \bigg(\frac{1}{\pi\varepsilon_k^2}\int_{\Xi([\alpha_k,\beta_k],\varepsilon_k)} \big|\nabla\Phi(x)\cdot\tau_k\big|\,dx + 2\Lambda\varepsilon_k\bigg)\,.\end{aligned}$$

Since Φ satisfies *(ii)*, we infer that for $k = 1, \ldots, n$ and a.e. $x \in \Xi([\alpha_k, \beta_k], \varepsilon_k)$,

$$|\nabla\Phi(x) \cdot \tau_k| = |(\mathrm{Cof}\, A(x))^{-1/2}\nabla\Phi(x) \cdot (\mathrm{Cof}\, A(x))^{1/2}\tau_k| \leq$$
$$\leq \sqrt{(\mathrm{Cof}\, A(x))^{-1}\nabla\Phi(x) \cdot \nabla\Phi(x)}\sqrt{\mathrm{Cof}\, A(x)\tau_k \cdot \tau_k} \leq \sqrt{\mathrm{Cof}\, A(x)\tau_k \cdot \tau_k}$$

and consequently,

$$\big|\Phi(P) - \Phi(Q)\big| \leq \sum_{k=1}^{n} \left(\frac{1}{\pi\varepsilon_k^2} \int_{\Xi([\alpha_k,\beta_k],\varepsilon_k)} \sqrt{\mathrm{Cof}\, A(x)\tau_k \cdot \tau_k}\, dx + 2\Lambda\varepsilon_k \right).$$

Taking the lim inf as $\varepsilon_k \to 0^+$ for each parameter ε_k, we derive that $|\Phi(P) - \Phi(Q)| \leq \ell_A(\mathcal{F})$. We obtain the result taking the infimum over all $\mathcal{F} \in \mathcal{P}_r(P, Q)$. Then we conclude that *(i)* holds in all $\overline{\Omega}_r$ by continuity. ■

As a consequence of Proposition 3.7, we may now show that, under a certain structure assumption on A, the distance d_A and $\bar{d}_A$ coincide on the plane. Once more, the following result, quite obvious in the continuous case, requires a specific analysis for a measurable field A.

Corollary 3.2. *Assume that $A \in \mathcal{A}_0$. Then*

$$\bar{d}_A(P, Q) = d_A(P, Q) \quad \forall P, Q \in \mathbb{R}^2 .$$

Proof. First we observe that it is enough to prove that for any $r > 0$, $d_A^r \equiv d_A$ on $\overline{\Omega}_r \times \overline{\Omega}_r$. Indeed, by Proposition 3.5, it would lead to $\bar{d}_A(P, Q) = \lim_{r\to 0} d_A^r(P, Q) = d_A(P, Q)$ for every $P, Q \in \mathbb{R}^2$. Now we fix $r > 0$ and $P \in \overline{\Omega}_r$ and we set for $x \in \overline{\Omega}_r$, $\Phi(x) = d_A^r(P, x)$. Obviously Φ is 1–Lipschitz in $\overline{\Omega}_r$ with respect to d_A^r so that by Proposition 3.7, $(\mathrm{Cof}\, A)^{-1}\nabla\Phi \cdot \nabla\Phi \leq 1$ a.e. in Ω_r. Next we extend Φ to $\overline{\Omega}_{2r}$ by setting $\Phi(x_1, x_2, x_3) = \Phi(x_1, x_2, 2r - x_3)$ for $r \leq x_3 \leq 2r$. Obviously, the resulting Φ is a Lipschitz function. Due to the structure assumption $A \in \mathcal{A}_0$, this extension satisfies now $(\mathrm{Cof}\, A)^{-1}\nabla\Phi \cdot \nabla\Phi \leq 1$ a.e. in Ω_{2r}. Then we extend Φ to the whole half space *by periodicity* into a $2r$–periodic function in the last variable, i.e., $\Phi(x_1, x_2, x_3 + 2kr) = \Phi(x_1, x_2, x_3)$ for any $x = (x_1, x_2, x_3) \in \overline{\Omega}_r$ and any integer $k \geq 0$. We easily check that Φ is Lipschitz and by the assumption on A, $(\mathrm{Cof}\, A)^{-1}\nabla\Phi \cdot \nabla\Phi \leq 1$ a.e. in $\mathbb{R}^3_+$. By Proposition 3.7, this implies that Φ is 1–Lipschitz in $\overline{\mathbb{R}^3_+}$ with respect to d_A. In particular, $d_A^r(P, Q) = \Phi(Q) - \Phi(P) \leq d_A(P, Q)$ for every $Q \in \overline{\Omega}_r$ and from the arbitrariness of P, we conclude that $d_A^r \leq d_A$ in $\overline{\Omega}_r \times \overline{\Omega}_r$. On the other hand, the reverse inequality comes from the definition of d_A and d_A^r so that the proof is complete. ■

A straightforward consequence of Proposition 3.7 and Corollary 3.2 is the following result.

Corollary 3.3. *Let $A \in \mathcal{A}_0$ and let $\varphi : \mathbb{R}^2 \to \mathbb{R}$. The following properties are equivalent:*

(i) $|\varphi(P) - \varphi(Q)| \leq \bar{d}_A(P, Q) \quad \forall P, Q \in \mathbb{R}^2$,

(ii) φ is Lipschitz continuous and $\Phi : \overline{\mathbb{R}^3_+} \to \mathbb{R}$, $\Phi(x_1, x_2, x_3) := \varphi(x_1, x_2)$ satisfies $(\mathrm{Cof}\, A)^{-1}\nabla\Phi \cdot \nabla\Phi \leq 1$ *a.e. in* $\mathbb{R}^3_+$.

4. Geometric distances vs energetic distances

In this section, we prove Theorem 1.1 using duality arguments similar to [31], [32] to establish the lower bound and a dipole-type construction based on [33] for the upper bound. As in the two papers cited above, the measurability of the matrix field forces to work in the context of length structures and a delicate and quite indirect dipole-type construction is required. This, combined with the theory of Hamilton-Jacobi (in)equations in Subsection 3.5, allows us to conclude.

The following lemma is the key point for proving the metric property of $\bar\rho_A$ and will be repeatedly used in the paper.

Lemma 4.1. *Let $f \in X$ and $\{f_n\}_{n\in\mathbb{N}} \subset X$ such that $f_n \to \alpha$ a.e. for some constant $\alpha \in \mathbb{S}^1$ as $n \to +\infty$ and $\sup_n |f_n|_{1/2} < +\infty$. Then*

$$\limsup_{n\to+\infty} \mathcal{E}_A(ff_n) \leq \mathcal{E}_A(f) + \limsup_{n\to+\infty} \mathcal{E}_A(f_n)\,.$$

Proof. Let u_n and u be the A–harmonic extensions of f_n and f respectively. By Proposition 2.7 and Proposition 2.8, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)$ and $\max\{|u_n|,|u|\} \leq 1$ a.e. in $\mathbb{R}^3_+$. Clearly, $f_n \rightharpoonup \alpha$ weakly in $\dot H^{1/2}$ by Theorem 2.5, so that $u_n \rightharpoonup \alpha$ weakly in $\dot H^1$ and locally uniformly as $n \to +\infty$ (again by Proposition 2.7 and Proposition 2.8). Due to the pointwise bounds on $|u_n|$ and $|u|$, we can apply Proposition 2.3 to derive that $uu_n \in \dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)$ with $\mathrm{Tr}\,(uu_n) = ff_n$ on $\mathbb{R}^2$. An easy computation leads to the identity

$$E_A(uu_n) = \frac{1}{2}\int_{\mathbb{R}^3_+} \Big\{ |u|^2 \,\mathrm{tr}\big(\nabla u_n A(\nabla u_n)^t\big) + |u_n|^2\,\mathrm{tr}\big(\nabla u A(\nabla u)^t\big)\Big\} + \tag{4.1}$$
$$+ \int_{\mathbb{R}^3_+} \Big\{ \mathrm{Re}(uu_n)\,\mathrm{tr}\big(\nabla \bar u_n A(\nabla u)^t\big) + \mathrm{Im}(uu_n)\,\mathrm{tr}\big(\nabla(i\bar u_n)A(\nabla u)^t\big)\Big\} = I_n + II_n\,.$$

Since $\mathrm{Tr}\,(uu_n) = ff_n$, formula (4.1) yields

$$\mathcal{E}_A(ff_n) \leq E_A(uu_n) \leq E_A(u) + E_A(u_n) + II_n = \mathcal{E}_A(f) + \mathcal{E}_A(f_n) + II_n\,.$$

Using that $u_n \to \alpha$ a.e. with $|u_n| \leq 1$ and $\nabla u_n \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^3_+)$ as $n \to +\infty$, we deduce that $II_n \to 0$. Taking the $\limsup$ in n in the previous inequality yields the announced result. ∎

Proof of Theorem 1.1. *Step 1.* We start by proving that $\bar\rho_A$ defines a distance on $\mathbb{R}^2$. Clearly $\bar\rho_A$ is nonnegative and symmetric. The nondegeneracy will follow from the equivalence with the Euclidean distance which will be proved in the next steps. Now we show the triangle inequality. Let $P, Q, N \in \mathbb{R}^2$ fixed. By definition of $\bar\rho_A$ and Remark 2.5, we may find sequences $\{f^{(1)}_m\}_{m\in\mathbb{N}}, \{f^{(2)}_n\}_{n\in\mathbb{N}} \subset X$ realizing $\bar\rho_A(P,Q)$ and $\bar\rho_A(Q,N)$ respectively. We denote by $\alpha_1 \in \mathbb{S}^1$ and $\alpha_2 \in \mathbb{S}^1$ the respective weak limits of $f^{(1)}_m$ and $f^{(2)}_n$. By Theorem 2.5, we may assume that for $i = 1,2$, $f^{(i)}_n \to \alpha_i$ a.e. in $\mathbb{R}^2$. Then we consider the sequence $h_{m,n} = f^{(1)}_m f^{(2)}_n$. By Proposition 2.6 and Lemma 4.1, $h_{m,n} \in X$, $\limsup_n \limsup_m \mathcal{E}_A(f_{n,m}) < +\infty$ and $T(h_{m,n}) = 2\pi(\delta_P - \delta_N)$. By Remark 2.5, we may find a diagonal sequence $h_k = h_{m_k,n_k}$ such that $\limsup_k \mathcal{E}_A(f_k) \leq \limsup_n \limsup_m \mathcal{E}_A(f_{n,m})$, $h_k \rightharpoonup \alpha_1\alpha_2$ as $k \to +\infty$ weakly in $\dot H^{1/2}$. Hence the sequence $\{h_k\}_{k\in\mathbb{N}}$ is admissible for computing $\bar\rho_A(P,N)$. Then we deduce from Lemma 4.1 that

$$\begin{aligned}\bar\rho_A(P,N) &\leq \liminf_{k\to+\infty} \mathcal{E}_A(h_{m_k,n_k}) \leq \limsup_{m\to+\infty}\limsup_{n\to+\infty} \mathcal{E}_A(f^{(1)}_m f^{(2)}_n) \leq \\ &\leq \limsup_{m\to+\infty} \mathcal{E}_A(f^{(1)}_m) + \limsup_{n\to+\infty} \mathcal{E}_A(f^{(2)}_n) \leq \bar\rho_A(P,Q) + \bar\rho_A(Q,N)\end{aligned}$$

and the proof of the triangle inequality is complete.

Step 2. Now we move on the proof of claim *(ii)*. We observe that $\bar\rho_A \geq \pi \bar d_A$ will imply the lower inequality in claim *(i)* between $\bar\rho_A$ and the Euclidean distance since $\bar d_A(P,Q) \geq \lambda|P-Q|$ for any $P, Q \in \mathbb{R}^2$ by (3.18).

To obtain the estimate involving ρ_A, we proceed as follows. Given two points $P, Q \in \mathbb{R}^2$, consider $f \in X$ arbitrary such that $T(f) = 2\pi(\delta_P - \delta_Q)$ and let $u \in \dot{H}^1$ be its A–harmonic extension to $\mathbb{R}^3_+$. A simple computation yields

$$\frac{1}{2}\,\mathrm{tr}\big(\nabla u A(\nabla u)^t\big) \geq \sqrt{\det(\nabla u A(\nabla u)^t)} = \frac{1}{2}\sqrt{(\mathrm{Cof}\,A)H(u)\cdot H(u)} \quad \text{a.e. in } \mathbb{R}^3_+. \tag{4.2}$$

For any $\Phi \in \mathrm{Lip}(\mathbb{R}^3_+)$ which is 1−Lipschitz with respect to d_A, we have $\sqrt{(\mathrm{Cof}\,A)^{-1}\nabla\Phi\cdot\nabla\Phi} \leq 1$ a.e. in $\mathbb{R}^3_+$ by Proposition 3.7. Therefore,

$$\begin{aligned}\mathcal{E}_A(f) = E_A(u) &\geq \frac{1}{2}\int_{\mathbb{R}^3_+}\sqrt{(\mathrm{Cof}\,A)H(u)\cdot H(u)}\\ &\geq \frac{1}{2}\int_{\mathbb{R}^3_+} H(u)\cdot\nabla\Phi \;=\; \pi\big(\Phi(Q)-\Phi(P)\big)\,.\end{aligned} \tag{4.3}$$

Choosing $\Phi(x) = d_A(P,x)$, we deduce that $\mathcal{E}_A(f) \geq \pi d_A(P,Q)$ and the conclusion follows taking the infimum over $f \in X$ satisfying $T(f) = 2\pi(\delta_P - \delta_Q)$.

In order to prove the inequality involving $\bar{\rho}_A$, we fix $0 < r < 1$ and we introduce a cut-off function $\chi \in C^\infty_0(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\mathrm{spt}\,\chi \subset (-r,r)$ and $\chi(t) \equiv 1$ for $|t| \leq r/2$. Let d^r_A be the geodesic distance corresponding to $\mathrm{Cof}\,A$ on the domain $\Omega_r = \mathbb{R}^2 \times (0,r)$ as constructed in Section 3. Given $P, Q \in \mathbb{R}^2$, we define for $x \in \overline{\Omega}_r$,

$$\Phi(x) = \left(\frac{1}{2}\,d^r_A(P,Q) - d^r_A(P,x)\right)^+ - \left(\frac{1}{2}\,d^r_A(P,Q) - d^r_A(Q,x)\right)^+ . \tag{4.4}$$

Clearly Φ is 1−Lipschitz in Ω_r with respect to d^r_A, so that, by Proposition 3.7, $\sqrt{(\mathrm{Cof}\,A)^{-1}\nabla\Phi\cdot\nabla\Phi} \leq 1$ a.e. in Ω_r. In addition, $K := \mathrm{spt}\,\Phi \subset \overline{\Omega}_r$ is a compact set. Now, consider $\{f_n\}_{n\in\mathbb{N}} \subset X$ such that $f_n \rightharpoonup \alpha \in \mathbb{S}^1$ weakly in $\dot{H}^{1/2}$ with $T(f_n) \equiv -2\pi(\delta_P - \delta_Q)$ for every n. Denote by u_n the corresponding A–harmonic extension. Arguing as in (4.3), we infer that

$$\begin{aligned}\mathcal{E}_A(f_n) &\geq \frac{1}{2}\int_{\mathbb{R}^3_+}\chi(x_3)\sqrt{(\mathrm{Cof}\,A)H(u_n)\cdot H(u_n)} \geq \frac{1}{2}\int_{\Omega_r}\chi(x_3)\,H(u_n)\cdot\nabla\Phi \;=\\ &= \pi(\Phi(P)-\Phi(Q)) - \frac{1}{2}\int_{\Omega_r}\Phi\,H(u_n)\cdot\nabla\chi\\ &= \pi d^r_A(P,Q) - \frac{1}{2}\int_{K\cap\{r/2<x_3<r\}}\Phi\,H(u_n)\cdot\nabla\chi\,.\end{aligned} \tag{4.5}$$

Since $u_n \rightharpoonup \alpha$ in $\dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$ as $n \to +\infty$, we derive from Proposition 2.8 that $\nabla u_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3_+)$ and hence $H(u_n) \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^3_+)$. Going back to (4.5), it yields $\liminf_{n\to+\infty}\mathcal{E}_A(f_n) \geq \pi d^r_A(P,Q)$. Now letting $r \to 0$, we recover $\liminf_{n\to+\infty}\mathcal{E}_A(f_n) \geq \pi\bar{d}_A(P,Q)$ thanks to Proposition 3.5. Then the conclusion follows from the arbitrariness of the sequence $\{f_n\}_{n\in\mathbb{N}}$.

Step 3. In order to show the upper bound $\bar{\rho}_A(P,Q) \leq \pi\Lambda|P-Q|$, we first observe that $E_A(u) \leq \Lambda E_{\mathrm{Id}}(u)$ for any $u \in \dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$ by the uniform ellipticity assumption on A. Thus $\mathcal{E}_A \leq \Lambda\mathcal{E}_{\mathrm{Id}}$ and consequently $\bar{\rho}_A(P,Q) \leq \Lambda\rho_{\mathrm{Id}}(P,Q)$. Then the conclusion follows from the explicit construction of an optimal dipole in Lemma 4.2 below. We present in this lemma additional results which will be of importance in Section 5. For a different construction of an optimal dipole see [25], Section 5.

Lemma 4.2. *(Euclidean dipole) Let P and Q be two distinct points in $\mathbb{R}^2$. There exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset X \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ such that $f_n \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{P,Q\})$, $T(f_n) = 2\pi(\delta_P - \delta_Q)$ for every n,*

$f_n \rightharpoonup (1,0)$ weakly in $\dot H^{1/2}$, $f_n \to (1,0)$ a.e. as $n\to+\infty$ and

$$\limsup_{n\to+\infty} \mathcal{E}_{\rm Id}(f_n) \leq \pi|P-Q|\,.$$

Moreover, $f_n = e^{i\psi_n}$ a.e. in $\mathbb{R}^2$ for some sequence of compactly supported functions $\{\psi_n\}_{n\in\mathbb{N}} \subset BV(\mathbb{R}^2;\mathbb{R})\cap {\rm Lip}_{\rm loc}(\mathbb{R}^2\setminus[P,Q])$ satisfying

(i) $\psi_n \to 0$ a.e. as $n\to+\infty$,

(ii) $|\psi_n|_{BV} \leq C|P-Q|$ for a constant C independent of P, Q and n,

(iii) the singular part of $D\psi_n$ is $D^s\psi_n = 2\pi\nu\,\mathcal{H}^1 \llcorner [P,Q]$ with $\nu^\perp = \frac{P-Q}{|P-Q|}$.

Proof. Due to the invariance and scaling properties of $E_{\rm Id}$, up to rotations, dilations and translations in $\mathbb{R}^3$, we may assume $Q=(1,0,0)$ and $P=(-1,0,0)$. This shows in particular that the constant C in claim *(ii)* does not depend on P and Q. We shall construct a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)$ such that $f_n = {\rm Tr}\, u_n \in X$ satisfies the conditions above and $E_{\rm Id}(u_n)\to 2\pi$ as $n\to+\infty$.

First, we introduce the conformal representation $v:\mathbb{R}^2_+\to\mathbb{D}$ defined by

$$v(z) = \frac{\bar z + i}{\bar z - i} = \left(\frac{x_2^2+x_3^2-1}{x_2^2+(1+x_3)^2}, \frac{2x_2}{x_2^2+(1+x_3)^2}\right),\ z = x_2+ix_3\in\mathbb{R}^2_+$$

so that $v\to 1$ as $|z|\to\infty$ and $v(\cdot,0)\in {\rm Lip}(\mathbb{R};\mathbb{S}^1\setminus\{(1,0)\})$. Moreover, for $x_2\neq 0$ we have $v(x_2,0) = e^{i\theta(x_2)}$ for $\theta(x_2) = 2\arctan(1/x_2)$. Then we introduce a sequence of deformations as follows, using the complex notation,

$$v_n(x_2,x_3) = \begin{cases} v(nx_2,nx_3) & \text{if } |x_2|^2+|x_3|^2\leq 1\\ 1 & \text{if } |x_2|^2+|x_3|^2 \geq 4,\\ e^{i(\theta(nx_2))(2-|x_2|)} & \text{if } 1<|x_2|<2 \text{ and } x_3 = 0\,,\\ \text{harmonic} & \text{if } 1<|x_2|^2+|x_3|^2<4 \text{ and } x_3>0\,,\end{cases} \tag{4.6}$$

i.e., we glue a pure dilation on the unit half disk with a constant map using two arcs on $\{x_3=0\}$ and an harmonic extension in the remaining half annulus. One easily check that $v_n\in{\rm Lip}(\mathbb{R}^2_+)$ and $v_n(\cdot,0)\in{\rm Lip}(\mathbb{R};\mathbb{S}^1)$. In addition, $v_n(x_2,0) = e^{i\theta_n(x_2)}$ for

$$\theta_n(x_2) = \begin{cases} 2\arctan\left(\dfrac{1}{nx_2}\right) & \text{if } 0<|x_2|<1\,,\\ 2(2-|x_2|)\arctan\left(\dfrac{1}{nx_2}\right) & \text{if } 1<|x_2|<2\,,\\ 0 & \text{otherwise.}\end{cases} \tag{4.7}$$

By construction, $\theta_n\in{\rm Lip}_{\rm loc}(\mathbb{R}\setminus\{0\})$, θ_n has compact support and finite pointwise variation in $\mathbb{R}$. Since its left and right limits at 0 are given by $-\pi$ and π respectively, the singular part of the distributional derivative of θ_n is given by $D^s\theta_n = 2\pi\delta_0$ for every n. Furthermore, we easily see that $\sup_n|\theta_n|_{BV}<+\infty$.

We observe that the map v_n has been constructed in such a way that

$$\int_{\mathbb{R}^2_+}|\nabla v_n|^2 \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2_+}|\nabla v|^2 = 2\pi$$

and $|v(nx_2,nx_3) - v_n(x_2,x_3)| \to 0$ uniformly in $\mathbb{R}^2_+$. Now we define for $x = (x_1,x_2,x_3) \in \overline{\mathbb{R}^3_+}$,

$$u_n(x) = \begin{cases} v_n\left(\dfrac{x_2}{\eta_n(x_1)}, \dfrac{x_3}{\eta_n(x_1)}\right) & \text{if } |x_1| < 1 \\ 1 & \text{if } |x_1| \geq 1 \end{cases} \tag{4.8}$$

where $\eta_n(x_1) = (1-|x_1|)/n$. Once more, one may check that $u_n \in \dot{H}^1(\mathbb{R}^2,\mathbb{R}^2)$, u_n is locally Lipschitz in $\overline{\mathbb{R}^3_+}$ away from $\{(-1,0,0),(1,0,0)\}$, and that the corresponding $f_n := u_{n|\mathbb{R}^2} \in X \cap W^{1,1}_{\rm loc}(\mathbb{R}^2)$, f_n is locally Lipschitz away from the points P and Q with degrees -1 and $+1$ respectively, i.e., $T(f_n) = 2\pi(\delta_P - \delta_Q)$. Moreover, $f_n \to (1,0)$ a.e. in $\mathbb{R}^2$ as $n \to +\infty$ since $\eta_n \to 0$. A direct computation, using the conformal invariance of $|\nabla_{2,3}\, u_n|^2$ yields

$$\begin{aligned} E_{\rm Id}(u_n) &= \frac{1}{2}\int_{\mathbb{R}^3_+} |\nabla u_n|^2 = \frac{1}{2}\int_{-1}^{1}\left(\int_{\mathbb{R}^2_+} |\nabla_{2,3}\, u_n|^2 + |\partial_1 u_n|^2\right)dx_1 \\ &= \int_{\mathbb{R}^2_+} |\nabla v_n|^2 + \frac{1}{2}\int_{-1}^{1}\left(\int_{\mathbb{R}^2_+} |\partial_1 u_n|^2\right)dx_1 \\ &= \int_{\mathbb{R}^2_+} |\nabla v_n|^2 + \frac{1}{2}\left(\int_{-1}^{1} |\dot{\eta}_n(x_1)|^2 dx_1\right)\left(\int_{\mathbb{R}^2_+} |\nabla v_n|^2(x_2^2+x_3^2)\right). \end{aligned}$$

Since by construction, $|\nabla v_n|^2(x_2^2+x_3^2) \leq 4|\nabla v_n|^2$ and $\int_{-1}^{1} |\dot{\eta}_n(x_1)|^2 dx_1 \to 0$ as $n \to +\infty$, we infer that

$$\limsup_{n\to+\infty} \mathcal{E}_{\rm Id}(f_n) \leq \lim_{n\to+\infty} E_{\rm Id}(u_n) = \lim_{n\to+\infty} \int_{\mathbb{R}^2_+} |\nabla v_n|^2 = 2\pi = \pi|P-Q|\,.$$

In particular, $\sup_n |f_n|_{1/2} < +\infty$, so that $f_n \rightharpoonup (1,0)$ weakly in $\dot{H}^{1/2}$ by Theorem 2.5.

Given θ_n as above, we now set

$$\psi_n(x_1,x_2) = \begin{cases} \theta_n\left(\dfrac{x_2}{\eta_n(x_1)}\right), & \text{if } |x_1| < 1\,,\ x_2 \neq 0 \\ 0 & \text{if } |x_1| > 1, \end{cases} \tag{4.9}$$

so that ψ_n has compact support and it satisfies $f_n = e^{i\psi_n}$ a.e. in $\mathbb{R}^2$. Then, taking (4.7) and (4.9) into account, straightforward computations yield claims *(i)*, *(ii)* and *(iii)* so the proof is complete. ■

Remark 4.1. Due to the invariance properties of $E_{\rm Id}$, up to rotation, dilation and translation in $\mathbb{R}^3$, the construction given in the previous lemma still holds if one replace $\mathbb{R}^3_+$ by any half space Ω with $0 \in \partial\Omega$. In this case, we consider $E_{\rm Id} : \dot{H}^1(\Omega;\mathbb{R}^2) \to \mathbb{R}_+$ defined as in (1.11) and then $\langle\cdot\rangle_{\rm Id}$ is the corresponding seminorm on $\dot{H}^{1/2}(\partial\Omega;\mathbb{R}^2)$. The distribution $T(f)$, for $f \in \dot{H}^{1/2}(\partial\Omega;\mathbb{S}^1)$, is defined as in (1.4) by integrating over Ω. As a consequence, Lemma 4.2 still holds if we consider $\mathcal{E}_A$ instead of $\mathcal{E}_{\rm Id}$ for a constant matrix $A \in \mathcal{S}^3_+$ replacing the Euclidean distance by the Riemannian distance

$$\bar{d}_A(P,Q) = d_A(P,Q) = \sqrt{(\operatorname{Cof} A)(\tilde{P}-\tilde{Q})\cdot(\tilde{P}-\tilde{Q})} \tag{4.10}$$

where $\tilde{P} = (P,0)$ and $\tilde{Q} = (Q,0)$ belong to $\partial\mathbb{R}^3_+$. Indeed, writing $A = RDR^t$ where D is a diagonal matrix made by the eigenvalues of A and $R \in SO(3)$, we change variables by setting $x = \Psi(y) := RD^{1/2}y$ and then, we apply the construction to the half space $\Omega = \Psi^{-1}(\mathbb{R}^3_+)$ and to the points $\Psi^{-1}(P)$ and $\Psi^{-1}(Q)$.

Remark 4.2. Applying additional regularization techniques, we could have constructed the sequence $\{f_n\}_{n\in\mathbb{N}}$ in Lemma 4.2, such that $f_n \in X \cap C^\infty_{\rm const}(\mathbb{R}^2 \setminus \{P,Q\})$ rather than Lipschitz away from the points P and Q.

Proof of Theorem 1.1. Step 4. Now we give the proof of claim *(iii)* which is based on Lemma 4.2 together with Remark 4.1. We recall that we assume here $A \in \mathcal{A}_0$ so $A(x) = A(\hat{x})$ where $x = (\hat{x}, x_3)$. Fix a point $Q_0 \in \mathbb{R}^2$ and let us define for an arbitrary point $P \in \mathbb{R}^2$, $\xi(P) = \bar{\rho}_A(P, Q_0)$ so that $\xi \in {\rm Lip}(\mathbb{R}^2)$ by claim *(i)*. Consider a standard mollifyer $\varrho \in C^\infty_0(\mathbb{R}^2;\mathbb{R})$, i.e., $\varrho \geq 0$, $\int_{\mathbb{R}^2} \varrho = 1$, and set, for $\varepsilon > 0$, $\varrho_\varepsilon(z) = \varepsilon^2 \varrho(z/\varepsilon)$. We define

$$\xi_\varepsilon(P) = \varrho_\varepsilon * \xi(P) = \int_{\mathbb{R}^2} \varrho_\varepsilon(-z)\xi(P+z)dz\,.$$

Clearly $\xi_\varepsilon \in C^\infty(\mathbb{R}^2) \cap {\rm Lip}(\mathbb{R}^2)$ and the family $\{\xi_\varepsilon\}_{\varepsilon>0}$ is equilipschitz. Moreover, $\xi_\varepsilon \to \xi$ locally uniformly and $\nabla\xi_\varepsilon \to \nabla\xi$ a.e. in $\mathbb{R}^2$ as $\varepsilon \to 0$. Similarly, let

$$A_\varepsilon(\hat{x}) = \varrho_\varepsilon * A(\hat{x}) = \int_{\mathbb{R}^2} \varrho_\varepsilon(-z)A(\hat{x}+z)dz,$$

so that $A_\varepsilon \to A$ a.e. as $\varepsilon \to 0$. One easily check that for every $\varepsilon > 0$, $A_\varepsilon \in \mathcal{A}_0$ and A_ε conserves the ellipticity bounds of A, i.e., $\lambda{\rm Id} \leq A_\varepsilon(\hat{x}) \leq \Lambda{\rm Id}$ (as quadratic forms) for every $\hat{x} \in \mathbb{R}^2$.

For any points $P, Q \in \mathbb{R}^2$, we have

$$\begin{aligned}|\xi_\varepsilon(P) - \xi_\varepsilon(Q)| &\leq \int_{\mathbb{R}^2} \varrho_\varepsilon(-z)\big|\xi(P+z) - \xi(Q+z)\big|dz \\ &\leq \int_{\mathbb{R}^2} \varrho_\varepsilon(-z)\bar{\rho}_A(P+z, Q+z)dz \end{aligned} \tag{4.11}$$

since ξ is 1-Lipschitz with respect to $\bar{\rho}_A$. Now we shall estimate the contribution of $\bar{\rho}_A(P+z,Q+z)$ in the integral above. We proceed as follows. Consider the sequences $\{f_n\}_{n\in\mathbb{N}} \subset X$ and $\{u_n\}_{n\in\mathbb{N}}$ constructed in Lemma 4.2 and Remark 4.1 relative to the points P, Q and to the constant matrix $A_\varepsilon(P)$. Then define the map $\tilde{u}_n(x) = u_n\big(x - (z,0)\big)$ so that $\tilde{f}_n := \tilde{u}_{n|\mathbb{R}^2} \in X$ satisfies $T(\tilde{f}_n) = 2\pi(\delta_{P+z} - \delta_{Q+z})$ and $\tilde{f}_n \rightharpoonup (1,0)$ weakly in $\dot{H}^{1/2}$ as $n \to +\infty$. By definition of $\bar{\rho}_A(P+z,Q+z)$ and $\mathcal{E}_A(\tilde{f}_n)$, we have

$$\bar{\rho}_A(P+z,Q+z) \leq \liminf_{n\to+\infty} \mathcal{E}_A(\tilde{f}_n) \leq \liminf_{n\to+\infty} E_A(\tilde{u}_n)\,. \tag{4.12}$$

Now we observe that a simple change of variables leads to

$$E_A(\tilde{u}_n) = \frac{1}{2}\int_{\mathbb{R}^3_+} {\rm tr}\big(\nabla\tilde{u}_n A(\hat{x})(\nabla\tilde{u}_n)^t\big)dx = \frac{1}{2}\int_{\mathbb{R}^3_+} {\rm tr}\big(\nabla u_n A(\hat{y}+z)(\nabla u_n)^t\big)dy \tag{4.13}$$

where $y = (\hat{y}, y_3)$. Inserting (4.12) and (4.13) in (4.11), invoking Fatou's lemma and changing the order of integration, we derive that

$$\begin{aligned}|\xi_\varepsilon(P) - \xi_\varepsilon(Q)| &\leq \liminf_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^3_+}\int_{\mathbb{R}^2} \varrho_\varepsilon(-z){\rm tr}\big(\nabla u_n(y)A(\hat{y}+z)(\nabla u_n(y))^t\big)dzdy \\ &= \liminf_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^3_+} {\rm tr}\bigg(\nabla u_n\bigg(\int_{\mathbb{R}^2} \varrho_\varepsilon(-z)A(\hat{y}+z)dz\bigg)(\nabla u_n)^t\bigg)dy \\ &= \liminf_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^3_+} {\rm tr}\big(\nabla u_n A_\varepsilon(\hat{y})(\nabla u_n)^t\big)dy \\ &= \liminf_{n\to+\infty} \frac{1}{2}\int_{B_{|P-Q|}(P)\times(0,+\infty)} {\rm tr}\big(\nabla u_n A_\varepsilon(\hat{y})(\nabla u_n)^t\big)dy\,,\end{aligned}$$

since u_n is constant outside $B_{|P-Q|}(P)\times(0,+\infty)$ for n large enough by construction. Using the partial order over quadratic forms, we easily see that for any $\hat{y}\in B_{|P-Q|}(P)$,

$$A_\varepsilon(\hat{y})\leq A_\varepsilon(P)+C_\varepsilon|P-Q|\,\mathrm{Id}\;\leq\left(1+\frac{C_\varepsilon}{\lambda}|P-Q|\right)A_\varepsilon(P)$$

for a constant C_ε which only depends on ε. Consequently,

$$|\xi_\varepsilon(P)-\xi_\varepsilon(Q)|\leq\left(1+\frac{C_\varepsilon}{\lambda}|P-Q|\right)\liminf_{n\to+\infty}\frac{1}{2}\int_{\mathbb{R}^3_+}\mathrm{tr}\big(\nabla u_n A_\varepsilon(P)(\nabla u_n)^t\big)dy$$

By construction of the sequence $\{u_n\}_{n\in\mathbb{N}}$, we have

$$\frac{1}{2}\int_{\mathbb{R}^3_+}\mathrm{tr}\big(\nabla u_n A_\varepsilon(P)(\nabla u_n)^t\big)dy\underset{n\to+\infty}{\longrightarrow}\pi\sqrt{(\mathrm{Cof}\,A_\varepsilon(P))(\tilde{P}-\tilde{Q})\cdot(\tilde{P}-\tilde{Q})}$$

where we used the notation of Remark 4.1, $\tilde{P}=(P,0)$ and $\tilde{Q}=(Q,0)$. Hence,

$$|\xi_\varepsilon(P)-\xi_\varepsilon(Q)|\leq\pi\sqrt{(\mathrm{Cof}\,A_\varepsilon(P))(\tilde{P}-\tilde{Q})\cdot(\tilde{P}-\tilde{Q})}+\frac{C_\varepsilon\Lambda}{\lambda}|P-Q|^2\,.$$

Defining $\Xi_\varepsilon(x_1,x_2,x_3):=\xi_\varepsilon(x_1,x_2)$ and setting $\tilde{Q}=\tilde{P}+th$ for some $h\in\mathbb{R}^2\times\{0\}$ and $t>0$, we infer from the above inequality,

$$\frac{|\Xi_\varepsilon(\tilde{P}+th)-\Xi_\varepsilon(\tilde{P})|}{t}\leq\pi\sqrt{(\mathrm{Cof}\,A_\varepsilon(P))h\cdot h}+\frac{C_\varepsilon\Lambda t|h|^2}{\lambda}$$

and we conclude, letting $t\to 0$, that

$$|\nabla\Xi_\varepsilon(\tilde{P})\cdot h|\leq\pi\sqrt{(\mathrm{Cof}\,A_\varepsilon(P))h\cdot h}\quad\forall h\in\mathbb{R}^2\times\{0\}\,.\tag{4.14}$$

Since $A_\varepsilon\in\mathcal{A}_0$, the linear map $A_\varepsilon(P):\mathbb{R}^2\times\{0\}\to\mathbb{R}^2\times\{0\}$ acts bijectively, and the same holds for $\mathrm{Cof}\,A_\varepsilon$. Then $\nabla\Xi_\varepsilon(\tilde{P})\in\mathbb{R}^2\times\{0\}$ because Ξ_ε is independent of the variable x_3. Hence, we may choose $h=(\mathrm{Cof}\,A_\varepsilon(P))^{-1}\nabla\Xi_\varepsilon(\tilde{P})$ in (4.14) which leads to $(\mathrm{Cof}\,A_\varepsilon(P))^{-1}\nabla\Xi_\varepsilon(\tilde{P})\cdot\nabla\Xi_\varepsilon(\tilde{P})\leq\pi^2$. Since A_ε and Ξ_ε does not depend on x_3 and P is arbitrary, we conclude that $(\mathrm{Cof}\,A_\varepsilon)^{-1}\nabla\Xi_\varepsilon\cdot\nabla\Xi_\varepsilon\leq\pi^2$ in $\mathbb{R}^3_+$. Now we observe that $\mathrm{Cof}\,A_\varepsilon\to\mathrm{Cof}\,A$ and $\nabla\Xi_\varepsilon\to\nabla\Xi$ a.e. in $\mathbb{R}^3_+$ as $\varepsilon\to 0$ where the function Ξ is given by $\Xi(x_1,x_2,x_3)=\xi(x_1,x_2)$. Then, passing to the limit $\varepsilon\to 0$ in the previous inequality yields

$$(\mathrm{Cof}\,A)^{-1}\nabla\Xi\cdot\nabla\Xi\leq\pi^2\quad\text{a.e. in }\mathbb{R}^3_+.$$

By Corollary 3.3, ξ is π-Lipschitz on $\mathbb{R}^2$ with respect to $\bar{d}_A$. Since $\xi(Q_0)=0$ and $\xi(\cdot)=\bar{\rho}_A(\cdot,Q_0)$, we get $\bar{\rho}_A(\cdot,Q_0)=\xi(\cdot)\leq\pi\bar{d}_A(\cdot,Q^0)$. Thus $\bar{\rho}_A\leq\pi\bar{d}_A$ because Q_0 can be chosen arbitrarily.

Due to the structure assumption $A\in\mathcal{A}_0$, we have $d_A=\bar{d}_A$ by Corollary 3.2. As $\rho_A\leq\bar{\rho}_A$, we infer from claim *(ii)* that $\pi d_A\leq\rho_A\leq\bar{\rho}_A\leq\pi\bar{d}_A=\pi d_A$, hence equality holds and ρ_A is a distance.

Step 5. We conclude the proof of Theorem 1.1 with claim *(iv)*. Let $P,Q\in\mathbb{R}^2$ be two distinct points and $\{f_n\}_{n\in\mathbb{N}}\subset X$ such that $T(f_n)=2\pi(\delta_P-\delta_Q)$ and $\rho_A(P,Q)=\pi d_A(P,Q)=\lim_{n\to+\infty}\mathcal{E}_A(f_n)$. Let $u_n\in\dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$ be the A–harmonic extension of f_n. Up to a subsequence, we may assume that $f_n\rightharpoonup f$ and $u_n\rightharpoonup u$ where u is the A–harmonic extension of f. We define $\Phi(x)$ as in (4.4) with $r=\infty$ so that Φ is clearly 1-Lipschitz with respect to d_A and has a compact support K. Arguing as in (4.5), we derive

$$\begin{aligned}\mathcal{E}_A(f_n) = E_A(u_n) &\geq \frac{1}{2}\int_{\mathbb{R}^3_+\setminus K} \mathrm{tr}\big(\nabla u_n A(\nabla u_n)^t\big) + \frac{1}{2}\int_K H(u_n)\cdot\nabla\Phi \\ &= \frac{1}{2}\int_{\mathbb{R}^3_+\setminus K} \mathrm{tr}\big(\nabla u_n A(\nabla u_n)^t\big) + \frac{1}{2} < T(f_n), \Phi_{|\mathbb{R}^2} > \\ &= \frac{1}{2}\int_{\mathbb{R}^3_+\setminus K} \mathrm{tr}\big(\nabla u_n A(\nabla u_n)^t\big) + \pi d_A(P,Q)\,.\end{aligned}$$

By lower semicontinuity, as $n\to+\infty$ we infer that

$$\pi d_A(P,Q) \geq \frac{1}{2}\int_{\mathbb{R}^3_+\setminus K} \mathrm{tr}\big(\nabla u A(\nabla u)^t\big) + \pi d_A(P,Q)\,,$$

thus u has to be constant in $\mathbb{R}^3_+\setminus K$. Since u in $A-$harmonic in the whole half space, $u\equiv\alpha$ for some constant $\alpha\in\mathbb{R}^2$ by Proposition 2.8 claim *(i)*, hence $f = u_{|\mathbb{R}^2}\equiv\alpha\in\mathbb{S}^1$. As a consequence, $\rho_A(P,Q)$ is not attained, $\rho_A(P,Q) = \bar\rho_A(P,Q)$ and the conclusion follows from claim *(ii)*. ■

We conclude this section with an example showing how gaps may occur between ρ_A and the metric distances when A depends on the x_3-variable.

Example 4.1. Given $0<\lambda<\Lambda$, we consider $\{A_n\}_{n\in\mathbb{N}}\subset\mathcal{A}$ defined as $A_n(x) = a_n(x_3)\mathrm{Id}$ with $a_n(x_3) = \max\{\lambda, \Lambda - nx_3\}$. One easily checks that for each $n\geq 1$ and each distint points $P,Q\in\mathbb{R}^2$,

$$\lambda|P-Q| \leq d_{A_n}(P,Q) < \bar d_{A_n}(P,Q) = \Lambda|P-Q|\,,$$

hence $\bar\rho_{A_n} = \pi\bar d_{A_n}$ by Theorem 1.1, claims *(i)* and *(ii)*. We claim that for n large enough, $\pi d_{A_n}(P,Q) < \rho_{A_n}(P,Q) < \bar\rho_{A_n}(P,Q) = \pi\bar d_{A_n}(P,Q)$ and

$$\lim_{n\to+\infty}\rho_{A_n}(P,Q) = \lim_{n\to+\infty}\pi d_{A_n}(P,,Q) = \pi\lambda|P-Q|\,,$$

for any $P,Q\in\mathbb{R}^2$ with $P\neq Q$. The first inequality is a consequence of Theorem 1.1, claims *(ii)* and *(iv)*. Therefore it suffices to show that $\limsup_{n\to+\infty}\rho_{A_n}(P,Q)\leq\pi\lambda|P-Q|$. An easy application of dominated convergence yields $\limsup_{n\to+\infty}\mathcal{E}_{A_n}(g) = \lambda\mathcal{E}_{\mathrm{Id}}(g)$ for every $g\in X$, so that $\limsup_{n\to+\infty}\rho_{A_n}(P,Q)\leq\lambda\rho_{\mathrm{Id}}(P,Q) = \pi\lambda|P-Q|$ by Theorem 1.1, claim *(iii)*.

5. Sobolev maps and graph currents

5.1 Pre-Jacobians and liftings of $\dot H^{1/2}(\mathbb{R}^2;\mathbb{S}^1)$–maps

In order to understand the concentration effects related to the minimization problems (1.6), (1.7) and (1.8), it is very useful to introduce for each $f\in X$ the pre-Jacobian of f as the 1–dimensional current $J(f)\in\mathcal{D}_1(\mathbb{R}^2)$ defined by

$$< J(f),\zeta >= \int_{\mathbb{R}^2}\left(-\zeta_2\, f\wedge\partial_1 f + \zeta_1\, f\wedge\partial_2 f\right) \tag{5.1}$$

for every $\zeta\in\mathcal{D}^1(\mathbb{R}^2)$ where $\zeta = \zeta_1 dx_1 + \zeta_2 dx_2$ and $\zeta_1,\zeta_2\in C^\infty_0(\mathbb{R}^2)$. The integral above will be always understood as an $\dot H^{1/2} - \dot H^{-1/2}$ duality according to Remark 2.2. One easily check that $J(f) = 0$ whenever f is constant and $X\ni f\mapsto J(f)\in\mathcal{D}_1(\mathbb{R}^2)$ is continuous under strong convergence in X because of the simple estimate

$$|< J(f_1) - J(f_2),\zeta >| \leq C(\mathrm{spt}\,\zeta)\|\zeta\|_{C^1}(\|f_1\|_{1/2} + \|f_2\|_{1/2})\|f_1 - f_2\|_{1/2}\,, \tag{5.2}$$

which is a consequence of Proposition 2.3. In addition $\partial J(f) = T(f)$ as distributions, according to (1.5).

The following elementary property of the operator J will be of importance in the sequel.

Lemma 5.1. *Let f_1 , $f_2 \in X$. We have*

$$< J(f_1\bar{f}_2), \zeta > = < J(f_1) - J(f_2), \zeta > \qquad \forall \zeta \in \mathcal{D}^1(\mathbb{R}^2)\,. \tag{5.3}$$

Proof. First we assume that $f_1, f_2 \in X \cap W^{1,1}_{\rm loc}(\mathbb{R}^2)$ and f_1, f_2 are constant outside a compact set. In this case, the conclusion comes from a straightforward computation. Then the general case follows by density, according to Theorem 2.6 and Proposition 2.6. ■

Our first goal in this section is to represent the current $J(f)$ in terms of suitable liftings of the map f. Concerning maps without topological singularities, we have the following result.

Proposition 5.1. *For every $f \in X$ such that $T(f) = 0$, there exist $\phi \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R})$ and $\psi \in \dot{BV}(\mathbb{R}^2;\mathbb{R})$ satisfying $f = e^{i(\phi+\psi)}$ a.e. in $\mathbb{R}^2$ and*

$$|\phi|_{1/2} \leq C|f|_{1/2}\,, \tag{5.4}$$

$$|\psi|_{BV} \leq C|f|^2_{1/2}\,, \tag{5.5}$$

for an absolute constant $C > 0$. In addition, for any $\zeta = \zeta_1 dx_1 + \zeta_2 dx_2 \in \mathcal{D}^1(\mathbb{R}^2)$, we have

$$< J(f), \zeta > = \int_{\mathbb{R}^2} \big(- \zeta_2\partial_1\phi + \zeta_1\partial_2\phi\big) + \int_{\mathbb{R}^2} \big(- \zeta_2 D_1\psi + \zeta_1 D_2\psi\big)\,. \tag{5.6}$$

Proof. Without loss of generality, we may assume $f^\infty = 1$ by Lemma 5.1. Since $T(f) = 0$, there exists $\{f_n\}_{n\in\mathbb{N}} \subset C^\infty_{\rm const}(\mathbb{R}^2;\mathbb{S}^1)$ such that $f_n \to f$ strongly in $\dot{H}^{1/2}$ and a.e. in $\mathbb{R}^2$ by Theorem 2.7. We may also assume that $f_n^\infty \equiv 1$ for each n. Therefore there exists $\{\eta_n\}_{n\in\mathbb{N}} \subset C^\infty_0(\mathbb{R}^2;\mathbb{R})$ such that for each n we have $f_n = e^{i\eta_n}$ everywhere in $\mathbb{R}^2$. Let $\{Q_k\}_{k\in\mathbb{N}}$ be an increasing sequence of open squares, $Q_k \subset \mathbb{R}^2$, $Q_k \subset\subset Q_{k+1}$ for each k, and $\cup_k Q_k = \mathbb{R}^2$, such that ${\rm spt}\,\eta_n \subset Q_n$ for each n.

According to [9], Theorem 3, we can write $\eta_n = \phi_n + \psi_n$ for two functions $\phi_n \in H^{1/2}(Q_n)$ and $\psi_n \in BV(Q_n)$ such that in terms of $f_n = e^{i(\phi_n+\psi_n)}$ we have the estimates

$$|\phi_n|_{1/2,Q_n} \leq C|f_n|_{1/2,Q_n} \leq C|f|_{1/2}\,, \tag{5.7}$$

$$|\psi_n|_{BV(Q_n)} \leq C|f_n|^2_{1/2,Q_n} \leq C|f|^2_{1/2}\,. \tag{5.8}$$

for some absolute constant $C > 0$ independent of f_n and Q_n (indeed, we stress that the constant $C > 0$ in [9], Theorem 3, is independent of the square). Thus, for a given $\zeta \in \mathcal{D}^1(\mathbb{R}^2)$ and $k_0 \in \mathbb{N}$ such that ${\rm spt}\,\zeta \subset Q_{k_0}$, we have for $n \geq k_0$,

$$\begin{aligned} < J(f_n), \zeta > = \int_{\mathbb{R}^2} (-\zeta_2\partial_1\eta_n + \zeta_1\partial_2\eta_n) = &\int_{Q_{k_0}} (-\zeta_2\partial_1\phi_n + \zeta_1\partial_2\phi_n) \\ &+ \int_{Q_{k_0}} (-\zeta_2 D_1\psi_n + \zeta_1 D_2\psi_n)\,. \end{aligned} \tag{5.9}$$

Taking (5.7) and (5.8) into account, by a standard diagonal argument (possibly subtracting suitable multiples of 2π) we may assume $\phi_n \to \phi$ (respectively $\psi_n \to \psi$) both in $L^1_{\rm loc}(\mathbb{R}^2)$ and a.e. in $\mathbb{R}^2$ and weakly in $H^{1/2}(Q_k)$ (respectively weakly-$\star$ in $BV(Q_k)$) for each $k \geq 1$ fixed, as $n \to +\infty$. This way $\phi \in \dot{H}^{1/2}(\mathbb{R}^2)$, $\phi \in \dot{BV}(\mathbb{R}^2)$ and the desired norm bounds (5.4), (5.5) follow by lower semicontinuity.

In addition, $f = e^{i(\phi+\psi)}$ a.e. since $f_n \to f$ a.e.. Finally, as $f_n \to f$ strongly in $\dot{H}^{1/2}$, (5.2) allows us to pass to the limit $n \to +\infty$ in the left hand side of (5.9) which yields (5.6). ∎

In order to deal with maps with topological singularities, we need the following lemma.

Lemma 5.2. *Let $f \in X$ be a dipole map as constructed in Lemma 4.2, i.e., such that $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{\mathrm{P}, \mathrm{Q}\})$ and $f = e^{i\psi}$ with $\psi \in BV(\mathbb{R}^2;\mathbb{R}) \cap \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^2 \setminus [\mathrm{P}, \mathrm{Q}])$ for some distinct points $P, Q \in \mathbb{R}^2$, satisfying :*

(i) $T(f) = 2\pi(\delta_p - \delta_Q)$,

(ii) $|f|^2_{1/2} + |\psi|_{BV} \le C|P - Q|$,

(iii) $(D^s\psi)^\perp = 2\pi \frac{P-Q}{|P-Q|}\, \mathcal{H}^1 \llcorner [P, Q] = 2\pi\overrightarrow{QP}$.

Then

$$< J(f), \zeta >= \int_{\mathbb{R}^2} (-\zeta_2 D_1\psi + \zeta_1 D_2\psi) + 2\pi < \overrightarrow{QP}, \zeta > , \tag{5.10}$$

for any $\zeta = \zeta_1 dx_1 + \zeta_2 dx_2 \in \mathcal{D}^1(\mathbb{R}^2)$.

Proof. As in the proof of Lemma 4.2, up to rotation, translation and dilation we may assume $Q = (1, 0) = -P$. Let $\zeta \in \mathcal{D}^1(\mathbb{R}^2)$ and let $\Omega = I_1 \times I_2 \subset \mathbb{R}^2$ a square containing P and Q such that $\mathrm{spt}\,\zeta \subset \Omega$. Clearly $f \in H^{1/2}(\Omega)$ and

$$< J(f), \zeta >= \int_\Omega (-\zeta_2 f \wedge \partial_1 f + \zeta_1 f \wedge \partial_2 f) \ .$$

In addition, since $f \in L^2(I_1; H^{1/2}(I_2; \mathbb{S}^1)) \cap L^2(I_2; H^{1/2}(I_1; \mathbb{S}^1))$ with equivalence of norms (see [35]), we have

$$< J(f), \zeta >= -\int_{I_2}\int_{I_1} \zeta_2 f \wedge \partial_1 f + \int_{I_1}\int_{I_2} \zeta_1 f \wedge \partial_2 f \, .$$

By construction $f(\cdot, x_2) \in \mathrm{Lip}(I_1; \mathbb{S}^1)$ for a.e. $x_2 \in I_2$ and $f(\cdot, x_2) = e^{i\psi(\cdot,x_2)}$ with $\psi(\cdot, x_2) \in \mathrm{Lip}(I_1; \mathbb{R})$ for a.e. $x_2 \in I_2$. Hence the standard chain rule for Lipschitz functions gives

$$-\int_{I_2}\int_{I_1} \zeta_2 f \wedge \partial_1 f = -\int_{I_2}\int_{I_1} \zeta_2 D_1\psi = -\int_{\mathbb{R}^2} \zeta_2 D_1\psi \, .$$

Now we recall that $\psi \in L^1(I_1; BV(I_2)) \cap L^1(I_2; BV(I_1))$ with equivalence of norms because $\psi \in BV(I_1 \times I_2)$ (see [3]). On the other hand, by construction, $f(x_1, \cdot) \in \mathrm{Lip}(I_2; \mathbb{S}^1)$ for a.e. $x_1 \in I_1$. Hence, for a.e. $x_1 \in I_1$, there exists a lifting function $\tilde{\psi}_{x_1} \in \mathrm{Lip}(I_2, \mathbb{R})$ such that $f(x_1, \cdot) = e^{i\tilde{\psi}_{x_1}(\cdot)} = e^{i\psi(x_1,\cdot)}$ a.e. in I_2. Arguing as above, we infer that

$$\int_{I_1}\int_{I_2} \zeta_1 f \wedge \partial_2 f = \int_{\mathbb{R}^2} \zeta_1 D_2\psi + \int_{I_1}\left(\int_{I_2} \zeta_1 D_2\big(\tilde{\psi}_{x_1}(x_2) - \psi(x_1, x_2)\big)dx_2\right)dx_1 \, ,$$

and the lemma is completely proved once we show that

$$\int_{I_1}\left(\int_{I_2} \zeta_1 D_2\big(\tilde{\psi}_{x_1}(x_2) - \psi(x_1, x_2)\big)dx_2\right)dx_1 = 2\pi < \overrightarrow{QP}, \zeta > \ .$$

Since $\tilde{\psi}_{x_1}(\cdot) - \psi(x_1, \cdot) \in BV(I_2; 2\pi\mathbb{Z})$, we may argue as in [33], eq. (3.23), taking into account the properties of ψ, to derive that, for a.e. $x_1 \in I_1$,

$$D_2\big(\tilde{\psi}_{x_1}(\cdot) - \psi(x_1, \cdot)\big) = \begin{cases} -2\pi\delta_0 & \text{if } |x_1| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

whence the conclusion. ■

The main result of this section is the following.

Theorem 5.1. *For every* $f \in X$*, there exist* $\phi \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R})$ *and* $\psi \in \dot{BV}(\mathbb{R}^2;\mathbb{R})$ *such that* $f = e^{i(\phi+\psi)}$ *a.e. in* $\mathbb{R}^2$ *and*

$$|\phi|_{1/2} \leq C|f|_{1/2}\,, \tag{5.11}$$

$$|\psi|_{BV} \leq C|f|^2_{1/2}\,, \tag{5.12}$$

for an absolute constant $C > 0$*. In addition, there exists an integer multiplicity* 1*-rectifiable current* $t \in \mathcal{R}_1(\mathbb{R}^2)$ *of finite mass* $\mathbf{M}(t) \leq C|f|^2_{1/2}$ *such that* $2\pi\partial t = T(f)$ *and for any* $\zeta = \zeta_1 dx_1 + \zeta_2 dx_2 \in \mathcal{D}^1(\mathbb{R}^2)$*, we have*

$$< J(f),\zeta >= \int_{\mathbb{R}^2} \big(-\zeta_2\partial_1\phi + \zeta_1\partial_2\phi\big) + \int_{\mathbb{R}^2} \big(-\zeta_2 D_1\psi + \zeta_1 D_2\psi\big) + 2\pi < t,\zeta > \,. \tag{5.13}$$

Proof. *Step 1.* First we consider the case $f \in X \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ with f smooth except at finitely many points $a_1,\dots,a_k$, and constant outside a compact set. According to Lemma 2.2, we have $T(f) = -2\pi\sum_{i=1}^k d_i\delta_{a_i}$, where $d_i = \deg(f,a_i)$. Since $\sum_{i=1}^k d_i = 0$, we may relabel the a_i's, taking the multiplicity $|d_i|$ into account, so that $T(f) = -2\pi\sum_{i=1}^N(\delta_{p_j} - \delta_{q_j})$ with $2N = \sum_{i=1}^k |d_i|$, where the p_j's (resp. the q_j's) correspond to singular points with positive (resp. negative) degree. In addition, we may also relabel the points $\{q_j\}$ in such a way that

$$\sum_{j=1}^N |p_j - q_j| = \operatorname*{Min}_{\sigma\in\mathcal{S}_N} \sum_{j=1}^N |p_j - q_{\sigma(j)}|\,.$$

where $\mathcal{S}_N$ denotes the set of all permutations of N indices. In view of (1.18) and a well known result in [13], we have

$$\operatorname*{Min}_{\sigma\in\mathcal{S}_N} \sum_{j=1}^N |p_j - q_{\sigma(j)}| = L_{\mathrm{Id}}(f)\,.$$

By an induction argument based on Lemma 4.1 and Lemma 4.2 (see also proof of Proposition 7.3), we can find for each $j = 1,\dots,N$, a dipole map f_j such that $T(f_j) = -2\pi(\delta_{p_j} - \delta_{q_j})$, $\mathcal{E}_{\mathrm{Id}}(f_j) \leq 2\pi|p_j - q_j|$ and $\mathcal{E}_{\mathrm{Id}}(\prod_{j=1}^N f_j) \leq 2\pi\sum_{j=1}^N |p_j - q_j|$. Setting $g = \prod_{j=1}^N f_j$, we infer from Lemma 4.2, Proposition 2.6 and Lemma 7.2 below, that $g \in X$ is constant outside a compact set, $g \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{a_1,\dots,a_N\})$, $T(g) = T(f)$ and $\mathcal{E}_{\mathrm{Id}}(g) \leq 2\pi L_{\mathrm{Id}}(f) \leq C\mathcal{E}_{\mathrm{Id}}(f)$ for some absolute constant $C > 0$. Now we consider $\tilde{f} = \bar{g}f$. By Proposition 2.6 and the above properties of g, we have $\tilde{f} \in X$, $T(\tilde{f}) = 0$, $f = g\tilde{f}$ a.e. and $|\tilde{f}|^2_{1/2} \leq C|f|^2_{1/2}$. In view of Lemma 5.1, Proposition 5.1 and the previous inequality, there exist $\tilde{\phi} \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R})$ and $\tilde{\psi} \in \dot{BV}(\mathbb{R}^2;\mathbb{R})$ such that $\tilde{f} = e^{(\tilde{\phi}+\tilde{\psi})}$ a.e., $|\tilde{\phi}|_{1/2} \leq C|f|_{1/2}$, $|\tilde{\psi}|_{BV} \leq C|f|^2_{1/2}$ and

$$< J(f),\zeta >=< J(g),\zeta > + \int_{\mathbb{R}^2}(-\zeta_2\partial_1\tilde{\phi} + \zeta_1\partial_2\tilde{\phi}) + \int_{\mathbb{R}^2}(-\zeta_2 D_1\tilde{\psi} + \zeta_1 D_2\tilde{\psi})\,, \tag{5.14}$$

for each $\zeta = \zeta_1 dx_1 + \zeta_2 dx_2 \in \mathcal{D}^1(\mathbb{R}^2)$. By Lemma 4.2, for each $j = 1,\dots,N$, we have $f_j = e^{i\psi_j}$, where $\psi_j \in BV(\mathbb{R}^2;\mathbb{R})$ has compact support and $|\psi_j|_{BV} \leq C|p_j - q_j|$. Thus, setting $\hat{\psi} = \sum_{j=1}^N \psi_j$, we have $\hat{\psi} \in \dot{BV}(\mathbb{R}^2;\mathbb{R})$, $g = e^{i\hat{\psi}}$ a.e. and

$$|\hat{\psi}|_{BV} \leq C\sum_{j=1}^N |p_j - q_j| \leq C|f|^2_{1/2}$$

Then we derive from Lemma 5.1 and Lemma 5.2 that

$$< J(g),\zeta >= \sum_{j=1}^{N} < J(f_j),\zeta >= \int_{\mathbb{R}^2} (-\zeta_2 D_1\hat{\psi} + \zeta_1 D_2\hat{\psi}) + 2\pi < \sum_{j=1}^{N} \overrightarrow{p_j q_j}, \zeta > \ . \tag{5.15}$$

Finally, we introduce $\phi = \tilde{\phi} \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R})$, $\psi = \tilde{\psi} + \hat{\psi} \in \dot{BV}(\mathbb{R}^2;\mathbb{R})$ and the current $t = \sum_{j=1}^{N} \overrightarrow{p_j q_j} \in \mathcal{R}_1(\mathbb{R}^2)$, we have $|\phi|_{1/2} \leq C|f|_{1/2}$, $|\psi|_{BV} \leq C|f|^2_{1/2}$ and $\mathbf{M}(t) \leq \sum_{j=1}^{N} \mathbf{M}(\overrightarrow{p_j q_j}) = L_{\mathrm{Id}}(f) \leq C|f|^2_{1/2}$. Combining (5.6) with (5.10) the conclusion follows.

Step 2. Now we consider the case of a general $f \in X$. By theorem 2.6, there exists $\{f_n\}_{n\in\mathbb{N}} \subset X \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ a sequence of functions, smooth outside finitely many points and constant outside compact sets, such that $f_n \to f$ strongly in $\dot{H}^{1/2}$ and a.e. as $n \to +\infty$. Since the product is continuous, up to subsequences, we may always assume that $|f_n \bar{f}|^2_{1/2} \leq 4^{-n}|f|^2_{1/2}$ for each $n \geq 1$. We define $g_1 = f_1$ and for $j \geq 2$, we set $g_j = f_j \bar{f}_{j-1}$. Clearly $g_j \in X \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$, g_j is constant outside a compact set and smooth except at finitely many points. In addition, $\prod_{j=1}^{n} g_j = f_n$ and we have $|g_j|^2_{1/2} \leq C4^{-j}|f|^2_{1/2}$ writing $g_j = (f_j\bar{f})(f\bar{f}_{j-1})$ and applying Proposition 2.6. To each function g_j, we may now apply Step 1 to obtain three sequences $\{\hat{\phi}_j\}_{j\in\mathbb{N}} \subset \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R})$, $\{\hat{\psi}_j\}_{j\in\mathbb{N}} \subset \dot{BV}(\mathbb{R}^2;\mathbb{R})$ and $\{\hat{t}_j\}_{j\in\mathbb{N}} \subset \mathcal{R}_1(\mathbb{R}^2)$ such that $g_j = e^{i(\hat{\phi}_j + \hat{\psi}_j)}$ a.e. in $\mathbb{R}^2$,

$$|\hat{\phi}_j|_{1/2} \leq C2^{-j}|f|_{1/2}\,, \quad |\hat{\psi}_j|_{BV} \leq C4^{-j}|f|^2_{1/2}\,, \quad \mathbf{M}(\hat{t}_j) \leq C4^{-j}|f|^2_{1/2} \tag{5.16}$$

and

$$< J(g_j),\zeta >= \int_{\mathbb{R}^2} \big(-\zeta_2\partial_1\hat{\phi}_j + \zeta_1\partial_2\hat{\phi}_j\big) + \int_{\mathbb{R}^2} \big(-\zeta_2 D_1\hat{\psi}_j + \zeta_1 D_2\hat{\psi}_j\big) + 2\pi < \hat{t}_j, \zeta > \tag{5.17}$$

for any $j \geq 1$ and any $\zeta \in \mathcal{D}^1(\mathbb{R}^2)$. Next we define

$$\phi_n = \sum_{j=1}^{n} \hat{\phi}_n \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R})\,, \quad \psi_n = \sum_{j=1}^{n} \hat{\psi}_n \in \dot{BV}(\mathbb{R}^2;\mathbb{R})\,, \quad t_n = \sum_{j=1}^{n} \hat{t}_n \in \mathcal{R}_1(\mathbb{R}^2)\,.$$

Clearly $f_n = e^{i(\phi_n + \psi_n)}$ a.e. and we derive from (5.16) that

$$|\phi_n|_{1/2} \leq C|f|_{1/2}\,, \quad |\psi_n|_{BV} \leq C|f|^2_{1/2}\,, \quad \mathbf{M}(t_n) \leq C|f|^2_{1/2}\,. \tag{5.18}$$

Summing up over j in (5.17) and applying Lemma 5.3, we obtain

$$< J(f_n),\zeta >= \int_{\mathbb{R}^2} \big(-\zeta_2\partial_1\phi_n + \zeta_1\partial_2\phi_n\big) + \int_{\mathbb{R}^2} \big(-\zeta_2 D_1\psi_n + \zeta_1 D_2\psi_n\big) + 2\pi < t_n, \zeta > \tag{5.19}$$

for any $n \geq 1$ and any $\zeta \in \mathcal{D}^1(\mathbb{R}^2)$. Subtracting suitable multiples of 2π if necessary and passing to subsequences, we infer from (5.16) that

$$\phi = \lim_{n\to+\infty} \phi_n = \sum_{j=1}^{\infty} \hat{\phi}_j \quad \text{and} \quad \psi = \lim_{n\to+\infty} \psi_n = \sum_{j=1}^{\infty} \hat{\psi}_j$$

exist in the weak $\dot{H}^{1/2}$–topology, respectively in the weak-$\star$ $\dot{BV}$–topology, and a.e. in $\mathbb{R}^2$. In addition, $f = e^{i(\phi+\psi)}$ a.e. in $\mathbb{R}^2$ and (5.11)–(5.12) follow from (5.18) by lower semicontinuity. Similarly, (5.16) and (5.18) yield the existence of

$$t = \lim_{n\to+\infty} t_n = \sum_{j=1}^{\infty} \hat{t}_j \ \in \mathcal{R}_1(\mathbb{R}^2)$$

as a weak limit of currents with $\mathbf{M}(t) \leq C|f|^2_{1/2}$. Finally, we deduce (5.13) letting $n \to +\infty$ in (5.19) taking into account the strong convergence of f_n together with (5.2), and the weak convergences of ϕ_n, ψ_n and t_n. ■

Using the previous representation formula it is possible to describe the behaviour of pre-Jacobians under weak convergence. We have the following.

Proposition 5.2. *Let $f \in X$ and $\{f_n\}_{n\in\mathbb{N}} \subset X$ such that $T(f_n) \equiv T_0 \in (\mathrm{Lip}(\mathbb{R}^2))'$ for all n and $f_n \rightharpoonup f$ weakly in $\dot{H}^{1/2}$. Set $M = \sup_n |f_n|_{1/2}$. Up to subsequences, there exists an integer multiplicity 1-rectifiable current $\Theta \in \mathcal{R}_1(\mathbb{R}^2)$ of finite mass, $\mathbf{M}(\Theta) \leq CM^2$ for an absolute constant $C > 0$, such that $2\pi\partial\Theta = T_0 - T(f)$ and for any $\zeta = \zeta_1 dx_1 + \zeta_2 dx_2 \in \mathcal{D}^1(\mathbb{R}^2)$,*

$$< J(f_n), \zeta > \underset{n\to+\infty}{\longrightarrow} < J(f), \zeta > +2\pi < \Theta, \zeta > \ . \tag{5.20}$$

Proof. Up to subsequence we may assume $f_n \to f$ a.e. in $\mathbb{R}^2$. Let $n \geq 1$ be fixed and write $f_n = (f_n\bar{f}_1)(f_1\bar{f})f$, so that $J(f_n) = J(f_n\bar{f}_1) + J(f_1\bar{f}) + J(f)$ by Lemma 5.3. Since $T(f_n) \equiv T_0$, using Proposition 2.6 we obtain $T(f_n\bar{f}_1) = 0$. By Proposition 5.1, there exist $\phi_n \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R})$, $|\phi_n|_{1/2} \leq CM$, and $\psi_n \in \dot{BV}(\mathbb{R}^2;\mathbb{R})$, $|\psi_n|_{BV} \leq CM^2$, such that $f_n\bar{f}_1 = e^{i(\phi_n+\psi_n)}$ a.e. and

$$< J(f_n\bar{f}_1), \zeta >= \int_{\mathbb{R}^2} \big(-\zeta_2\partial_1\phi_n + \zeta_1\partial_2\phi_n \big) + \int_{\mathbb{R}^2} \big(-\zeta_2 D_1\psi_n + \zeta_1 D_2\psi_n \big) \tag{5.21}$$

for any $\zeta \in \mathcal{D}^1(\mathbb{R}^2)$. On the other hand, by Theorem 5.1, there exists $\phi \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R})$, $|\phi|_{1/2} \leq CM$, and $\psi \in \dot{BV}(\mathbb{R}^2;\mathbb{R})$, $|\psi|_{BV} \leq CM^2$, and a current $t \in \mathcal{R}_1(\mathbb{R}^2)$, $\mathbf{M}(t) \leq CM^2$, such that $f_1\bar{f} = e^{i(\phi+\psi)}$ a.e. and

$$< J(f_1\bar{f}), \zeta >= \int_{\mathbb{R}^2} \big(-\zeta_2\partial_1\phi + \zeta_1\partial_2\phi \big) + \int_{\mathbb{R}^2} \big(-\zeta_2 D_1\psi + \zeta_1 D_2\psi \big) + 2\pi < t, \zeta > \tag{5.22}$$

for any $\zeta \in \mathcal{D}^1(\mathbb{R}^2)$. Up to subsequences (and possibly subtracting suitable multiples of 2π), we may assume $\phi_n \rightharpoonup \tilde{\phi}$ weakly in $\dot{H}^{1/2}$ and a.e. in $\mathbb{R}^2$. Similarly, we may assume $\psi_n \overset{\star}{\rightharpoonup} \tilde{\psi}$ weakly-$\star$ in $\dot{BV}$ and a.e. in $\mathbb{R}^2$. Obviously, $f\bar{f}_1 = e^{i(\tilde{\phi}+\tilde{\psi})}$ a.e. and by lower semicontinuity, we have the norm bounds $|\tilde{\phi}|_{1/2} \leq CM$, and $|\tilde{\psi}|_{BV} \leq CM^2$. Combining the decomposition of $J(f_n)$ given by Lemma 5.1 with (5.21) and (5.22), we deduce that

$$\begin{aligned} \lim_{n\to+\infty} < J(f_n), \zeta > = \ & < J(f), \zeta > + \int_{\mathbb{R}^2} \big(-\zeta_2\partial_1(\phi+\tilde{\phi}) + \zeta_1\partial_2(\phi+\tilde{\phi}) \big) + \\ & + \int_{\mathbb{R}^2} \big(-\zeta_2 D_1(\psi+\tilde{\psi}) + \zeta_1 D_2(\psi+\tilde{\psi}) \big) + 2\pi < t, \zeta > \end{aligned} \tag{5.23}$$

for any $\zeta \in \mathcal{D}^1(\mathbb{R}^2)$. Since $1 = (f_1\bar{f})(f\bar{f}_1) = e^{i(\phi+\psi)}e^{i(\tilde{\phi}+\tilde{\psi})}$ a.e. in $\mathbb{R}^2$, we have $(\phi+\tilde{\phi}) + (\psi+\tilde{\psi}) \in \big(\dot{H}^{1/2} + \dot{BV}\big)(\mathbb{R}^2; 2\pi\mathbb{Z})$ with the norm bounds $|\phi+\tilde{\phi}|_{1/2} \leq CM$, $|\psi+\tilde{\psi}|_{BV} \leq CM^2$. Then, the conclusion follows from Lemma 5.3 below and (5.23) for $\Theta = t + \bar{t}$. ■

Lemma 5.3. *Let $\phi \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R})$ and $\psi \in \dot{BV}(\mathbb{R}^2;\mathbb{R})$ such that $\big(\phi+\psi\big)(\cdot) \in \mathbb{Z}$ a.e. in $\mathbb{R}^2$. Then there exist $\theta \in \dot{BV}(\mathbb{R}^2;\mathbb{Z})$ and an integer multiplicity rectifiable current $\bar{t} \in \mathcal{R}_1(\mathbb{R}^2)$ of finite mass without boundary such that $\phi + \psi = \theta$ a.e. in $\mathbb{R}^2$ and $|\theta|_{BV} \leq |\psi|_{BV}$, $\mathbf{M}(\bar{t}) \leq C|\psi|_{BV}$ for an absolute constant $C > 0$, and*

$$\int_{\mathbb{R}^2} \big(-\zeta_2\partial_1\phi + \zeta_1\partial_2\phi \big) + \int_{\mathbb{R}^2} \big(-\zeta_2 D_1\psi + \zeta_1 D_2\psi \big) = < \bar{t}, \zeta > \tag{5.24}$$

for any $\zeta = \zeta_1 dx_1 + \zeta_2 dx_2 \in \mathcal{D}^1(\mathbb{R}^2)$.

Proof. Let us set $\theta = \phi + \psi$, so that $\theta(x_1, x_2) \in \mathbb{Z}$ a.e. (x_1, x_2), and fix an arbitrary square $Q = I \times J \subset \mathbb{R}^2$. Clearly $\phi \in H^{1/2}(Q)$ and $\psi \in BV(Q)$ with uniform seminorm bound $|\phi|_{1/2,Q} \leq |\phi|_{1/2}$ and $|\psi|_{BV(Q)} \leq |\psi|_{BV}$. Clearly $\theta \in L^1(Q)$ and we claim that $\theta \in BV(Q)$ and $|\theta|_{BV(Q)} \leq |\psi|_{BV(Q)}$, whence $\theta \in \dot{BV}(\mathbb{R}^2; \mathbb{Z})$ with the desired bound. To prove the claim we argue by slicing. Recall that (see [35] and [3]) $\phi \in L^2(I; H^{1/2}(J)) \cap L^2(J; H^{1/2}(I))$ with equivalence of norms and similarly $\psi \in L^1(I; BV(J)) \cap L^1(J; BV(I))$ with equivalence of norms. Therefore, it is enough to show that $\theta \in L^1(I; BV(J)) \cap L^1(J; BV(I))$, i.e., that the slices $\theta(x_1, \cdot)$ and $\theta(\cdot, x_2)$ satisfy the seminorm bounds $\int_I |\theta(x_1, \cdot)|_{BV(J)} dx_1 \leq \int_I |\psi(x_1, \cdot)|_{BV(J)}$ and $\int_J |\theta(\cdot, x_2)|_{BV(I)} dx_2 \leq \int_J |\psi(\cdot, x_2)|_{BV(J)}$. These inequalities hold pointwise under integral signs as follows from [33], pag. 265-267, so the claim holds, i.e., $\theta \in \dot{BV}$ and $|\theta|_{BV} \leq |\psi|_{BV}$.

Since $\theta = \phi + \psi$ a.e., taking the derivatives in the sense of distributions in (5.24), it remains to prove that for each $\zeta \in \mathcal{D}^1(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \big(-\zeta_2 D_1 \theta + \zeta_1 D_2 \theta \big) = -\int_{\mathbb{R}^2} \zeta D^\perp \theta = < \bar{t}, \zeta >$$

for some integer multiplicity 1-rectifiable current $\bar{t} \in \mathcal{R}_1(\mathbb{R}^2)$ of finite mass $\mathbf{M}(\bar{t}) \leq C|\theta|_{BV}$. We observe that such a current must have zero boundary, as follows from the previous formula taking derivatives in the sense of distributions.

First, we recall that $|\theta|_{BV} = \int_{\mathbb{R}} |\chi_{\{\theta > t\}}|_{BV} dt < \infty$. Hence if we set for each $j \in \mathbb{Z}$, $\Omega_j = \{|\theta - j| < 1\}$ and $\theta_j = j \chi_{\Omega_j}$, then $\theta_j \in \dot{BV}(\mathbb{R}^2)$, $\theta \equiv \theta_j$ a.e. in Ω_j, $\theta = \sum_{j=-\infty}^{\infty} \theta_j$ a.e. in $\mathbb{R}^2$ and $\sum_{j=-\infty}^{\infty} |\theta_j|_{BV} = |\theta|_{BV}$. On the other hand, we have $D\theta_j = j\, n_j \mathcal{H}^1 \llcorner \partial_* \Omega_j$, where n_j is the inward measure theoretic normal at the points of the reduced boundary. Therefore, $t_j \equiv D^\perp \theta_j = j\, n_j^\perp \mathcal{H}^1 \llcorner \partial_* \Omega_j$ is a 1-dimensional integer multiplicity rectifiable current of finite mass $t_j \in \mathcal{R}_1(\mathbb{R}^2)$ with $\mathbf{M}(t_j) = |\theta_j|_{BV}$. As a consequence $\bar{t} = \lim_{k \to +\infty} \sum_{j=-k}^{k} t_j$ exists as a weak limit of currents and it satisfies $\mathbf{M}(\bar{t}) \leq |\theta|_{BV}$ and $\bar{t} = \sum_{j=-\infty}^{\infty} t_j = \sum_{j=-\infty}^{\infty} D^\perp \theta_j = D^\perp \theta$ as $\mathbb{R}^2-$valued measures, which ends the proof. ■

5.2 Graph currents of $\dot{H}^{1/2}(\mathbb{R}^2; \mathbb{S}^1)$–maps

In order to interpret lack of compactness of minimizing sequences in terms of bubbling-off of vertical current, it is very convenient to consider the graphs of a sequence of maps $\{f_n\}_{n \in \mathbb{N}}$ as two dimensional currents in the product space $\mathbb{R}^2 \times \mathbb{S}^1$, as already pursued in [22], [25], in the spirit or the general theory of Cartesian currents developed in [21]. Our approach here is more direct and essentially based on Theorem 5.1. For another approach regarding graph currents as integral flat chains see e.g. [25].

Given $f \in C^\infty(\mathbb{R}^2; \mathbb{S}^1)$, the graph of f is a 2-dimensional smooth submanifold without boundary, $G_f \subset \mathbb{R}^2 \times \mathbb{S}^1$, with the natural orientation induced by the parametrization $(x_1, x_2) \in \mathbb{R}^2 \mapsto \big(x_1, x_2, f(x_1, x_2)\big)$. The *graph current* G_f associated to f is defined by its action on smooth compactly supported 2-forms $\vartheta \in \mathcal{D}^2(\mathbb{R}^2_s \times \mathbb{S}^1_{s'})$, $s = (x_1, x_2)$, through the formula

$$< G_f, \vartheta > = \int_{G_f} \vartheta \,. \tag{5.25}$$

Clearly, by Stokes theorem, we have

$$\int_{G_f} d\beta = \int_{\partial G_f} \beta = 0 \,, \qquad \forall\ \beta\ \in \mathcal{D}^1(\mathbb{R}^2 \times \mathbb{S}^1) \,, \tag{5.26}$$

since G_f has no boundary in $\mathbb{R}^2 \times \mathbb{S}^1$.

Let ω be the standard volume form on $\mathbb{S}^1$. Then every 2-form $\vartheta \in \mathcal{D}^2(\mathbb{R}^2_s \times \mathbb{S}^1_{s'})$ can be uniquely and globally written as

$$\begin{aligned}\vartheta(s,s') &= F_0(s,s')dx_1 \wedge dx_2 + \big(\, F_1(s,s')dx_1 + F_2(s,s')dx_2 \,\big) \wedge \omega(s') \\ &= \vartheta^{2,0}(s,s') + \vartheta^{1,1}(s,s')\,,\end{aligned} \tag{5.27}$$

for suitable smooth functions $F_0, F_1, F_2 \in C^\infty_0(\mathbb{R}^2 \times \mathbb{S}^1; \mathbb{R})$. In other words, we have the direct sum decomposition $\mathcal{D}^2(\mathbb{R}^2_s \times \mathbb{S}^1_{s'}) = \mathcal{D}^{2,0}(\mathbb{R}^2_s \times \mathbb{S}^1_{s'}) \bigoplus \mathcal{D}^{1,1}(\mathbb{R}^2_s \times \mathbb{S}^1_{s'})$, with obvious meaning of the summands.

Using decomposition (5.27), we may rewrite (5.25) as

$$\begin{aligned}< G_f, \vartheta >= &\int_{\mathbb{R}^2} F_0\big(s, f(s)\big)ds + \\ &+ \int_{\mathbb{R}^2} \Big[F_1\big(s, f(s)\big)\, f(s) \wedge \partial_2 f(s) - F_2\big(s, f(s)\big)\, f(s) \wedge \partial_1 f(s)\Big] ds\,.\end{aligned} \tag{5.28}$$

Clearly, whenever f is smooth, the right hand side of (5.28) defines a current, i.e.,

$$G_f \in \mathcal{D}_2(\mathbb{R}^2 \times \mathbb{S}^1) = \big(\mathcal{D}^2(\mathbb{R}^2 \times \mathbb{S}^1)\big)'.$$

Indeed, if $\vartheta_n \to \vartheta$ in $\mathcal{D}^2(\mathbb{R}^2 \times \mathbb{S}^1)$, then the corresponding densities F^n_0, F^n_1 and F^n_2 satisfy $F^n_j \to F_j$ in $C^\infty_0(\mathbb{R}^2 \times \mathbb{S}^1)$, $j = 0, 1, 2$. Thus, $< G_f, \vartheta_n > \to < G_f, \vartheta >$, by uniform convergence of the integrands. Moreover, by construction, this current coincides with the integration over the graph, i.e., with (5.25).

Since the F_j's, $j = 0, 1, 2$, are compactly supported smooth functions, for each $f \in X$ we can take (5.28) as the *definition of the graph current* associated to f. Indeed, the first term in the right hand side of (5.28) is an integral of a bounded measurable function with compact support. We shall see that the second term can be interpreted as an $\dot{H}^{1/2} - \dot{H}^{-1/2}$ duality in the sense of Remark 2.2. To this purpose, we introduce for each $F \in C^\infty_0(\mathbb{R}^2 \times \mathbb{S}^1)$, the superposition operator $\mathcal{F}$ defined by

$$\mathcal{F} :\ f \in X \mapsto \mathcal{F}(f)(\cdot) = F\big(\cdot, f(\cdot)\big) \in \dot{H}^{1/2}(\mathbb{R}^2; \mathbb{R})\,.$$

Since F is globally Lipschitz in the s'-variable, uniformly with respect to $s \in \mathbb{R}^2$, we infer from (1.2) that the operator $\mathcal{F}$ is well-defined and bounded on bounded sets. Indeed, one has the straightforward estimate

$$\|\mathcal{F}(f)\|_{1/2} \leq \|F\|_{C^1}(1 + \|f\|_{1/2})\,, \tag{5.29}$$

whence the $\dot{H}^{1/2}$–continuity of $\mathcal{F}(f)(s) = F(s, f(s))$ with respect to F follows. On the other hand, continuity with respect to $f \in X$ is a well known consequence of estimate (5.29) (see e.g. [2]). According to Proposition 2.3, it is clear that the products $F_j\big(\cdot, f(\cdot)\big)f$, $j = 1, 2$, in (5.28) belong to $\dot{H}^{1/2}(\mathbb{R}^2; \mathbb{R}^2)$ and they are continuous with respect to F_1 and F_2 respectively. By Remark 2.2, the second integral in the right hand side of (5.28) is well defined and continuous with respect to F_1 and F_2. Therefore, (5.28) defines a current for any $f \in X$.

By (5.28) and the continuity of the superposition operators with respect to f, we also deduce that, if $f_n \to f$ strongly in $\dot{H}^{1/2}$, then $G_{f_n} \rightharpoonup G_f$ as currents.

In contrast with the one dimensional case treated in [33], a graph current G_f for $f \in X$ arbitrary, may have boundary. Indeed, we have the following description.

Lemma 5.4. *Let $f \in X$ and let G_f be its corresponding graph current. For any $\beta \in \mathcal{D}^1(\mathbb{R}^2 \times \mathbb{S}^1)$, we have*

$$< \partial G_f, \beta >=< T(f), \beta_0 > \tag{5.30}$$

where $\beta_0(x_1,x_2) := \fint_{\mathbb{S}^1} \beta(x_1,x_2,\cdot)$, $\beta_0 \in C^\infty_0(\mathbb{R}^2;\mathbb{R})$.

Proof. Let $\beta \in \mathcal{D}^1(\mathbb{R}^2\times\mathbb{S}^1)$, i.e.,

$$\beta(s,s') = \big(B_1(s,s')dx_1 + B_2(s,s')dx_2\big) + B_0(s,s')\omega(s')$$

for suitable smooth functions B_0, B_1 and $B_2 \in C^\infty_0(\mathbb{R}^2\times\mathbb{S}^1)$. In other words, $\beta = \beta^{1,0}+\beta^{0,1} \in \mathcal{D}^1(\mathbb{R}^2\times\mathbb{S}^1) = \mathcal{D}^{1,0}(\mathbb{R}^2\times\mathbb{S}^1)\bigoplus\mathcal{D}^{0,1}(\mathbb{R}^2\times\mathbb{S}^1)$ with obvious meaning of the summands. A trivial calculation gives

$$d\beta = (\partial_1 B_2 - \partial_2 B_1)dx_1\wedge dx_2 + (\partial_1 B_0 - \partial_{s'}B_1)dx_1\wedge\omega(s') + (\partial_2 B_0 - \partial_{s'}B_2)dx_2\wedge\omega(s')\,,$$

where $\partial_{s'}$ denotes the differentiation with respect to the unit tangent field $\tau_{s'}$ on $\mathbb{S}^1$ oriented counterclockwise. Taking (5.28) into account we have

$$\begin{aligned}<\partial G_f,\beta> = < G_f, d\beta> &= \int_{\mathbb{R}^2}\big(\partial_1 B_2(x_1,x_2,f) + \partial_{s'}B_2(x_1,x_2,f)f\wedge\partial_1 f\big) + \\ &\quad - \int_{\mathbb{R}^2}\big(\partial_2 B_1(x_1,x_2,f) + \partial_{s'}B_1(x_1,x_2,f)f\wedge\partial_2 f\big) + \\ &\quad + \int_{\mathbb{R}^2}\big(\partial_1 B_0(x_1,x_2,f)f\wedge\partial_2 f - \partial_2 B_0(x_1,x_2,f)f\wedge\partial_1 f\big) \\ &= I + II + III\,.\end{aligned}$$

As $f\in\dot H^{1/2}(\mathbb{R}^2;\mathbb{S}^1)$, slicing in the x_1 or in the x_2 direction, we have have $f\in L^2_{x_1}(\dot H^{1/2}_{x_2}(\mathbb{R};\mathbb{S}^1))$ and similarly $f\in L^2_{x_2}(\dot H^{1/2}_{x_1}(\mathbb{R};\mathbb{S}^1))$. For $f(\cdot,x_2)\in\dot H^{1/2}(\mathbb{R};\mathbb{S}^1)$, using the strong density of smooth maps (see e.g. [33]) in one dimension, we obtain

$$I = \int_{\mathbb{R}}\bigg(\int_{\mathbb{R}}\big(\partial_1 B_2(x_1,x_2,f) + \partial_{s'}B_2(x_1,x_2,f)f\wedge\partial_1 f\big)dx_1\bigg)dx_2 = 0\,,$$

and similarly $II=0$, because both the integrands appearing in the inner integrals (more precisely, in the inner $\dot H^{1/2}-\dot H^{-1/2}$ dualities) are exact derivatives, so they vanish on smooth maps. As a consequence, we obtain $<\partial G_f,\beta>=0$ for any $\beta=\beta^{1,0}\in\mathcal{D}^{1,0}(\mathbb{R}^2\times\mathbb{S}^1)$. Then for any $\beta=\beta^{0,1} = B_0(x_1,x_2,s')\omega(s')\in\mathcal{D}^{0,1}(\mathbb{R}^2\times\mathbb{S}^1)$, we have

$$<\partial G_f,\beta^{0,1}> = \int_{\mathbb{R}^2}\big(\partial_1 B_0(x_1,x_2,f)f\wedge\partial_2 f - \partial_2 B_0(x_1,x_2,f)f\wedge\partial_1 f\big)\,. \tag{5.31}$$

Finally, we recall that $H^1_{dR}(\mathbb{S}^1)=\mathbb{R}$ and it is generated by the volume form ω on $\mathbb{S}^1$. Using the Hodge decomposition in $\mathbb{S}^1_{s'}$, we may write $\beta=\beta^{0,1}+\beta^{1,0} = \beta_0(x_1,x_2)\omega(s') + d\eta(x_1,x_2,s') + (\beta^{1,0} - d_s\eta(x_1,x_2,s'))$ for some $\eta\in C^\infty_0(\mathbb{R}^2\times\mathbb{S}^1)$. Hence,

$$\begin{aligned}<\partial G_f,\beta> &= <\partial G_f,\beta_0(x_1,x_2)\omega(s')> + <\partial G_f, d\eta> + <\partial G_f,\beta^{1,0}-d_s\eta> \\ &= <\partial G_f,\beta_0(x_1,x_2)\omega(s')>\end{aligned}$$

because $\beta^{1,0}-d_s\eta\in\mathcal{D}^{1,0}(\mathbb{R}^2\times\mathbb{S}^1)$ and $<\partial G_f,d\eta> = < G_f, d^2\eta> = 0$. Then the conclusion follows from (5.31) together with (1.5) since β_0 does not depend on the s'-variable. ∎

The following result, which parallels Proposition 5.2, describes change of topological singularities in terms of graph currents.

Proposition 5.3. *Let $f \in X$ and $\{f_n\}_{n\in\mathbb{N}} \subset X$ such that $T(f_n) \equiv T_0$ for each n and $f_n \rightharpoonup f$ weakly in $\dot{H}^{1/2}$. Then, up to subsequences, there is an integer multiplicity 1-rectifiable current $\Theta \in \mathcal{R}_1(\mathbb{R}^2)$ of finite mass such that $2\pi\partial\Theta = T_0 - T(f)$ and*

$$G_{f_n} \underset{n\to+\infty}{\rightharpoonup} G_f + \Theta \times [\mathbb{S}^1] \quad \text{in } \mathcal{D}_2(\mathbb{R}^2 \times \mathbb{S}^1)\,. \tag{5.32}$$

Proof. In order to prove (5.32), we rely on Proposition 5.2. Let $\vartheta \in \mathcal{D}^2(\mathbb{R}^2_s \times \mathbb{S}^1_{s'})$, $\vartheta(s,s') = \vartheta^{2,0}(s,s') + \vartheta^{1,1}(s,s')$ as in (5.27). Define $\bar\vartheta \in \mathcal{D}^1(\mathbb{R}^2)$ as $\bar\vartheta(x_1,x_2) = \bar F_1(x_1,x_2)dx_1 + \bar F_2(x_1,x_2)dx_2$ with

$$\bar F_1(x_1,x_2) = \fint_{\mathbb{S}^1} F_1(x_1,x_2,s')ds' \quad \text{and} \quad \bar F_2(x_1,x_2) = \fint_{\mathbb{S}^1} F_2(x_1,x_2,s')ds'\,, \tag{5.33}$$

so that for each $f \in X$,

$$< G_f, \bar\vartheta \wedge \omega(s') >= \int_{\mathbb{R}^2} \big(\bar F_1(x_1,x_2) f \wedge \partial_2 f - \bar F_2(x_1,x_2) f \wedge \partial_1 f\big) =< J(f), \bar\vartheta >\,. \tag{5.34}$$

On the other hand, as in the previous lemma, the Hodge decomposition gives us two functions $H_1, H_2 \in C_0^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$ such that

$$\begin{aligned} F_1(x_1,x_2,s')dx_1 \wedge \omega(s') = \bar F_1(x_1,x_2)dx_1 \wedge \omega(s') + d\big(-H_1(x_1,x_2,s')dx_1\big) \\ - \partial_2 H_1(x_1,x_2,s')dx_1 \wedge dx_2\,, \end{aligned}$$

and

$$\begin{aligned} F_2(x_1,x_2,s')dx_2 \wedge \omega(s') = \bar F_2(x_1,x_2)dx_2 \wedge \omega(s') + d\big(-H_2(x_1,x_2,s')dx_2\big) \\ + \partial_1 H_2(x_1,x_2,s')dx_1 \wedge dx_2\,. \end{aligned}$$

Combining these formulas with (5.27), we conclude that

$$\vartheta = \bar\vartheta \wedge \omega(s') + d\beta + \eta\,, \tag{5.35}$$

for some $\beta \in \mathcal{D}^{1,0}(\mathbb{R}^2 \times \mathbb{S}^1)$ and $\eta \in \mathcal{D}^{2,0}(\mathbb{R}^2 \times \mathbb{S}^1)$. Since $\eta = P(x_1,x_2,s')dx_1 \wedge dx_2$ for some $P \in C_0^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$, we derive by dominated convergence that

$$\begin{aligned} < G_{f_n}, \eta >= \int_{\mathbb{R}^2} P(x_1,x_2,f_n(x_1,x_2)) \underset{n\to+\infty}{\longrightarrow} \\ \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2} P(x_1,x_2,f(x_1,x_2)) =< G_f + t \times [\mathbb{S}^1], \eta > \end{aligned} \tag{5.36}$$

for any $t \in \mathcal{R}_1(\mathbb{R}^2)$ of finite mass. Then, by (5.34) and Proposition 5.2, up to subsequences, we have

$$\begin{aligned} < G_{f_n}, \bar\vartheta \wedge \omega(s') >=< J(f_n), \bar\vartheta > \underset{n\to+\infty}{\longrightarrow} < J(f), \bar\vartheta > +2\pi < \Theta, \bar\vartheta >= \\ =< G_f, \bar\vartheta \wedge \omega(s') > + < \Theta \times [\mathbb{S}^1], \bar\vartheta \wedge \omega(s') > \\ =< G_f + \Theta \times [\mathbb{S}^1], \bar\vartheta \wedge \omega(s') >\,, \end{aligned} \tag{5.37}$$

for some 1-dimensional integer multiplicity rectifiable current $\Theta \in \mathcal{R}_1(\mathbb{R}^2)$ of finite mass such that $2\pi\partial\Theta = T_0 - T(f)$. Finally, Lemma 5.4 yields

$$< G_{f_n}, d\beta >=< \partial G_{f_n}, \beta >= 0 \quad \forall n \tag{5.38}$$

and

$$< \partial G_f, \beta >= 0 =< \partial G_f + \partial\Theta \times [\mathbb{S}^1], \beta >=< G_f + \Theta \times [\mathbb{S}^1], d\beta > \tag{5.39}$$

because $\beta \in \mathcal{D}^{1,0}(\mathbb{R}^2 \times \mathbb{S}^1)$. Combining (5.35) to the conclusion follows. ■

Remark 5.1. We emphasize that the limiting current Θ obtained in Proposition 5.3 is precisely the one given by Proposition 5.2.

6. Concentration effects and quantization of Jacobians

We begin the section with some propositions concerning concentration effects both of topological and energetic nature. The following quantization property for Jacobians relies on the results of the previous section.

Proposition 6.1. *Let $f \in X$ and $\{f_n\}_{n\in\mathbb{N}} \subset X$ such that $T(f_n) = T_0 \in (\mathrm{Lip}(\mathbb{R}^2))'$ for each n and $f_n \rightharpoonup f$ weakly in $\dot{H}^{1/2}$. Let $u \in \dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$ and $\{u_n\}_{n\in\mathbb{N}} \subset \dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$ such that $f = \mathrm{Tr}\, u$, $f_n = \mathrm{Tr}\, u_n$ for each n, and $u_n \rightharpoonup u$ weakly in $\dot{H}^1$. Then, up to subsequences, there exists a 1-rectifiable current $\Theta \in \mathcal{R}_1(\mathbb{R}^2)$ of finite mass such that $2\pi\partial\Theta = T_0 - T(f)$ and for any $\overrightarrow{\varphi} \in C^0_0(\mathbb{R}^2;\mathbb{R}^2)$ and $\overrightarrow{\Phi} \in C^0_0(\mathbb{R}^3;\mathbb{R}^3)$ such that $\overrightarrow{\Phi}_{|\mathbb{R}^2} = (\overrightarrow{\varphi}, 0)$, we have*

$$\int_{\mathbb{R}^3_+} \big(H(u_n) - H(u)\big) \cdot \overrightarrow{\Phi} \underset{n\to+\infty}{\longrightarrow} 2\pi < \Theta, \overrightarrow{\varphi} > . \tag{6.1}$$

Proof. Clearly we may assume $\overrightarrow{\varphi}$ and $\overrightarrow{\Phi}$ smooth since the general case follows by uniform approximation, as $H(u) \in L^1(\mathbb{R}^3_+)$ and $\{H(u_n)\}_{n\in\mathbb{N}}$ is bounded in $L^1(\mathbb{R}^3_+)$. We write $\overrightarrow{\varphi} = (\varphi_1, \varphi_2)$ and $\overrightarrow{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$ and we set $\zeta = \varphi_1 dx_1 + \varphi_2 dx_2$. Taking (2.14) into account, a simple integration by parts (which can be justified by density, due to Theorem 2.2) gives for any $g \in X$ and any extension of g to the half space, $v \in \dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$,

$$\int_{\mathbb{R}^3_+} H(v) \cdot \overrightarrow{\Phi} = < J(g), \zeta > + \int_{\mathbb{R}^3_+} \big(v \wedge \nabla v\big) \cdot \mathrm{curl}\, \overrightarrow{\Phi} \, .$$

Since $u_n \rightharpoonup u$ weakly in $\dot{H}^1$, we infer that

$$\int_{\mathbb{R}^3_+} \big(u_n \wedge \nabla u_n\big) \cdot \mathrm{curl}\, \overrightarrow{\Phi} \longrightarrow \int_{\mathbb{R}^3_+} \big(u \wedge \nabla u\big) \cdot \mathrm{curl}\, \overrightarrow{\Phi}$$

as $n \to +\infty$ and hence

$$\int_{\mathbb{R}^3_+} \big(H(u_n) - H(u)\big) \cdot \overrightarrow{\Phi} = < J(f_n) - J(f), \zeta > + o(1) \quad \text{as } n \to +\infty.$$

Thus, the conclusion follows from Proposition 5.2. ■

Remark 6.1. We emphasize that the limiting current Θ obtained in Proposition 6.1 is precisely the one given by Proposition 5.2.

The following simple proposition deals with energy minimizing sequences for $\bar{\rho}_A$.

Proposition 6.2. *Let $A \in \mathcal{A}$ and $P, Q \in \mathbb{R}^2$ two distinct points. Let $\{f_n\}_{n\in\mathbb{N}} \subset X$ be an optimal sequence for $\bar{\rho}_A(P,Q)$ and $\{u_n\}_{n\in\mathbb{N}} \subset \dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$ the corresponding A–harmonic extensions. Then, up to a subsequence, there exists $\mu \in \mathcal{M}^+(\overline{\mathbb{R}^3_+})$ with $\mathrm{spt}\,\mu \subset \partial\mathbb{R}^3_+$ such that*

$$\frac{1}{2}\, \mathrm{tr}\, \big(\nabla u_n A (\nabla u_n)^t\big) dx \overset{\star}{\rightharpoonup} \mu \quad \text{as } n \to +\infty \tag{6.2}$$

weakly-$\star$ in the sense of measures. Moreover, if $\bar{\rho}_A(P,Q) \leq \pi \bar{d}_A(P,Q)$, then μ has compact support.

Proof. Extracting a subsequence if necessary, we may clearly assume that (6.2) holds for some $\mu \in \mathcal{M}^+(\overline{\mathbb{R}^3_+})$. Since $f_n \rightharpoonup \alpha$ weakly for some $\alpha \in \mathbb{S}^1$, we infer that $u_n \rightharpoonup \alpha$ weakly in $\dot{H}^1$. By Proposition 2.8, $\nabla u_n \to 0$ in $L^2_{\rm loc}(\mathbb{R}^3_+)$ and consequently, $\operatorname{spt}\mu \subset \partial\mathbb{R}^3_+$. To prove the last statement, we shall use similar arguments to those in the proof of Theorem 1.1, steps 2 and 5. Let us consider the compact set

$$K = \left\{ x \in \overline{\mathbb{R}^3_+} \;:\; \max\big(|x-P|, |x-Q|\big) \leq \frac{\Lambda}{\lambda}|P-Q| \right\}.$$

By weak-$\star$ convergence, we have

$$\liminf_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^3_+\setminus K} \operatorname{tr}\big(\nabla u_n A(\nabla u_n)^t\big)dx \geq \mu\big(\overline{\mathbb{R}^3_+}\setminus K\big)$$

and therefore

$$\pi \bar{d}_A(P,Q) \geq \bar{\rho}_A(P,Q) \geq \mu\big(\overline{\mathbb{R}^3_+}\setminus K\big) + \liminf_{n\to+\infty}\frac{1}{2}\int_K \operatorname{tr}(\nabla u_n A(\nabla u_n)^t)dx\,. \tag{6.3}$$

Then the conclusion follows once we prove that

$$\liminf_{n\to+\infty}\frac{1}{2}\int_K \operatorname{tr}(\nabla u_n A(\nabla u_n)^t)dx \geq \pi\bar{d}_A(P,Q)\,. \tag{6.4}$$

In order to prove (6.4), we fix $0 < r < 1$ and we introduce a cut-off function $\chi \in C^\infty_0(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\operatorname{spt}\chi \subset (-r,r)$, $\chi(t) \equiv 1$ for $|t| \leq r/2$. Consider Φ defined on $\overline{\Omega}_r$ by (4.4). Since $\operatorname{spt}\Phi \subset K \cap \overline{\Omega}_r$, we may argue as in the proof of (4.5) to obtain

$$\frac{1}{2}\int_K \operatorname{tr}\big(\nabla u_n A(\nabla u_n)^t\big)dx \geq \pi d^r_A(P,Q) - \frac{1}{2}\int_{K\cap\{r/2<x_3<r\}} \Phi\, H(u_n)\cdot\nabla\chi\,.$$

Since $\nabla u_n \to 0$ in $L^2_{\rm loc}(\mathbb{R}^3_+)$ the last term in the right hand side vanishes as $n \to +\infty$. Then we recover (6.4) letting $r \to 0$ by Proposition 3.5. ∎

Proof of Theorem 1.2. *Step 1.* Since A is continuous in $\overline{\mathbb{R}^3_+}$, $\bar{d}_A$ coincides with the Riemannian distance on $\mathbb{R}^2 \simeq \partial\mathbb{R}^3_+$ induced by the (continuous) matrix field $\operatorname{Cof} A(x_1,x_2,0)$. We start with the proof of claim *(i)*. Observe that it is enough to show $\bar{\rho}_A \leq \pi\bar{d}_A$ since the converse inequality holds by Theorem 1.1, claim *(ii)*. We shall use a convolution argument with respect to the (x_1,x_2)-variables, as in the proof of Theorem 1.1, step 4. We fix $P_0, Q_0 \in \mathbb{R}^2 \simeq \partial\mathbb{R}^3_+$, and define for $P \in \mathbb{R}^2$, $\xi(P) = \bar{\rho}_A(P,Q_0)$. We introduce a standard mollifyer $\varrho \in C^\infty_0(\mathbb{R}^2;\mathbb{R})$, i.e., $\varrho \geq 0$, $\int_{\mathbb{R}^2}\varrho = 1$, and set, for $\varepsilon > 0$, $\varrho_\varepsilon(z) = \varepsilon^{-2}\varrho(z/\varepsilon)$. We define $\xi_\varepsilon = \varrho_\varepsilon * \xi$ and

$$A_\varepsilon(x_1,x_2,x_3) = \int_{\mathbb{R}^2}\varrho_\varepsilon(-z_1,-z_2)A(x_1+z_1,x_2+z_2,x_3)dz_1dz_2,$$

so that $A_\varepsilon \in \mathcal{A}$, A_ε is continuous in $\overline{\mathbb{R}^3_+}$ and $A_\varepsilon \to A$ locally uniformly in $\overline{\mathbb{R}^3_+}$ as $\varepsilon \to 0$. Arguing exactly as in the proof of Theorem 1.1, step 4, we obtain

$$|\nabla\Xi_\varepsilon(x)\cdot h| \leq \pi\sqrt{(\operatorname{Cof}A_\varepsilon(x))h\cdot h} \quad \forall h \in \mathbb{R}^2\times\{0\}, \forall x \in \partial\mathbb{R}^3_+$$

where $\Xi_\varepsilon(x_1,x_2,x_3) := \xi_\varepsilon(x_1,x_2)$. Consequently, for any $\gamma \in \operatorname{Lip}_{P_0,Q_0}([0,1];\partial\mathbb{R}^3_+)$, we have

$$|\xi_\varepsilon(P_0) - \xi_\varepsilon(Q_0)| \leq \int_0^1 |\nabla\Xi_\varepsilon(\gamma(t))\cdot\dot{\gamma}(t)|dt \leq \pi\int_0^1 \mathcal{L}_{A_\varepsilon}\big(\gamma(t),\dot{\gamma}(t)\big)\,dt\,.$$

Taking the infimum over γ, we derive $|\xi_\varepsilon(P_0) - \xi_\varepsilon(Q_0)| \leq \pi \bar{d}_{A_\varepsilon}(P_0, Q_0)$ and the conclusion follows letting $\varepsilon \to 0$ in view of Proposition 3.6. ■

Remark 6.2. We emphasize that the assumption $\bar{A} \in \mathcal{A}_0$ is not used in the above proof. In other words, claim *(i)* in Theorem 1.2 holds for any $A \in \mathcal{A} \cap C^0(\overline{\mathbb{R}^3_+})$.

Remark 6.3. We observe that $\bar{\rho}_A$ only depends on the trace of A on $\partial\mathbb{R}^3_+$, i.e., $\bar{\rho}_A = \bar{\rho}_{A'}$ whenever $A, A' \in \mathcal{A} \cap C^0(\overline{\mathbb{R}^3_+})$ with $A_{|\mathbb{R}^2} \equiv A'_{|\mathbb{R}^2}$. Indeed, in such a case $\mathrm{Cof}\, A_{|\mathbb{R}^2} = \mathrm{Cof}\, A'_{|\mathbb{R}^2}$ and the distances $\bar{d}_A$, $\bar{d}_{A'}$ are respectively equal to the induced Riemannian distances on $\mathbb{R}^2$, so they coincide.

Proof of Theorem 1.2. Step 2. First, we provide a lower bound for the energetic distance $\bar{\rho}_A(P, Q)$ in terms of the Jacobians of the optimal sequence. Let f_n and u_n as in the assumption with $f_n \rightharpoonup \alpha \in \mathbb{S}^1$. Clearly $u_n \rightharpoonup \alpha$ weakly in $\dot{H}^1$. Arguing as in (4.3), we obtain

$$\bar{\rho}_A(P, Q) = \lim_{n\to+\infty} E_A(u_n) \geq \limsup_{n\to+\infty} \frac{1}{2} \int_{\mathbb{R}^3_+} H(u_n) \cdot \overrightarrow{\Phi}\,, \tag{6.5}$$

for any vector field $\overrightarrow{\Phi} \in C^0_0(\overline{\mathbb{R}^3_+}; \mathbb{R}^3)$ satisfying $(\mathrm{Cof}\, A)^{-1}\overrightarrow{\Phi} \cdot \overrightarrow{\Phi} \leq 1$ in $\mathbb{R}^3_+$. In order to choose $\overrightarrow{\Phi}$ in (6.5), we claim that, given $\overrightarrow{\varphi} \in C^0_0(\mathbb{R}^2; \mathbb{R}^2 \times \{0\})$ such that $(\mathrm{Cof}\, A)^{-1}\overrightarrow{\varphi} \cdot \overrightarrow{\varphi} \leq 1$ in $\mathbb{R}^2$, there exists $\overrightarrow{\Phi} \in C^0_0(\overline{\mathbb{R}^3_+}; \mathbb{R}^2 \times \{0\})$ such that $\overrightarrow{\Phi}_{|\mathbb{R}^2} = \overrightarrow{\varphi}$ and $(\mathrm{Cof}\, A)^{-1}\overrightarrow{\Phi} \cdot \overrightarrow{\Phi} \leq 1$ in $\mathbb{R}^3_+$. This is an easy consequence of the celebrated Michael's selection theorem (see [4], Theorem 9.1.2 and Corollary 9.1.3). Indeed for each $r > 0$ and $B_r \subset \mathbb{R}^2$ such that $\mathrm{spt}\, \overrightarrow{\varphi} \subset B_r$, the sets $K' = \overline{B_{2r}}$ and $K = K' \times [0, 2r]$ are compact and the set-valued map

$$K \ni P \rightsquigarrow C_P = \big\{\xi \in \mathbb{R}^2 \times \{0\} \ : \ (\mathrm{Cof}\, A(P))^{-1}\xi \cdot \xi \leq 1\big\}$$

is a nonempty lower semicontinuous compact convex valued map. The map $K' \ni P \mapsto \overrightarrow{\varphi}(P)$ is a continuous selection and hence, it can be extended to a continuous selection $\overrightarrow{\Phi}$ defined on the whole K. Multiplying $\overrightarrow{\Phi}$ by a cut-off function $\chi \in C^0_0(\mathbb{R}^3)$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $K' \times [0, r]$, the claim follows.

Since $T(f_n) = 2\pi(\delta_P - \delta_Q)$ for every n, applying Proposition 6.1 (with $f \equiv \alpha$ and $u \equiv \alpha$ so that $T(f) = 0$ and $H(u) \equiv 0$), the previous claim and inequality (6.5), we obtain a limiting current $\Theta \in \mathcal{R}_1(\mathbb{R}^2)$ such that $\partial\Theta = \delta_P - \delta_Q$ and

$$\bar{\rho}_A(P, Q) \geq \pi < \Theta, \overrightarrow{\varphi} > \tag{6.6}$$

for any $\overrightarrow{\varphi} \in C^0_0(\mathbb{R}^2; \mathbb{R}^2)$ such that $(\mathrm{Cof}\, A)^{-1}(\overrightarrow{\varphi}, 0) \cdot (\overrightarrow{\varphi}, 0) \leq 1$ in $\mathbb{R}^2$.

Since $\overline{\rho}_A(P, Q) = \pi \overline{d}_A(P, Q)$ and $\bar{A} \in \mathcal{A}_0$, taking the supremum in (6.6) over all admissible $\overrightarrow{\varphi}$, we conclude $\overline{M}_A(\Theta) \leq \overline{d}_A(P, Q)$, where $\overline{M}_A$ is the mass of the current $\Theta \in \mathcal{R}_1(\mathbb{R}^2)$ with respect to the Riemannian structure on $\mathbb{R}^2$ induced by $\mathrm{Cof}\, A_{|\mathbb{R}^2}$ restricted to the tangent space. Since $\Theta \in \mathcal{R}_1(\mathbb{R}^2)$ and $\partial\Theta = (\delta_P - \delta_Q)$, we also have the lower bound $\overline{M}_A(\Theta) \geq \overline{d}_A(P, Q)$ by standard polyhedral approximation. Therefore $\overline{M}_A(\Theta) = \overline{d}_A(P, Q)$ and consequently, Θ is (the image of) a minimizing geodesic running from Q to P, i.e., there is an injective curve $\gamma \in \mathrm{Lip}_{Q,P}([0, 1]; \partial\mathbb{R}^3_+)$ with $\mathbb{L}_A(\gamma) = \bar{d}_A(P, Q)$ such that $\Theta = \overrightarrow{\Gamma}$ where $\overrightarrow{\Gamma}$ is the 1-rectifiable current relative to oriented curve $\Gamma := \gamma([0, 1])$ running from Q to P. Then, claim *(iii)* follows as a consequence of Proposition 6.1, Proposition 5.3, Remark 5.1, Remark 6.1 and the explicit form of the limiting current Θ.

Step 3. Now we move on the proof of claim *(ii)*. By Proposition 6.2 and Step 1, we may assume

$$\mu_n := \frac{1}{2}\mathrm{tr}\,(\nabla u_n A (\nabla u_n)^t) dx \overset{*}{\rightharpoonup} \mu \quad \text{as } n \to +\infty,$$

weakly-$\star$ in the sense of measures for some compactly supported measure $\mu \in \mathcal{M}^+(\overline{\mathbb{R}^3_+})$ such that $\operatorname{spt}\mu \subset \partial\mathbb{R}^3_+$. Without loss of generality, we may also assume that $|\dot\gamma| > 1/2$ a.e. in $(0,1)$ (otherwise we reparametrize Γ). Then , for $\mathcal{H}^1$-a.e. $x \in \Gamma$, the oriented unit tangent vector to Γ at x is well defined and given by

$$\tau(x) = \frac{\dot\gamma(\gamma^{-1}(x))}{|\dot\gamma(\gamma^{-1}(x))|}\,.$$

Arguing as in Step 2, we derive that for any $\psi \in C^0_0(\overline{\mathbb{R}^3_+})$ with $0 \le \psi \le 1$ and any $\overrightarrow{\varphi} \in C^0_0(\mathbb{R}^2;\mathbb{R}^2)$ such that $(\operatorname{Cof} A)^{-1}(\overrightarrow{\varphi},0)\cdot(\overrightarrow{\varphi},0) \le 1$ in $\mathbb{R}^2 \simeq \partial\mathbb{R}^3_+$, we have

$$\bar\rho_A(P,Q) \ge \int \psi\, d\mu = \lim_{n\to+\infty} \int \psi\, d\mu_n \ge \pi < \overrightarrow{\Gamma}, \psi\overrightarrow{\varphi} > = \pi \int_\Gamma \psi\, \overrightarrow{\varphi}\cdot\tau\, d\mathcal{H}^1\,. \tag{6.7}$$

We claim that there is an admissible sequence $\{\overrightarrow{\varphi}_k\}_{k\in\mathbb{N}}$ such that

$$\overrightarrow{\varphi}_k\cdot\tau \to \sqrt{(\operatorname{Cof} A)\tau\cdot\tau} \quad \mathcal{H}^1\text{-a.e. on } \Gamma \text{ as } k \to +\infty. \tag{6.8}$$

To construct such a sequence, we shall use a regularization procedure. First, we extend γ to $[-1,2]$ into a Lipschitz curve satisfying $|\dot\gamma| > 1/2$ a.e. in $(-1,2)$. Then we consider a sequence $\delta_k \downarrow 0$. From the uniform ellipticity of A, we infer that the map $B : t \in [-1,2] \mapsto \big(\operatorname{Cof} A(\gamma(t))\big)^{-1/2}$ is uniformly continuous. Hence, we can find $\varepsilon_k \downarrow 0$ such that $\|B(t) - B(s)\| \le \delta_k$ whenever $t,s \in [-1,2]$ with $|t-s| \le \varepsilon_k$. We define on $(-1,2)$ the $\partial\mathbb{R}^3_+$-valued function

$$\nu(t) = \frac{(\operatorname{Cof} A(\gamma(t)))\dot\gamma(t)}{\sqrt{(\operatorname{Cof} A(\gamma(t)))\dot\gamma(t)\cdot\dot\gamma(t)}}\,.$$

We easily check that $|\nu| \le \sqrt{\Lambda}$ and $|B\nu| = 1$ a.e. in $(-1,2)$, and also

$$\nu(\gamma^{-1}(x))\cdot\tau(x) = \sqrt{(\operatorname{Cof} A(x))\tau(x)\cdot\tau(x)} \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \Gamma. \tag{6.9}$$

Next we introduce a standard mollifyer $\varrho \in C^\infty_0(\mathbb{R};\mathbb{R})$, i.e., $\varrho \ge 0$, $\operatorname{spt}\varrho \subset (-1,1)$, $\int_\mathbb{R} \varrho = 1$, and set $\varrho_k(t) = \varepsilon_k^{-1}\varrho(t/\varepsilon_k)$. We define for $t \in [0,1]$, $\nu_k(t) = \varrho_k * \nu(t)$. Writing

$$B(t)\nu_k(t) = \int_{t-\varepsilon_k}^{t+\varepsilon_k} \varrho_k(s-t)B(s)\nu(s)ds + \int_{t-\varepsilon_k}^{t+\varepsilon_k} \varrho_k(s-t)\big(B(t)-B(s)\big)\nu(s)ds\,,$$

we easily obtain the estimate $|B(t)\nu_k(t)| \le 1 + \delta_k\sqrt{\Lambda}$ for every $t \in [0,1]$. Then we consider for $x \in \Gamma$, $\overrightarrow{\varphi}_k(x) = (1+\delta_k\sqrt{\Lambda})^{-1}\nu_k(\gamma^{-1}(x))$. By construction, we have $(\operatorname{Cof} A)^{-1}(\overrightarrow{\varphi}_k,0)\cdot(\overrightarrow{\varphi}_k,0) \le 1$ on Γ. By a similar extension procedure to the one used in Step 2, we may now extend $\overrightarrow{\varphi}_k$ to $\mathbb{R}^2$ in such a way that the resulting function is continuous, has compact support and satisfies the required constraint $(\operatorname{Cof} A)^{-1}(\overrightarrow{\varphi}_k,0)\cdot(\overrightarrow{\varphi}_k,0) \le 1$ in $\mathbb{R}^2$. Then, since $\nu_k \to \nu$ a.e. in $(0,1)$, we conclude that (6.8) holds in view of (6.9).

Plugging the function $\overrightarrow{\varphi}_k$ in (6.7) and letting $k \to +\infty$, we deduce that

$$\bar\rho_A(P,Q) \ge \int \psi\, d\mu \ge \pi \int_\Gamma \psi\mathcal{L}_A(x,\tau_x)d\mathcal{H}^1 \quad \text{for any } \psi \in C^0(\overline{\mathbb{R}^3_+}) \text{ wit } 0 \le \psi \le 1.$$

Therefore $\mu \ge \pi\mathcal{L}_A(\cdot,\tau(\cdot))\mathcal{H}^1 \llcorner \Gamma$. On the other hand, the length formula and Step 1 yield

$$\bar\rho_A(P,Q) \ge \mu(\overline{\mathbb{R}^3_+}) \ge \big(\pi\mathcal{L}_A(\cdot,\tau(\cdot))\mathcal{H}^1 \llcorner \Gamma\big)(\overline{\mathbb{R}^3_+}) = \pi\mathbb{L}_{d_A}(\gamma) = \pi\bar d_A(P,Q) = \bar\rho_A(P,Q)$$

so that the two measures have the same mass. Hence $\mu \equiv \pi\mathcal{L}_A(\cdot,\tau(\cdot))\mathcal{H}^1 \llcorner \Gamma$.

Step 4. To complete the proof of Theorem 1.2, we shall need two auxiliary results. The first one gives a coarea type formula for Jacobians in the spirit of [1] and [27].

Proposition 6.3. *Let $f \in X$ and $u \in \dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$ its A–harmonic extension with $A \in \mathcal{A}$ continuous in $\overline{\mathbb{R}^3_+}$. For any $b \in C^0_0(\mathbb{D};\mathbb{R})$, any $\varphi \in \mathrm{Lip}(\mathbb{R}^2;\mathbb{R})$ and any $\Phi \in \mathrm{Lip}(\mathbb{R}^3_+;\mathbb{R})$ such that $\Phi_{|\mathbb{R}^2} = \varphi$, we have*

$$\int_{\mathbb{R}^3_+} b(u)H(u)\cdot\nabla\Phi = \left(\fint_{\mathbb{D}} b\right) < T(f),\varphi > \ . \tag{6.10}$$

Proof. Clearly we may assume that both φ and Φ are smooth and compactly supported, by standard approximation and weak-$\star$ convergence in L^∞, and that $f \not\equiv \alpha \in \mathbb{S}^1$ is nonconstant, otherwise $u \equiv \alpha$ and (6.10) trivially holds. We also observe that it suffices to prove (6.10) under the extra assumption $A \in C^\infty(\mathbb{R}^3_+;\mathcal{S}^+)$. Indeed, let us assume that (6.10) holds under this assumption. Since $A \in \mathcal{A}$ is continuous in $\overline{\mathbb{R}^3_+}$, a usual convolution argument gives a sequence of smooth matrices $\{A_m\}_{m\in\mathbb{N}} \subset \mathcal{A}$ with the ellipticity bounds of A, such that $A_m \to A$ in $C^0_{\mathrm{loc}}(\overline{\mathbb{R}^3_+})$. According to Propositions 2.7 and Proposition 2.8, the corresponding sequence $\{u_m\}_{m\in\mathbb{N}}$ of A_m–harmonic extensions of f satisfies $\nabla u_m \to \nabla u$ in $L^2_{\mathrm{loc}}(\overline{\mathbb{R}^3_+};\mathbb{R}^2)$ and (up to subsequences) $u_m \to u$ a.e. as $m \to +\infty$. Since (6.10) holds for u_m and every m, the conclusion follows letting $m \to +\infty$.

To prove (6.10) for A smooth, we combine the results of [1] and [27]. First, we observe that by Propositions 2.7 and Proposition 2.8, we have $u \in C^\infty(\mathbb{R}^3_+;\mathbb{R}^2)$, $|u| < 1$ a.e. in $\mathbb{R}^3_+$, $\|\nabla u\|_2 \le C|f|_{1/2}$ and $\|u(\cdot,x_3) - f(\cdot)\|^2_{L^2(\mathbb{R}^2)} = o(x_3)$ as $x_3 \to 0$. Under these assumptions, we can apply the argument in [27], Section 3, to conclude

$$\frac{1}{2\pi} < T(f),\varphi >= \int_{u^{-1}(y)} d\Phi$$

for a.e. $y \in \mathbb{D}$. Here and in [27], with a slight abuse of notation, $u^{-1}(y)$ represents the integer multiplicity 1-rectifiable current of integration over the the fiber $u^{-1}(y)$ (generically a smooth curve by Sard's Theorem). Combining this relation with the oriented coarea formula of [1] (see Section 2, formula 2.6), we obtain

$$\int_{\mathbb{R}^3_+} b(u)H(u)\cdot\nabla\Phi = 2\int_{\mathbb{D}} b(y)\left(\int_{u^{-1}(y)} d\Phi\right)dy = \left(\fint_{\mathbb{D}} b\right) < T(f),\varphi > ,$$

which is the desired formula. ∎

The following simple lemma gives a first description of the behaviour of the vorticity sets of minimal extensions under weak convergence.

Lemma 6.1. *Let $\{f_n\}_{n\in\mathbb{N}} \subset X$ such that $f_n \rightharpoonup \alpha \in \mathbb{S}^1$ weakly in $\dot{H}^{1/2}$ and let $\{u_n\}_{n\in\mathbb{N}} \subset \dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$ be the corresponding A–harmonic extensions. We set $V(u_n,R) = \big\{x \in \mathbb{R}^3_+\,;\ |u_n(x)| \le R\big\}$ for $0 < R < 1$. Then, for any compact set $K \subset \mathbb{R}^3_+$, there exists $n_K \ge 1$ such that $V(u_n,R)\cap K = \emptyset$ for every $n \ge n_K$.*

Proof. We argue by contradiction. Assume there exists a sequence of positive integers $n_k \to +\infty$ as $k \to +\infty$ and $x_k \in V(u_{n_k},R)\cap K$ for every k. By Proposition 2.8, the sequence $\{u_{n_k}\}$ is compact in $C^0_{\mathrm{loc}}(\mathbb{R}^3_+;\mathbb{R}^2)$. Therefore, up to a subsequence, $u_{n_k} \to \alpha \in \mathbb{S}^1$ uniformly on K. Hence, $|u_{n_k}(x_k)| \to |\alpha| = 1$ which contradicts $|u_{n_k}(x_k)| \le R$ for every k. ∎

Proof of Theorem 1.2. Step 5. Let $\{f_n\}_{n\in\mathbb{N}} \subset X$ be an optimal sequence for $\bar{\rho}_A(P,Q)$, i.e., $f_n \rightharpoonup f \equiv \alpha \in \mathbb{S}^1$, $T(f_n) = 2\pi(\delta_P - \delta_Q)$ and $\mathcal{E}_A(f_n) \to \bar{\rho}_A(P,Q)$ as $n \to +\infty$. We claim that, for every

$0 < R < 1$, we have

$$\liminf_{n\to+\infty} \frac{1}{2}\int_{\{|u_n(x)|<R\}} \mathrm{tr}\big(\nabla u_n A(\nabla u_n)^t\big) \geq \pi R^2 \bar{d}_A(P,Q)\,, \tag{6.11}$$

$$\liminf_{n\to+\infty} \frac{1}{2}\int_{\{R\leq|u_n(x)|\}} \mathrm{tr}\big(\nabla u_n A(\nabla u_n)^t\big) \geq \pi(1-R^2) \bar{d}_A(P,Q)\,. \tag{6.12}$$

Then the conclusion easily follows from (6.11) and (6.12) together with the equality $\bar{\rho}_A(P,Q) = \pi\bar{d}_A(P,Q)$. In order to prove (6.11) ((6.12) can be proved in the same way), we fix $b \in C^0_0(\mathbb{D};\mathbb{R})$ such that $0 \leq b \leq 1$ and $\mathrm{spt}\, b \subset \big\{ y \in \mathbb{D};\, |y| < R\big\}$. Arguing as in the proof of Theorem 1.1, Step 2, we introduce for each $r > 0$, the function Φ defined by (4.4) and a cut-off function $\chi \in C^\infty_0(\mathbb{R};\mathbb{R})$, $0 \leq \chi \leq 1$, $\mathrm{spt}\,\chi \subset (-r,r)$ and $\chi(t) = 1$ for $|t| \leq r/2$. Arguing as in (4.5), we infer that

$$\frac{1}{2}\int_{\{|u_n(x)|<R\}} \mathrm{tr}\big(\nabla u_n A(\nabla u_n)^t\big) \geq \frac{1}{2}\int_{\Omega_r} b(u_n)\chi(x_3)\, H(u_n)\cdot\nabla\Phi \;=$$
$$= \frac{1}{2}\int_{\mathbb{R}^3_+} b(u_n)\, H(u_n)\cdot\nabla(\chi\Phi) \;-\; \frac{1}{2}\int_{K\cap\{r/2<x_3<r\}} b(u_n)\Phi\, H(u_n)\cdot\nabla\chi$$

where $K = \mathrm{spt}\,\Phi$ is compact. Taking Lemma 6.1 into account, we may take n so large that, $\mathrm{spt}\, b(u_n)\cap K\cap\{r/2 < x_3 < r\} = \emptyset$ for every n. Then we derive from Proposition 6.3 that

$$\frac{1}{2}\int_{\{|u_n(x)|<R\}} \mathrm{tr}\big(\nabla u_n A(\nabla u_n)^t\big) \geq \frac{1}{2}\int_{\mathbb{R}^3_+} b(u_n)\, H(u_n)\cdot\nabla(\chi\Phi) \;=$$
$$= \frac{1}{2}\bigg(\,\fint_{\mathbb{D}} b\bigg) < T(f_n), \Phi_{|\mathbb{R}^2} >= \pi\bigg(\,\fint_{\mathbb{D}} b\bigg) d^r_A(P,Q)\,.$$

Taking the supremum over all admissible b's and letting $n \to +\infty$, we deduce

$$\liminf_{n\to+\infty} \int_{\{|u_n(x)|<R\}} \mathrm{tr}\big(\nabla u_n A(\nabla u_n)^t\big) \geq \pi R^2 d^r_A(P,Q)\,.$$

Now we recover (6.11) letting $r \to 0$ by Proposition 3.5. ∎

7. Minimal connections and relaxed energies

This section is devoted to the proof of Theorem 1.3 and is divided into three parts. First we prove the lower bound of the relaxed energy $\overline{\mathcal{E}}_A(f)$ using the duality argument of Section 4 in combination with a method developed in [6] and [32]. Next we apply a *dipole removing technique* (by analogy with [5]) to obtain upper bounds in terms of the energetic distance. We conclude the proof of Theorem 1.3 in the third part.

7.1 Lower bound for $\overline{\mathcal{E}}_A$ by lower semicontinuity

In the sequel, we shall denote by $\mathcal{F}_A$ the expected lower bound for $\overline{\mathcal{E}}_A$, i.e., the functional defined for maps $f \in X$ by

$$\mathcal{F}_A(f) = \mathcal{E}_A(f) + \pi L_A(f).$$

As we will see in Corollary 7.1, the proof of the lower bound in (1.20) basically reduces to show the sequential lower semicontinuity of the functional $\mathcal{F}_A$. We start with this later fact.

Theorem 7.1. *The functional $\mathcal{F}_A$ is sequentially lower semicontinuous on X with respect to the weak $\dot{H}^{1/2}$–topology.*

Proof. We introduce the auxiliary functional $\mathcal{F}_A^r$ on X defined for $0 < r \leq \infty$ by

$$\mathcal{F}_A^r(f) = \mathcal{E}_A(f) + \pi L_A^r(f)$$

where $L_A^r(f)$ denotes the length of minimal connection relative to d_A^r, i.e.,

$$L_A^r(f) = \frac{1}{2\pi} \operatorname{Sup} \Big\{ < T(f), \Phi_{|\mathbb{R}^2} >; \ \Phi \in \operatorname{Lip}(\overline{\Omega}_r, \mathbb{R}), \quad |\Phi(P) - \Phi(Q)| \leq d_A^r(P,Q) \ \forall P, Q \in \overline{\Omega}_r \Big\}. \tag{7.1}$$

By the results in Section 3, the distance $\bar{d}_A$ is the increasing limit of the distances d_A^r as $r \to 0$. In view of the definition of L_A and L_A^r, one can expected to recover $\mathcal{F}_A$ as the pointwise increasing limit of $\mathcal{F}_A^r$ as $r \to 0$. We shall see in Proposition 7.2 that it is indeed the case and as a consequence, proving the lower semicontinuity of $\mathcal{F}_A$ reduces to prove it for $\mathcal{F}_A^r$.

Proposition 7.1. *For every $0 < r \leq \infty$, the functional $\mathcal{F}_A^r$ is sequentially lower semicontinuous on X with respect to the weak $\dot{H}^{1/2}$–topology.*

Proof. We begin the proof with a very useful lemma.

Lemma 7.1. *Let $\operatorname{Lip}_0(\overline{\Omega}_r, \mathbb{R})$ be the set of all functions $\Phi \in \operatorname{Lip}(\overline{\Omega}_r, \mathbb{R})$ with compact support in $\overline{\Omega}_r$. We have*

$$L_A^r(f) = \frac{1}{2\pi} \operatorname{Sup} \Big\{ < T(f), \Phi_{|\mathbb{R}^2} >; \ \Phi \in \operatorname{Lip}_0(\overline{\Omega}_r, \mathbb{R}), \quad |\Phi(P) - \Phi(Q)| \leq d_A^r(P,Q) \ \forall P, Q \in \overline{\Omega}_r \Big\}. \tag{7.2}$$

Proof. For $0 < r < \infty$, we extend any function $\Phi \in \operatorname{Lip}(\overline{\Omega}_r, \mathbb{R})$ to $\overline{\mathbb{R}^3_+}$, by setting $\Phi(x_1, x_2, x_3) = \Phi(x_1, x_2, r)$ if $x_3 \geq r$. Obviously, we obtain by this process, a globally Lipschitz function Φ on the half space and $\|\nabla\Phi\|_{L^\infty(\mathbb{R}^3_+)} = \|\nabla\Phi\|_{L^\infty(\Omega_r)}$. To prove Lemma 7.1, it suffices to find, for any $\Phi \in \operatorname{Lip}(\Omega_r, \mathbb{R})$ which is 1–Lipschitz with respect to d_A^r, a sequence $\{\Phi_n\}_{n\in\mathbb{N}} \subset \operatorname{Lip}(\mathbb{R}^3_+, \mathbb{R})$ such that: Φ_n is 1–Lipschitz with respect to d_A^r in $\overline{\Omega}_r$, Φ_n has a compact support in $\overline{\mathbb{R}^3_+}$, $\|\nabla\Phi_n\|_{L^\infty(\mathbb{R}^3_+)} \leq C$ for a constant C independent of n, $\Phi_n \to \Phi$ and $\nabla\Phi_n \to \nabla\Phi$ a.e. in $\mathbb{R}^3_+$ as $n \to +\infty$. Indeed, for such a sequence, we easily obtain by dominated convergence that

$$\lim_{n\to+\infty} < T(f), \Phi_{n|\mathbb{R}^2} > = \lim_{n+\to\infty} \int_{\mathbb{R}^3_+} H(u)\cdot\nabla\Phi_n = \int_{\mathbb{R}^3_+} H(u)\cdot\nabla\Phi = < T(f), \Phi_{|\mathbb{R}^2} > .$$

Given a function $\Phi \in \operatorname{Lip}(\Omega_r, \mathbb{R})$ which is 1–Lipschitz with respect to d_A^r, we construct the sequence $\{\Phi_n\}_{n\in\mathbb{N}}$ as follows. We consider for each $n \in \mathbb{N}$, the truncated function $\Phi^{(n)}$ defined by

$$\Phi^{(n)}(x) = \begin{cases} \Phi(x) & \text{if } |\Phi(x)| \leq n, \\ \operatorname{sign}(\Phi(x))n & \text{otherwise.} \end{cases}$$

Obviously, $\Phi^{(n)} \in L^\infty(\mathbb{R}^3_+)$, $\Phi^{(n)}$ is globally Lipschitz, $\Phi^{(n)} \to \Phi$ and $\nabla\Phi^{(n)} \to \nabla\Phi$ a.e. in $\mathbb{R}^3_+$. Since for every $x, y \in \overline{\mathbb{R}^3_+}$, we have $|\Phi^{(n)}(x) - \Phi^{(n)}(y)| \leq |\Phi(x) - \Phi(y)|$, we deduce that $\|\nabla\Phi^{(n)}\|_{L^\infty(\mathbb{R}^3_+)} \leq \|\nabla\Phi\|_{L^\infty(\mathbb{R}^3_+)}$ and $\Phi^{(n)}$ is 1–Lipschitz with respect to d_A^r in $\overline{\Omega}_r$. Now we consider a sequence of positive numbers $\theta_n < 1$ such that $\theta_n \to 1$ as $n \to +\infty$ and a function $\chi : \mathbb{R} \to \mathbb{R}$ defined by $\chi(r) = 1$ if $|r| \leq 1$, $\chi(r) = 2 - |r|$ if $1 \leq |r| \leq 2$ and $\chi(r) = 0$ otherwise. For a sequence of positive numbers $R_n \geq 1$ that we shall choose later satisfying $R_n \to +\infty$ as $n \to +\infty$, we define for $x \in \overline{\mathbb{R}^3_+}$, $\zeta_n(x) = \theta_n \chi\big(R_n^{-1}|x|\big)$ and $\Phi_n(x) = \zeta_n(x)\Phi^{(n)}(x)$. Clearly, the function Φ_n is globally Lipschitz and has a compact support in $\overline{\mathbb{R}^3_+}$. Since $\Phi^{(n)}$ is 1–Lipschitz with respect to d^r_A in $\overline{\Omega}_r$, we infer from Proposition 3.7 that a.e. in Ω_r,

$$\begin{aligned}\big(\mathrm{Cof}\,A\big)^{-1}\nabla\Phi_n \cdot \nabla\Phi_n =& \zeta_n^2\big(\mathrm{Cof}\,A\big)^{-1}\nabla\Phi^{(n)} \cdot \nabla\Phi^{(n)} + (\Phi^{(n)})^2\big(\mathrm{Cof}\,A\big)^{-1}\nabla\zeta_n \cdot \nabla\zeta_n \\ &+ 2\zeta_n\Phi^{(n)}\big(\mathrm{Cof}\,A\big)^{-1}\nabla\Phi^{(n)} \cdot \nabla\zeta_n \\ \leq& \,\theta_n^2 + n^2\Lambda\lambda^{-3}R_n^{-2} + 2n\Lambda\lambda^{-3}R_n^{-1}\|\nabla\Phi\|_{L^\infty}.\end{aligned}$$

We also observe that

$$\|\nabla\Phi_n\|_{L^\infty(\mathbb{R}^3_+)} \leq \|\nabla\Phi^{(n)}\|_{L^\infty(\mathbb{R}^3_+)} + R_n^{-1}\|\Phi^{(n)}\|_{L^\infty(\mathbb{R}^3_+)} \leq \|\nabla\Phi\|_{L^\infty(\mathbb{R}^3_+)} + nR_n^{-1}\,.$$

Choosing R_n such that

$$R_n \geq \max\left\{n+1,\ \frac{n^2\Lambda\lambda^{-3} + 2n\Lambda\lambda^{-3}\|\nabla\Phi\|_{L^\infty}}{1-\theta_n^2}\right\}$$

we derive that $\|\nabla\Phi_n\|_{L^\infty(\mathbb{R}^3_+)} \leq \|\nabla\Phi\|_{L^\infty(\mathbb{R}^3_+)} + 1 \leq C$ for a constant C independent of n, and

$$\big(\mathrm{Cof}\,A\big)^{-1}\nabla\Phi_n \cdot \nabla\Phi_n \leq 1 \quad \text{a.e. in } \Omega_r,$$

so that Φ_n is 1–Lipschitz with respect to d^r_A in $\overline{\Omega}_r$ by Proposition 3.7. Since $\zeta_n \to 1$ and $\nabla\zeta_n \to 0$ a.e. as $n \to +\infty$, we trivially have $\Phi_n \to \Phi$ and $\nabla\Phi_n \to \nabla\Phi$ a.e. in $\mathbb{R}^3_+$ as $n \to +\infty$ and we conclude that the sequence $\{\Phi_n\}_{n\in\mathbb{N}}$ meets the requirement. ■

Proof of Proposition 7.1 completed. For $0 < r < \infty$, we extend any function $\Phi \in \mathrm{Lip}_0(\overline{\Omega}_r, \mathbb{R})$ to $\overline{\mathbb{R}^3_+}$ by setting $\Phi(x_1, x_2, x_3) = [1 + r - x_3]_+\Phi(x_1, x_2, r)$ if $x_3 \geq r$. Trivially, we obtain a globally Lipschitz function Φ with compact support in $\overline{\mathbb{R}^3_+}$. By Lemma 7.1, for any $0 < r \leq \infty$ and any $f \in X$, we have (using the extension convention above)

$$\mathcal{F}^r_A(f) = \mathrm{Sup}\bigg\{\mathcal{E}_A(f) + \frac{1}{2}\int_{\mathbb{R}^3_+} H(u_f)\cdot\nabla\Phi\ ;\ \Phi \in \mathrm{Lip}_0(\overline{\Omega}_r, \mathbb{R})\,, \\ |\Phi(P) - \Phi(Q)| \leq d^r_A(P,Q)\ \forall P, Q \in \overline{\Omega}_r\bigg\}\,.$$

Since the supremum of a family of sequentially lower semicontinuous functionals is still lower semi-continuous, it suffices to show that for any function $\Phi \in \mathrm{Lip}_0(\overline{\Omega}_r, \mathbb{R})$ which is 1–Lipschitz with respect to d^r_A, the functional

$$\mathcal{G}^r_A[\Phi]\ :\ f \in X \mapsto \ \mathrm{Max}\bigg\{\mathcal{E}_A(f) + \frac{1}{2}\int_{\mathbb{R}^3_+} H(u_f)\cdot\nabla\Phi\ ,\ \mathcal{E}_A(f) + \frac{1}{2}\int_{\mathbb{R}^3_+} H(u_f)\cdot\nabla(-\Phi)\bigg\}$$

is sequentially lower semi-continuous with respect to the weak $\dot H^{1/2}$–topology. Consider a sequence $\{f_n\}_{n\in\mathbb{N}} \subset X$ and $f \in X$ such that $f_n \rightharpoonup f$ weakly in $\dot H^{1/2}$ as $n \to +\infty$. Without loss of generality, we may assume that

$$\liminf_{n\to+\infty}\ \mathcal{G}^r_A[\Phi](f_n) = \lim_{n\to+\infty}\ \mathcal{G}^r_A[\Phi](f_n) < +\infty.$$

We denote by u_n and u the respective A–harmonic extensions of f_n and f. Obviously, we have

$$\frac{\lambda}{2}\int_{\mathbb{R}^3_+}|\nabla u_n|^2 \leq E_A(u_n) = \mathcal{E}_A(f_n) \leq \mathcal{G}^r_A[\Phi](f_n) \leq C.$$

Then Proposition 2.8 tell us that, up to a subsequence, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\mathbb{R}^3_+)$ and $u_n \to u$ strongly in $H^1(K)$ for any compact $K \subset \mathbb{R}^3_+$ since $f_n \rightharpoonup f$. We set $v_n := u_n - u$. Assuming that $\int_{\mathbb{R}^3_+} H(u)\cdot\nabla\Phi \geq 0$ (otherwise we use $-\Phi$ instead of Φ), an easy computation leads to

$$\mathcal{G}^r_A[\Phi](f_n) \geq \mathcal{G}^r_A[\Phi](f) + E_A(v_n) + \frac{1}{2}\int_{\mathbb{R}^3_+} H(v_n)\cdot\nabla\Phi + I_n + II_n + III_n \tag{7.3}$$

where

$$\begin{aligned} I_n &= \int_{\mathbb{R}^3_+} \mathrm{tr}\big(\nabla u A(\nabla v_n)^t\big)\,,\\ II_n &= \int_{\mathbb{R}^3_+}\Big\{\big(\partial_2 u\wedge\partial_3 v_n\big)\partial_1\Phi + \big(\partial_3 u\wedge\partial_1 v_n\big)\partial_2\Phi + \big(\partial_1 u\wedge\partial_2 v_n\big)\partial_3\Phi\Big\}\,,\\ III_n &= \int_{\mathbb{R}^3_+}\Big\{\big(\partial_2 v_n\wedge\partial_3 u\big)\partial_1\Phi + \big(\partial_3 v_n\wedge\partial_1 u\big)\partial_2\Phi + \big(\partial_1 v_n\wedge\partial_2 u\big)\partial_3\Phi\Big\}\,. \end{aligned}$$

Since $\nabla v_n \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^3_+)$, we infer that

$$\lim_{n\to+\infty} I_n = \lim_{n\to+\infty} II_n = \lim_{n\to+\infty} III_n = 0. \tag{7.4}$$

On the other hand, $v_n \to 0$ strongly in $H^1(K)$ for any compact set $K \subset \mathbb{R}^3_+$ and consequently (since $\operatorname{spt}\Phi$ is compact),

$$\int_{\mathbb{R}^3_+\cap\{x_3>r\}} H(v_n)\cdot\nabla\Phi \longrightarrow 0 \quad \text{as } n\to+\infty. \tag{7.5}$$

Since Φ is 1–Lipschitz with respect to d^r_A in $\overline{\Omega}_r$, we derive that a.e. in Ω_r,

$$\big|H(u_n)\cdot\nabla\Phi\big| \leq \sqrt{(\mathrm{Cof}\,A)H(u_n)\cdot H(u_n)} \leq \mathrm{tr}\big(\nabla u_n A(\nabla u_n)^t\big) \tag{7.6}$$

using Cauchy-Schwartz inequality, Proposition 3.7 and (4.2). Hence

$$E_A(v_n) + \frac{1}{2}\int_{\Omega_r} H(v_n)\cdot\nabla\Phi \geq 0\,. \tag{7.7}$$

Combining (7.3) to (7.7), we conclude that $\lim\limits_{n\to+\infty} \mathcal{G}^r_A[\Phi](f_n) \geq \mathcal{G}^r_A[\Phi](f)$ which ends the proof. ■

Proposition 7.2. *For every $f \in X$, we have $L^r_A(f) \to L_A(f)$ as $r \to 0$.*

Proof. *Step 1.* First we prove Proposition 7.2 for $f \in X \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ such that f is smooth except at finitely many points $a_1,\dots,a_k$. In this case, the distribution $T(f)$ can be written as $T(f) = -2\pi\sum_{i=1}^k d_i\delta_{a_i}$ where $d_i = \deg(f,a_i)$ is the topological degree of f around its singularity a_i and since $|f|_{1/2} < \infty$, we have $\sum_{i=1}^k d_i = 0$ (see Lemma 2.2). Hence we can relabel the a_i's, taking into account their multiplicity $|d_i|$, as two lists $(p_1,\dots,p_N)$ and $(q_1,\dots,q_N)$ of respectively positive and negative points. In this way, we rewrite $T(f)$ as

$$T(f) = -2\pi\sum_{i=1}^N(\delta_{p_i} - \delta_{q_i})\,. \tag{7.8}$$

Consequently, we obtain from (7.1),

$$L_A^r(f) = \mathrm{Sup}\left\{ \sum_{i=1}^N \Phi(p_i) - \Phi(q_i)\,;\, \Phi \in \mathrm{Lip}(\Omega_r,\mathbb{R})\,,\right.$$
$$\left. |\Phi(P)-\Phi(Q)| \le d_A^r(P,Q)\ \forall P,Q \in \overline{\Omega}_r \right\}.$$

By a well known result in [13], we derive that $L_A^r(f) = \underset{\sigma\in\mathcal{S}_N}{\mathrm{Min}} \sum_{i=1}^N d_A^r(p_i,q_{\sigma(i)})$ where $\mathcal{S}_N$ denotes the set of all permutations of N indices. Taking an arbitrary sequence of positive numbers $r_n \to 0$, we deduce that, for each $n\in\mathbb{N}$, there exists $\sigma_n \in \mathcal{S}_N$ such that $L_A^{r_n}(f) = \sum_{i=1}^N d_A^{r_n}(p_i,q_{\sigma_n(i)})$. Extracting a subsequence if necessary, we may assume that $\sigma_n = \sigma_\star$ for every $n\in\mathbb{N}$ and some $\sigma_\star \in \mathcal{S}_N$. Then we infer from Proposition 3.5 that

$$L_A^{r_n}(f) = \sum_{i=1}^N d_A^{r_n}(p_i,q_{\sigma_\star(i)}) \underset{n\to+\infty}{\longrightarrow} \sum_{i=1}^N \bar{d}_A(p_i,q_{\sigma_\star(i)})$$

which yields by the same result in [13] and (7.8),

$$\begin{aligned}
&\lim_{n\to+\infty} L_A^{r_n}(f) \ge \underset{\sigma\in\mathcal{S}_N}{\mathrm{Min}} \sum_{i=1}^N \bar{d}_A(p_i,q_{\sigma(i)}) = \\
&=\mathrm{Sup}\left\{\sum_{i=1}^N \varphi(p_i)-\varphi(q_i);\ \varphi\in\mathrm{Lip}(\mathbb{R}^2,\mathbb{R})\,,\ |\varphi(P)-\varphi(Q)|\le \bar{d}_A(P,Q)\ \forall P,Q\in\mathbb{R}^2\right\}\\
&=\frac{1}{2\pi}\mathrm{Sup}\left\{<T(f),\varphi>;\ \varphi\in\mathrm{Lip}(\mathbb{R}^2,\mathbb{R})\,,\ |\varphi(P)-\varphi(Q)|\le \bar{d}_A(P,Q)\ \forall P,Q\in\mathbb{R}^2\right\}\\
&= L_A(f)\,.
\end{aligned}$$

On the other hand $d_A^{r_n} \le \bar{d}_A$ on $\mathbb{R}^2\times\mathbb{R}^2$ by Remark 3.4 and hence $L_A^{r_n}(f) \le L_A(f)$ for any $n\in\mathbb{N}$. Consequently, $\lim_{n\to+\infty} L_A^{r_n}(f) = L_A(f)$. Then the result follows from the standard argument on the uniqueness of the limit.

Step 2. To obtain the result for a general map $f\in X$, we shall require the following stability property.

Lemma 7.2. *For any $f_1,f_2\in X$, we have*

$$\left|L_A^r(f_1)-L_A^r(f_2)\right| \le C\Lambda\big(|f_1|_{1/2}+|f_2|_{1/2}\big)|f_1-f_2|_{1/2}$$

for some positive constant C independent of $r\in[0,\infty]$ (here we set $L_A^0 := L_A$).

Proof. *Step 1.* For $r\in(0,\infty]$ and $f_1,f_2\in X$, we introduce

$$L_A^r(f_1,f_2) = \frac{1}{2\pi}\mathrm{Sup}\big\{ <T(f_1)-T(f_2),\Phi_{|\mathbb{R}^2}>;\ \Phi\in\mathrm{Lip}(\Omega_r,\mathbb{R})\,,$$
$$|\Phi(P)-\Phi(Q)|\le d_A^r(P,Q)\ \forall P,Q\in\overline{\Omega}_r\,\big\}$$

and we easily check that

$$\left|L_A^r(f_1)-L_A^r(f_2)\right| \le L_A^r(f_1,f_2). \tag{7.9}$$

Using (1.4), we derive that for any $\Phi \in \mathrm{Lip}(\Omega_r, \mathbb{R})$,

$$< T(f_1) - T(f_2), \Phi_{|\mathbb{R}^2} >= \int_{\mathbb{R}^3_+} \big(H(u_1) - H(u_2)\big) \cdot \nabla\Phi \leq$$
$$\leq \|\nabla\Phi\|_{L^\infty(\mathbb{R}^3_+)} \int_{\mathbb{R}^3_+} \big|H(u_1) - H(u_2)\big|$$

where u_1 and u_2 denote the respective harmonic extensions of f_1 and f_2 and here, Φ is extended to the whole half space by setting $\Phi(x_1, x_2, x_3) = \Phi(x_1, x_2, r)$ if $x_3 \geq r$. If the function Φ is chosen to be 1–Lipschitz in Ω_r with respect to d^r_A, then Φ is Λ–Lipschitz in Ω_r with respect to the Euclidean distance by (3.4). Hence $\|\nabla\Phi\|_{L^\infty(\mathbb{R}^3_+)} \leq \|\nabla\Phi\|_{L^\infty(\Omega^r)} \leq \Lambda$ and consequently

$$L^r_A(f_1, f_2) \leq \Lambda \int_{\mathbb{R}^3_+} \big|H(u_1) - H(u_2)\big|\,.$$

Writing $u_j = (u^1_j, u^2_j)$, we observe that

$$H(u_1) - H(u_2) = 2\nabla(u^1_1 - u^1_2) \wedge \nabla u^2_1 + 2\nabla u^1_2 \wedge \nabla(u^2_1 - u^2_2)\,.$$

From Cauchy-Schwartz inequality and Proposition 2.7, we infer that

$$L^r_A(f_1, f_2) \leq C\Lambda\big(|f_1|_{1/2} + |f_2|_{1/2}\big)|f_1 - f_2|_{1/2}\,.$$

Combining this estimate with (7.9) we obtain the announced result.

Step 2. For $r = 0$ and $f_1, f_2 \in X$, we introduce as in Step 1,

$$L_A(f_1, f_2) = \frac{1}{2\pi}\,\mathrm{Sup}\big\{ < T(f_1) - T(f_2), \varphi >;\ \varphi \in \mathrm{Lip}(\mathbb{R}^2, \mathbb{R})\,,$$
$$|\varphi(P) - \varphi(Q)| \leq \bar{d}_A(P, Q)\ \forall P, Q \in \mathbb{R}^2 \big\}$$

and then $\big|L_A(f_1) - L_A(f_2)\big| \leq L_A(f_1, f_2)$. By (3.18), any φ which is 1–Lipschitz in $\mathbb{R}^2$ with respect to $\bar{d}_A$ is also Λ–Lipschitz in $\mathbb{R}^2$ with respect to the Euclidean distance. Hence such φ can be extended to the half space into a Λ–Lipschitz function Φ with respect to the Euclidean distance by setting $\Phi(x_1, x_2, x_3) = \varphi(x_1, x_2)$. Consequently, we infer from Step 1,

$$L_A(f_1, f_2) \leq L^\infty_{\Lambda \mathrm{Id}}(f_1, f_2) \leq C\Lambda\big(|f_1|_{1/2} + |f_2|_{1/2}\big)|f_1 - f_2|_{1/2}$$

which ends the proof. ■

Proof of Proposition 7.2 completed. Let f be an arbitrary map in X. By Theorem 2.6, there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset X \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ such that f_n is smooth except at finitely many points and $|f_n - f|_{1/2} \to 0$ as $n \to +\infty$. Then we infer from Lemma 7.2 that for any $r \in (0, \infty]$ and $n \in \mathbb{N}$,

$$L^r_A(f_n) - C\Lambda\big(|f_n|_{1/2} + |f|_{1/2}\big)|f_n - f|_{1/2} \leq L^r_A(f) \leq$$
$$\leq L^r_A(f_n) + C\Lambda\big(|f_n|_{1/2} + |f|_{1/2}\big)|f_n - f|_{1/2}\,.$$

Letting $r \to 0$ with n fixed, we deduce from Step 1 and Lemma 7.2 that

$$\liminf_{r\to 0} L^r_A(f) \geq L_A(f_n) - C\Lambda\big(|f_n|_{1/2} + |f|_{1/2}\big)|f_n - f|_{1/2}$$
$$\geq L_A(f) - 2C\Lambda\big(|f_n|_{1/2} + |f|_{1/2}\big)|f_n - f|_{1/2}$$

and similarly,

$$\limsup_{r\to 0} L^r_A(f) \leq L_A(f) + 2C\Lambda\big(|f_n|_{1/2} + |f|_{1/2}\big)|f_n - f|_{1/2}\,.$$

Now letting $n \to +\infty$ in the previous inequalities, we conclude that $\lim_{r\to 0} L^r_A(f) = L_A(f)$ and the proof is complete. ■

Proof of Theorem 7.1 completed. We infer from Remark 3.4 that $L^{r_1}_A(f) \leq L^{r_2}_A(f) \leq L_A(f)$ for any $f \in X$ and $0 < r_2 \leq r_1 \leq \infty$. On the other hand $L^r_A(f) \to L_A(f)$ as $r \to 0$ for every $f \in X$ by Proposition 7.2, so that $L_A(f) = \sup_{r>0} L^r_A(f)$. Hence $\mathcal{F}_A(f) = \sup_{r>0} \mathcal{F}^r_A(f)$ for all $f \in X$. Since $\mathcal{F}^r_A$ is sequentially lower semicontinuous with respect to the weak $\dot{H}^{1/2}$–topology for every $r > 0$, the same property holds for $\mathcal{F}_A$. ■

Corollary 7.1. *For every $f \in X$, we have*

$$\overline{\mathcal{E}}_A(f) \geq \mathcal{E}_A(f) + \pi L_A(f)\,.$$

Proof. Let $f \in X$ and consider an arbitrary sequence of smooth maps $\{f_n\}_{n\in\mathbb{N}} \subset X$ such that $f_n \rightharpoonup f$ weakly in $\dot{H}^{1/2}$ as $n \to +\infty$. Since f_n is smooth, we have $T(f_n) = 0$ so that $L_A(f_n) = 0$ for every n. Then we infer from Theorem 7.1,

$$\liminf_{n\to+\infty} \mathcal{E}_A(f_n) = \liminf_{n\to+\infty} \mathcal{F}_A(f_n) \geq \mathcal{F}_A(f) = \mathcal{E}_A(f) + \pi L_A(f)\,.$$

Taking the infimum over all such $\{f_n\}_{n\in\mathbb{N}}$, we obtain the announced result. ■

7.2 Upper bound for $\overline{\mathcal{E}}_A$ by a dipole removing technique

In this subsection, we build the recovery sequence required to prove the upper bound of $\overline{\mathcal{E}}_A(f)$ stated in Theorem 1.3. Our main result here is the following.

Theorem 7.2. *For every $f \in X$, there exists a sequence of smooth maps $\{f_n\}_{n\in\mathbb{N}} \subset X$ such that $f_n \rightharpoonup f$ weakly in $\dot{H}^{1/2}$, $f_n \to f$ a.e. in $\mathbb{R}^2$ as $n \to +\infty$ and*

$$\limsup_{n\to+\infty} \mathcal{E}_A(f_n) \leq \mathcal{E}_A(f) + \pi \tilde{L}_A(f)\,.$$

The proof of this theorem is based on the preliminary propostion below asserting that maps with trivial $T(f)$ can be approximated strongly. This fact has been proved first in [34] for maps defined on the two-dimensional sphere. Here we follow the method of [9].

Proposition 7.3. *Let $f \in X$ such that $T(f) = 0$. Then there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset X \cap C^\infty_{\mathrm{const}}(\mathbb{R}^2)$ such that $f_n \to f$ a.e. in $\mathbb{R}^2$ and $|f_n - f|_{1/2} \to 0$ as $n \to +\infty$.*

Proof. Let $f \in X$ such that $T(f) = 0$. By Theorem 2.6, there exists a sequence $\{\tilde{f}_n\}_{n\in\mathbb{N}} \subset X \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ such that $\tilde{f}_n$ is smooth except at finitely many points, $\tilde{f}_n$ is constant outside a compact set, $\tilde{f}_n \to f$ a.e. in $\mathbb{R}^2$ and $|\tilde{f}_n - f|_{1/2} \to 0$ as $n \to +\infty$. Since $T(f) = 0$, we have $L_{\mathrm{Id}}(f) = 0$ and we infer from Lemma 7.2 that $L_{\mathrm{Id}}(\tilde{f}_n) = |L_{\mathrm{Id}}(\tilde{f}_n) - L_{\mathrm{Id}}(f)| \leq C|\tilde{f}_n - f|_{1/2}$. Hence

$$\lim_{n\to+\infty} L_{\mathrm{Id}}(\tilde{f}_n) = 0\,. \tag{7.10}$$

Since $\tilde{f}_n$ is smooth except at finitely many points, we may proceed as in the proof of Proposition 7.2 to write $T(\tilde{f}_n) = -2\pi \sum_{i=1}^N (\delta_{p_i} - \delta_{q_i})$ for some $N = N(n)$ and relabeling the q_i's if necessary, we may assume that

$$\sum_{i=1}^N |p_i - q_i| = \operatorname*{Min}_{\sigma\in\mathcal{S}_N} \sum_{i=1}^N |p_i - q_{\sigma(i)}| = L_{\mathrm{Id}}(\tilde{f}_n)\,.$$

We set $\tilde{f}_{n,0} = \tilde{f}_n$ and we define by induction on $i \in \{1,\dots,N\}$ a map $\tilde{f}_{n,i}$ as follows. By the construction given in the proof of Lemma 4.2 with Remark 4.2 and by Lemma 4.1, we can find $h_{n,i} \in X \cap W^{1,1}_{\rm loc}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \{p_i, q_i\})$ such that $T(h_{n,i}) = 2\pi(\delta_{p_i} - \delta_{q_i})$, $h_{n,i}$ is equal to 1 outside a neighborhood of $[p_i, q_i]$ of measure less than $2^{-n}N^{-1}$ and $\mathcal{E}_{\rm Id}(h_{n,i}\tilde{f}_{n,i-1}) \leq \mathcal{E}_{\rm Id}(\tilde{f}_{n,i-1}) + 2\pi|p_i - q_i|$. We set $\tilde{f}_{n,i} = h_{n,i}\tilde{f}_{n,i-1} \in X \cap W^{1,1}_{\rm loc}(\mathbb{R}^2)$ and at the final step, we relabel the resulting map $\hat{f}_n = \tilde{f}_{n,N}$. We claim that, up to a subsequence, $\hat{f}_n \to f$ a.e. and $|\hat{f}_n - f|_{1/2} \to 0$ as $n \to +\infty$. The convergence a.e. (up to a subsequence) of $\hat{f}_n$ to f is clear since $\mathcal{L}^2(\{\hat{f}_n \neq \tilde{f}_n\}) \leq 2^{-n}$ and $\tilde{f}_n \to f$ a.e. as $n \to +\infty$. Then observe that, by construction and (7.10),

$$\mathcal{E}_{\rm Id}(\hat{f}_n) \leq \mathcal{E}_{\rm Id}(\tilde{f}_n) + 2\pi L_{\rm Id}(\tilde{f}_n) \underset{n\to+\infty}{\longrightarrow} \mathcal{E}_{\rm Id}(f)\,.$$

On the other hand, we infer from Theorem 7.1 that $\liminf\limits_{n\to+\infty} \mathcal{E}_{\rm Id}(\hat{f}_n) \geq \mathcal{F}_{\rm Id}(f) = \mathcal{E}_{\rm Id}(f)$ so that $\mathcal{E}_{\rm Id}(\hat{f}_n) \to \mathcal{E}_{\rm Id}(f)$ as $n \to +\infty$. Setting u_n and u to be the respective harmonic extensions to the half space of $\hat{f}_n$ and f, we have by classical results, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\mathbb{R}^3_+)$. Together with

$$\mathcal{E}_{\rm Id}(\hat{f}_n) = \frac{1}{2}\int_{\mathbb{R}^3_+} |\nabla u_n|^2 \underset{n\to+\infty}{\longrightarrow} \frac{1}{2}\int_{\mathbb{R}^3_+} |\nabla u|^2 = \mathcal{E}_{\rm Id}(f)$$

it implies that $\nabla u_n \to \nabla u$ strongly in $L^2(\mathbb{R}^3_+)$, i.e., $|\hat{f}_n - f|_{1/2} \to 0$ as $n \to +\infty$.

Now observe that the map $\hat{f}_n \in X \cap W^{1,1}_{\rm loc}(\mathbb{R}^2)$ is smooth away from the points p_i and q_i, it is constant outside a compact set, and since $T(\hat{f}_n) = T(f) - \sum_i T(h_{n,i}) = 0$ by Proposition 2.6, the topological degree of $\hat{f}_n$ around a point p_i or q_i is equal to 0. Hence the singularities p_i and q_i can be removed by standard techniques (see e.g. [9,16]), i.e., one can find a map smooth map $f_n \in C^\infty_{\rm const}(\mathbb{R}^2; \mathbb{S}^1)$ such that f_n agrees with $\hat{f}_n$ outside a neighborhood of $\cup_i\{p_i, q_i\}$ of measure less then 2^{-n} and $|f_n - \hat{f}_n|_{1/2} \leq 2^{-n}$. Then, up to a sequence, $f_n \to f$ a.e. in $\mathbb{R}^2$ and $|f_n - f|_{1/2} \to 0$ as $n \to +\infty$ so the proposition is proved. ■

Proof of Theorem 7.2. *Step 1.* We start by proving Theorem 7.2 for $f \in X \cap W^{1,1}_{\rm loc}(\mathbb{R}^2)$ such that f is smooth except at finitely many points. Then we proceed as in the previous proof to write $T(f) = -2\pi\sum_{i=1}^N(\delta_{p_i} - \delta_{q_i})$ and relabeling the q_i's if necessary, we have

$$\sum_{i=1}^N \bar{\rho}_A(p_i, q_i) = \operatorname*{Min}_{\sigma\in\mathcal{S}_N} \sum_{i=1}^N \bar{\rho}_A(p_i, q_{\sigma(i)})\,. \tag{7.11}$$

Since $\bar{\rho}_A$ is a distance, we derive from (7.11) and the results in [13],

$$\tilde{L}_A(f) = \operatorname{Sup}\Big\{\sum_{i=1}^N \varphi(p_i) - \varphi(q_i)\,,\ \varphi \in {\rm Lip}(\mathbb{R}^2,\mathbb{R})\,,$$
$$|\varphi(P) - \varphi(Q)| \leq \pi^{-1}\bar{\rho}_A(P,Q)\ \forall P,Q \in \mathbb{R}^2\Big\} = \frac{1}{\pi}\sum_{i=1}^N \bar{\rho}_A(p_i, q_i)\,.$$

By the definition of energetic distance $\bar{\rho}_A$, for every $i \in \{1,\dots,N\}$ there exists a sequence $\{h_{i,n_i}\}_{n_i\in\mathbb{N}} \subset X$ such that (without loss of generality) $h_{i,n_i} \rightharpoonup 1$ weakly in $\dot{H}^{1/2}$ as $n_i \to +\infty$, $T(h_{i,n_i}) = 2\pi(\delta_{p_i} - \delta_{q_i})$ and $\lim\limits_{n_i\to+\infty} \mathcal{E}_A(h_{i,n_i}) = \bar{\rho}_A(p_i, q_i)$. Up to subsequences, we may assume that $h_{i,n_i} \to 1$ a.e. for every i by Theorem 2.5. From Proposition 2.6, we infer that $\Pi_{i=1}^N h_{i,n_i} \in X$ and

$$T(\Pi_{i=1}^N h_{i,n_i}) = \sum_{i=1}^N T(h_{i,n_i}) = -T(f)\,.$$

Then a straightforward consequence of Lemma 4.1 yields

$$\limsup_{n_N\to+\infty}\dots\limsup_{n_2\to+\infty}\limsup_{n_1\to+\infty}\mathcal{E}_A\big(\Pi_{i=1}^N h_{i,n_i}\big)\leq\sum_{i=1}^N\lim_{n_i\to+\infty}\mathcal{E}_A(h_{i,n_i})=\sum_{i=1}^N\bar\rho_A(p_i,q_i)\,.$$

By Remark 2.5 in Section 2, we may find a diagonal sequence $h_k:=\Pi_{i=1}^N h_{i,n_i(k)}$ which satisfies $T(h_k)=-T(f)$, $h_k\rightharpoonup 1$ weakly in $\dot H^{1/2}$, $h_k\to 1$ a.e. as $k\to+\infty$ and $\limsup_{k\to+\infty}\mathcal{E}_A(h_k)\leq\pi\tilde L_A(f)$. Next we consider $\tilde f_k=h_kf\in X$ and we infer from Lemma 4.1 that

$$\limsup_{k\to+\infty}\mathcal{E}_A(\tilde f_k)\leq\mathcal{E}_A(f)+\pi\tilde L_A(f)\,.$$

Since $T(\tilde f_k)=T(h_k)+T(f)=0$ and $f_k\to f$ a.e. as $k\to+\infty$, we may now apply Proposition 7.3 to $\tilde f_k$ and then the diagonalization procedure in Remark 2.5 to obtain the desired sequence of smooth maps approximating f.

Step 2. To treat the case of a general map $f\in X$, we shall require the following version Lemma 7.2 for the minimal connection relative to the energetic distance $\bar\rho_A$.

Lemma 7.3. *For any $f_1,f_2\in X$, we have*

$$\big|\tilde L_A(f_1)-\tilde L_A(f_2)\big|\leq C\Lambda\big(|f_1|_{1/2}+|f_2|_{1/2}\big)|f_1-f_2|_{1/2}$$

for some positive constant C.

Proof. By Theorem 1.1, any function φ which is 1–Lipschitz in $\mathbb{R}^2$ with respect to $\pi^{-1}\bar\rho_A$ is also Λ–Lipschitz in $\mathbb{R}^2$ with respect to the Euclidean distance and hence we can proceed exactly as in Step 2, proof of Lemma 7.2. ∎

Proof of Theorem 7.2 completed. Let f be an arbitrary map in X. By Lemma 2.6, there exists a sequence $\{\tilde f_n\}_{n\in\mathbb{N}}\subset X\cap W^{1,1}_{\rm loc}(\mathbb{R}^2)$ such that $\tilde f_n$ is smooth except at finitely many points, $\tilde f_n\to f$ a.e. in $\mathbb{R}^2$ and $|\tilde f_n-f|_{1/2}\to 0$ as $n\to+\infty$. In particular $\mathcal{E}_A(\tilde f_n)\to\mathcal{E}_A(\tilde f)$ and by Lemma 7.3, $\tilde L_A(\tilde f_n)\to\tilde L_A(f)$ as $n\to+\infty$. By Step 1, for every n, there exists a sequence of smooth maps $\{f_{n,m}\}_{m\in\mathbb{N}}\subset X$ such that $f_{n,m}\to\tilde f_n$ a.e. in $\mathbb{R}^2$ as $m\to+\infty$ and $\limsup_{m\to+\infty}\mathcal{E}_A(f_{n,m})\leq\mathcal{E}_A(\tilde f_n)+\pi\tilde L_A(\tilde f_n)$. Then we just have to apply the diagonalization procedure in Remark 2.5 to obtain the required sequence. ∎

As a direct consequence of Theorem 7.2, we obtain the upper bound of $\overline{\mathcal{E}}_A(f)$.

Corollary 7.2. *For every $f\in X$, we have*

$$\overline{\mathcal{E}}_A(f)\leq\mathcal{E}_A(f)+\pi\tilde L_A(f)\,.$$

We close this subsection with the existence of admissible and optimal sequences in the definition of $\bar m_A(T(f))$.

Proposition 7.4. *For every $f\in X$, there exists a sequence $\{h_n\}_{n\in\mathbb{N}}\subset X$ such that $T(h_n)=T(f)$ for every n, $h_n\rightharpoonup\alpha$ weakly in $\dot H^{1/2}$ and $h_n\to\alpha$ a.e. in $\mathbb{R}^2$ as $n\to+\infty$ for some constant $\alpha\in\mathbb{S}^1$. In particular, $\bar m_A(T(f))$ is well defined for every $f\in X$ and moreover, the infimum defining $\bar m_A(T(f))$ is achieved.*

Proof. Let $f\in X$. By Theorem 7.2, there exists a sequence of smooth maps $\{f_n\}_{n\in\mathbb{N}}\subset X$ such that $f_n\to f$ a.e. and $\sup_n\mathcal{E}_A(f_n)<+\infty$. We set $h_n=\bar f_nf$. By Proposition 2.6, $h_n\in X$ and

$|h_n|_{1/2} \leq |f_n|_{1/2} + |f|_{1/2}$ so that $\mathcal{E}_A(h_n) \leq C$. Obviously $h_n \to 1$ a.e. in $\mathbb{R}^2$. Extracting a subsequence if necessary, we have $h_n \rightharpoonup 1$ weakly in $\dot{H}^{1/2}$ as $n \to +\infty$ by Theorem 2.5. The existence of an optimal sequence realizing $\bar{m}_A(T(f))$ follows from a standard diagonalization argument together with Theorem 2.5 and Remark 2.5. ■

7.3 Proof of Theorem 1.3.

We start with the following proposition.

Proposition 7.5. *For every $f \in X$, we have*

$$\overline{\mathcal{E}}_A(f) \leq \mathcal{E}_A(f) + \bar{m}_A(T(f))\,.$$

Proof. Let $f \in X$ and consider a sequence $\{h_n\}_{n\in\mathbb{N}} \subset X$ such that $T(h_n) = T(f)$, $h_n \rightharpoonup \alpha \in \mathbb{S}^1$ weakly in $\dot{H}^{1/2}$ and $\lim\limits_{n\to+\infty} \mathcal{E}_A(h_n) = \bar{m}_A(T(f))$. Such a sequence exists by Proposition 7.4 and we may assume $h_n \to \alpha$ a.e. by Theorem 2.5. Without loss of generality, we may also assume that $\alpha = 1$. Then we set $\tilde{f}_n = \bar{h}_n f \in X$ so that $\tilde{f}_n \to f$ a.e. in $\mathbb{R}^2$, $T(\tilde{f}_n) = T(f) - T(h_n) = 0$ by Proposition 2.6. We infer from Lemma 4.1,

$$\limsup_{n\to+\infty} \mathcal{E}_A(\tilde{f}_n) \leq \mathcal{E}_A(f) + \lim_{n\to+\infty} \mathcal{E}_A(h_n) = \mathcal{E}_A(f) + \bar{m}_A(T(f))\,.$$

Since $T(\tilde{f}_n) = 0$, we may apply Proposition 7.3 to $\tilde{f}_n$ and then the diagonalization procedure in Remark 2.5 to obtain a sequence of smooth maps $\{f_n\}_{n\in\mathbb{N}} \subset X$ which satisfies $f_n \rightharpoonup f$ weakly in $\dot{H}^{1/2}$ and $\limsup\limits_{n\to+\infty} \mathcal{E}_A(f_n) \leq \mathcal{E}_A(f) + \bar{m}_A(T(f))$. Then the conclusion follows from the definition of $\overline{\mathcal{E}}_A(f)$. ■

Proposition 7.6. *For every $f \in X$, we have*

$$\bar{m}_A(T(f)) \leq \pi \tilde{L}_A(f)\,. \tag{7.12}$$

Proof. *Step 1.* In the case $f \in X \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ such that f is smooth except at finitely many points, we may construct as in the proof of Theorem 7.2, Step 1, a sequence $\{h_n\}_{n\in\mathbb{N}} \subset X$ satisfying $T(h_n) = T(f)$ for every n, $h_n \rightharpoonup 1$ weakly in $\dot{H}^{1/2}$ and $\limsup\limits_{n\to+\infty} \mathcal{E}_A(h_n) \leq \pi \tilde{L}_A(f)$. Hence (7.12) holds by definition of $\bar{m}_A(T(f))$.

Step 2. To treat the case of an arbitrary map f in X, we shall require the following lemma.

Lemma 7.4. *Let $f \in X$. For any sequence $\{f_n\}_{n\in\mathbb{N}} \subset X$ such that $f_n \to f$ a.e. and $|f_n - f|_{1/2} \to 0$ as $n \to +\infty$, we have*

$$\bar{m}_A(T(f)) \leq \liminf_{n\to+\infty} \bar{m}_A(T(f_n))\,.$$

Proof. Without loss of generality, we may assume that

$$\liminf_{n\to+\infty} \bar{m}_A(T(f_n)) = \lim_{n\to+\infty} \bar{m}_A(T(f_n)) < +\infty\,.$$

Let $g_n = \bar{f}_n f \in X$ by Proposition 2.6. Clearly $|g_n|_{1/2} \to 0$ as $n \to \infty$, because the product in X is strongly continuous. Now, for every $n \in \mathbb{N}$, let $h_{n,m} \in X$ such that $h_{n,m} \to 1$ a.e. as $m \to +\infty$, $T(h_{n,m}) = T(f_n)$ and $\lim\limits_{m\to+\infty} \mathcal{E}_A(h_{n,m}) = \bar{m}_A(T(f_n))$ (as in the proof of Proposition 7.5). We consider $\tilde{h}_{n,m} = h_{n,m}\, g_n \in X$. By Proposition 2.6 we have $T(\tilde{h}_{n,m}) = T(h_{n,m}) + T(g_n) = T(f)$ and Lemma

4.1 yields $\limsup_{m\to+\infty} \mathcal{E}_A(\tilde{h}_{n,m}) \leq \mathcal{E}_A(g_n) + \bar{m}_A(T(f_n))$. Since $|g_n|_{1/2} \to 0$, we infer that $\mathcal{E}_A(g_n) \to 0$ and hence

$$\limsup_{n\to+\infty} \limsup_{m\to+\infty} \mathcal{E}_A(\tilde{h}_{n,m}) \leq \lim_{n\to+\infty} \bar{m}_A(T(f_n)) .$$

Since $h_{n,m} \to 1$ a.e. as $m \to +\infty$ and $g_n \to 1$ a.e. as $n \to +\infty$, we may apply the procedure in Remark 2.5 to extract a diagonal sequence $\tilde{h}_k := \tilde{h}_{n_k,m_k}$ such that $\tilde{h}_k \rightharpoonup 1$ weakly in $\dot{H}^{1/2}$ as $k \to +\infty$ and $\limsup_{k\to+\infty} \mathcal{E}_A(\tilde{h}_k) \leq \lim_{n\to+\infty} \bar{m}_A(T(f_n))$. Then, by definition of $\bar{m}_A(T(f))$, we have

$$\bar{m}_A(T(f)) \leq \limsup_{k\to+\infty} \mathcal{E}_A(\tilde{h}_k) \leq \lim_{n\to+\infty} \bar{m}_A(T(f_n))$$

which completes the proof. ■

Proof of Proposition 7.6 completed. Let f be an arbitrary map in X. By Lemma 2.6, there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset X \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ such that f_n is smooth except at a finite number of point, $f_n \to f$ a.e. and $|f_n - f|_{1/2} \to 0$ as $n \to +\infty$. By Step 1, for every n we have

$$\bar{m}_A(T(f_n)) \leq \pi\tilde{L}_A(f_n) \leq \pi\tilde{L}_A(f) + \pi|\tilde{L}_A(f_n) - \tilde{L}_A(f)| .$$

Letting $n \to +\infty$ in the previous inequality, we conclude from Lemma 7.3 and Lemma 7.4 that $\bar{m}_A(T(f)) \leq \pi\tilde{L}_A(f)$. ■

Proof of Theorem 1.3 completed. First we observe that (1.20) comes from the combination of Corollary 7.1 and Corollary 7.2. From Corollary 7.1 and Proposition 7.5, we also deduce that $\bar{m}_A(T(f)) \geq \pi L_A(f)$ for every $f \in X$. The upper inequality in (1.21) is given in Proposition 7.6.

Now if we assume that $\bar{\rho}_A = \pi\bar{d}_A$ then $\tilde{L}_A \equiv L_A$. Hence $\bar{m}_A(T(\cdot)) \equiv L_A(\cdot)$ and (1.22) trivially follow from (1.20). In order to prove the reverse implication, let us assume that (1.22) holds and fix $P, Q \in \mathbb{R}^2$ two arbitrary distinct points. We consider $f \in X$ such that $T(f) = 2\pi(\delta_P - \delta_Q)$ (see e.g. Lemma 4.2) and $\{f_n\}_{n\in\mathbb{N}} \subset X$ a sequence of smooth maps such that $f_n \rightharpoonup f$ weakly in $\dot{H}^{1/2}$, $f_n \to f$ a.e. and $\mathcal{E}_A(f_n) \to \overline{\mathcal{E}}_A(f) = \mathcal{E}_A(f) + \pi L_A(f) = \mathcal{E}_A(f) + \pi\bar{d}_A(P,Q)$ as $n \to +\infty$ (such a sequence exists by a standard diagonalization argument together with Theorem 2.5 and Remark 2.5). We denote by u and u_n the respective A–harmonic extensions of f and f_n. By Proposition 2.8, $u_n \to u$ locally uniformly in $\mathbb{R}^3_+$, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\mathbb{R}^3_+)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^3_+)$ as $n \to +\infty$. By definition of $\mathcal{E}_A(f_n)$ and $\mathcal{E}_A(f)$, we have

$$E_A(u_n) \underset{n\to+\infty}{\longrightarrow} E_A(u) + \pi\bar{d}_A(P,Q) . \tag{7.13}$$

For any $R > 0$, we introduce the localized energies

$$E_A(v, \Omega_R) = \frac{1}{2}\int_{\Omega_R} \mathrm{tr}\big(\nabla v A (\nabla v)^t\big) \ , \quad E_A(v, \mathbb{R}^3_+ \setminus \overline{\Omega}_R) = E_A(v) - E_A(v, \Omega_R) ,$$

for $v = u$ or $v = u_n$ and $\Omega_R = \mathbb{R}^2 \times (0, R)$. We claim that for every $R > 0$,

$$E_A(u_n, \Omega_R) \underset{n\to+\infty}{\longrightarrow} E_A(u, \Omega_R) + \pi\bar{d}_A(P,Q) , \tag{7.14}$$

$$E_A(u_n, \mathbb{R}^3_+ \setminus \overline{\Omega}_R) \underset{n\to+\infty}{\longrightarrow} E_A(u, \mathbb{R}^3_+ \setminus \overline{\Omega}_R) . \tag{7.15}$$

Indeed, arguing as in the proof of Proposition 7.1, we obtain that for every $r > 0$,

$$\liminf_{n\to+\infty} E_A(u_n, \Omega_R) \geq E_A(u, \Omega_R) + \pi L^r_A(f) = E_A(u, \Omega_R) + \pi d^r_A(P,Q) .$$

Letting $r \to 0$ in this inequality, we derive from Proposition 3.5 that $\liminf_{n\to+\infty} E_A(u_n,\Omega_R) \geq E_A(u,\Omega_R)+\pi\bar{d}_A(P,Q)$. Hence $\limsup_{n\to+\infty} E_A(u_n,\mathbb{R}^3_+\setminus\Omega_R) \leq E_A(u,\mathbb{R}^3_+\setminus\Omega_R)$ by (7.13) and the reverse inequality with the lim inf easily follows by lower semicontinuity. Therefore (7.15) holds and (7.14) is deduced from (7.13) and (7.15).

Next we consider $h_n = f\bar{f}_n$. Exactly as in the proof of Proposition 7.4, we have $h_n \in X$, $\mathcal{E}_A(h_n) \leq C$, $h_n \to 1$ a.e. as $n \to +\infty$ and $T(h_n) = T(f) = 2\pi(\delta_P - \delta_Q)$. By Theorem 2.5, we may assume that $h_n \rightharpoonup 1$ weakly in $\dot{H}^{1/2}$ (extracting a subsequence if necessary). Hence $\bar{\rho}_A(P,Q) \leq \liminf_{n\to+\infty} \mathcal{E}_A(h_n)$ by definition of the energetic distance $\bar{\rho}_A$. We claim that

$$\limsup_{n\to+\infty} \mathcal{E}_A(h_n) \leq \pi L_A(f) = \pi\bar{d}_A(P,Q). \tag{7.16}$$

Once the claim is proved, we would get $\bar{\rho}_A(P,Q) \leq \pi\bar{d}_A(P,Q)$ so that $\bar{\rho}_A \leq \pi\bar{d}_A$ by the arbitrariness of the points P and Q. Then the conclusion follows since the reverse inequality holds by Theorem 1.1, claim *(ii)*.

Proof of (7.16). For every $R > 0$, we construct a comparison map $v_{n,R}$ as follows:

$$v_{n,R} = \begin{cases} u\bar{u}_n & \text{in } \Omega_R, \\ w_{n,R} & \text{in } \mathbb{R}^3_+ \setminus \overline{\Omega}_R, \end{cases}$$

where $w_{n,R}$ is the (finite energy) harmonic extension of $u\bar{u}_n$ to $\mathbb{R}^3_+\setminus\overline{\Omega}_R$. Since $v_{n,R|\mathbb{R}^2} = h_n$, we clearly have

$$\mathcal{E}_A(h_n) \leq E_A(v_{n,R}) = E_A(u\bar{u}_n,\Omega_R) + E_A(w_{n,R},\mathbb{R}^3_+\setminus\overline{\Omega}_R)\,. \tag{7.17}$$

Since $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\mathbb{R}^3_+)$ we infer from (7.15) that $\nabla u_n \to \nabla u$ strongly in $L^2(\mathbb{R}^3_+\setminus\overline{\Omega}_R)$ as $n \to +\infty$. Then we easily deduce by dominated convergence that $\nabla(u\bar{u}_n) \to \nabla|u|^2$ strongly in $L^2(\mathbb{R}^3_+\setminus\overline{\Omega}_R)$ as $n \to +\infty$. Setting $g_{n,R}$ and g_R to be the respective traces of $u\bar{u}_n$ and $|u|^2$ on the plane $\mathbb{R}^2\times\{R\}$, it yields

$$E_A(w_{n,R},\mathbb{R}^3_+\setminus\overline{\Omega}_R) \leq \Lambda E_{\mathrm{Id}}(w_{n,R},\mathbb{R}^3_+\setminus\overline{\Omega}_R) = C\Lambda|g_{n,R}|^2_{1/2} \underset{n\to+\infty}{\longrightarrow} C\Lambda|g_R|^2_{1/2}\,.$$

Since $|u|^2 \leq 1$ and its gradient is square integrable, it turns out that $|g_R|_{1/2}$ is a continuous function of R and $|g_R|_{1/2} \to 0$ as $R \to 0$ because $|u|^2$ has a constant trace equal to 1 on $\mathbb{R}^2\times\{0\}$. Therefore, letting first $n \to +\infty$ and then $R \to 0$ in (7.17), we obtain

$$\limsup_{n\to+\infty} \mathcal{E}_A(h_n) \leq \limsup_{R\to 0}\limsup_{n\to+\infty} E_A(u\bar{u}_n,\Omega_R)$$

so that it remains to prove that

$$\limsup_{R\to 0}\limsup_{n\to+\infty} E_A(u\bar{u}_n,\Omega_R) \leq \pi\bar{d}_A(P,Q)\,.$$

Arguing as in (4.1), we have

$$\begin{aligned} E_A(u\bar{u}_n,\Omega_R) &= \frac{1}{2}\int_{\Omega_R}\Big\{|u|^2\,\mathrm{tr}\big(\nabla u_n A(\nabla u_n)^t\big) + |u_n|^2\,\mathrm{tr}\big(\nabla u A(\nabla u)^t\big)+ \\ &+ 2\mathrm{Re}(uu_n)\,\mathrm{tr}\big(\nabla\bar{u}_n A(\nabla u)^t\big) + 2\mathrm{Im}(uu_n)\,\mathrm{tr}\big(\nabla(i\bar{u}_n)A(\nabla u)^t\big)\Big\} \leq E_A(u_n,\Omega_R)+ \\ &+ E_A(u,\Omega_R) + \int_{\Omega_R}\Big\{\mathrm{Re}(uu_n)\,\mathrm{tr}\big(\nabla\bar{u}_n A(\nabla u)^t\big) + \mathrm{Im}(uu_n)\,\mathrm{tr}\big(\nabla(i\bar{u}_n)A(\nabla u)^t\big)\Big\} = \\ &= E_A(u_n,\Omega_R) + E_A(u,\Omega_R) + I_{n,R} \end{aligned}$$

because $|u| \leq 1$ and $|u_n| \leq 1$ a.e.. Clearly (7.14) yields

$$\limsup_{R\to 0} \limsup_{n\to+\infty} \big(E_A(u_n,\Omega_R) + E_A(u,\Omega_R)\big) = \\ = \lim_{R\to 0} \big(\pi \bar{d}_A(P,Q) + 2E_A(u,\Omega_R)\big) = \pi \bar{d}_A(P,Q)$$

which is the desired contribution. On the other hand , since $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega_R)$ and $u_n \to u$ a.e. with $|u_n| \leq 1$, we infer that

$$\limsup_{R\to 0} \limsup_{n\to+\infty} I_{n,R} = \limsup_{R\to 0} \lim_{n\to+\infty} I_{n,R} = \\ = \limsup_{R\to 0} \int_{\Omega_R} \Big\{ \mathrm{Re}(u^2)\,\mathrm{tr}\big(\nabla \overline{u} A (\nabla u)^t\big) + \mathrm{Im}(u^2)\,\mathrm{tr}\big(\nabla(i\bar{u}) A (\nabla u)^t\big) \Big\} \\ \leq \limsup_{R\to 0} C E_A(u,\Omega_R) = 0$$

and the proof is complete. ■

Appendix A.

Proof of Theorem 2.6. *Step 1.* Consider $\varrho : \mathbb{R}^2 \to \mathbb{R}$ a nonnegative radial, smooth function such that $\int_{\mathbb{R}^2} \varrho = 1$ and $\operatorname{spt} \varrho \subset B_1(0)$. For $\varepsilon > 0$, we define $\varrho_\varepsilon(x) = \varepsilon^{-2}\varrho(\varepsilon^{-1}x)$ and $\tilde{f}_\varepsilon = \varrho_\varepsilon * f$. Then $\tilde{f}_\varepsilon$ defines a smooth function and we easily check that $|\tilde{f}_\varepsilon| \leq 1$ in $\mathbb{R}^2$. Then standard estimates yield

$$\|\tilde{f}_\varepsilon - f\|_{L^2(\mathbb{R}^2)} \leq o(\varepsilon^{1/2})\,, \tag{A.1}$$

$$\|\nabla \tilde{f}_\varepsilon\|_{L^2(\mathbb{R}^2)} \leq o(\varepsilon^{-1/2})\,, \tag{A.2}$$

$$|\tilde{f}_\varepsilon - f|_{1/2} \leq o(1) \quad \text{as } \varepsilon \to 0, \tag{A.3}$$

$$\big|\{|\tilde{f}_\varepsilon| < 3/4\}\big| \leq o(\varepsilon)\,, \tag{A.4}$$

$$|\nabla \tilde{f}_\varepsilon(x)| \leq C_0\,\varepsilon^{-1} \quad \text{for every } x \in \mathbb{R}^2 \text{ and for some constant } C_0. \tag{A.5}$$

Step 2. Let (ε_k) be a sequence of positive numbers such that $\varepsilon_k \to 0$ and set $\tilde{f}_k := \tilde{f}_{\varepsilon_k}$. We claim that for k large enough, $\{|\tilde{f}_k| < 1/2\}$ is bounded. Indeed, assume that it is unbounded. Then we can find a sequence of points $\{x_m\}_{m\in\mathbb{N}}$ such that $|x_{m+1}| \geq |x_m| + 1$ and $|\tilde{f}_k(x_m)| < 1/2$ for every m. By (A.5), we infer that $|\tilde{f}_k| < 3/4$ in $B_{R_k}(x_m)$ with $R_k = \varepsilon_k/(4C_0)$. Since $|x_i - x_j| \geq 1$ for $i \neq j$, we have $B_{R_k}(x_i) \cap B_{R_k}(x_j) = \emptyset$ for any $i \neq j$ and k large enough so that

$$\Big| \bigcup_{m\in\mathbb{N}} B_{R_k}(x_m) \Big| = \sum_{m\in\mathbb{N}} |B_{R_k}(x_m)| = +\infty\,.$$

On the other hand $\bigcup_{m\in\mathbb{N}} B_{R_k}(x_m) \subset \{|\tilde{f}_k| < 3/4\}$ which has finite measure by (A.4) and we are led to a contradiction. Hence we may now assume without loss of generality that $\{|\tilde{f}_k| < 1/2\}$ is bounded.

Step 3. Now we proceed as in [9]. Given $a \in \mathbb{R}^2$ with $|a| < 1/10$, we consider the projection $\pi_a : \mathbb{R}^2 \setminus \{a\} \to \mathbb{S}^1$ defined by $\pi_a(\xi) = a + \theta(\xi - a)$ where $\theta > 0$ is determined by $|a + \theta(\xi - a)| = 1$. Note that $\pi_a(\xi) = \xi$ whenever $\xi \in \mathbb{S}^1$ and

$$|\nabla \pi_a(\xi)| \leq \frac{C}{|\xi - a|} \quad \forall \xi \in \mathbb{R}^2 \setminus \{a\}$$

so that π_a is lipschitzian on $\{|\xi| \geq 1/2\}$ with a Lipschitz constant independent of a. Since $\tilde{f}_k$ is smooth, we can choose for every $k \in \mathbb{N}$, $a = a_k \in B_{1/10}$ to be a regular value of $\tilde{f}_k$ and then

$$\Sigma_k^a := \{x \in \mathbb{R}^2\,;\; \tilde{f}_k(x) = a\}$$

is a locally finite set. Since $\Sigma_k^a \subset \{|\tilde{f}_k| < 1/4\}$, we deduce from Step 2 that Σ_k^a is bounded and therefore finite. Hence the map

$$f_{a,k} := \pi_a \circ \tilde{f}_k : \mathbb{R}^2 \to \mathbb{S}^1$$

is smooth on $\mathbb{R}^2 \setminus \Sigma_k^a$ with Σ_k^a finite and in a neighborhood of each singular point $\sigma \in \Sigma_k^a$, we have

$$|\nabla f_{a,k}(x)| \leq \frac{C_k}{|x-\sigma|}\,.$$

Therefore $f_{a,k} \in W^{1,p}_{\rm loc}(\mathbb{R}^2, \mathbb{S}^1)$ for any $p < 2$ and every $k \in \mathbb{N}$.

Now we introduce a smooth function $\psi : [0, +\infty) \to [0, 1]$ such that $\psi(t) = 0$ if $t \leq 1/4$ and $\psi(t) = 1$ if $t \geq 1/2$. We write

$$f_{a,k} = \big(1 - \psi(|\tilde{f}_k|)\big) f_{a,k} + \psi(|\tilde{f}_k|) f_{a,k} =: f^s_{a,k} + f^r_{a,k}\,.$$

We claim that, for every $k \in \mathbb{N}$, we can find a regular value $a_k \in B_{1/10}$ of $\tilde{f}_k$ such that $|f_{a_k,k} - f|_{1/2} \to 0$ and $f_{a_k,k} \to f$ a.e. as $k \to +\infty$. Obviously, it suffices to find such a point a_k such that

$$|f^s_{a_k,k}|_{1/2} \to 0 \quad \text{as } k \to +\infty, \tag{A.6}$$

$$|f^r_{a_k,k} - f|_{1/2} \to 0 \quad \text{as } k \to +\infty. \tag{A.7}$$

Since we clearly have $f^s_{a_k,k} \to 0$ and $f^r_{a_k,k} \to f$ a.e. in $\mathbb{R}^2$ as $k \to +\infty$, the map $\hat{f}_k := f_{a_k,k}$ will satisfy $|\hat{f}_k - f|_{1/2} \to 0$ and $\hat{f}_k \to f$ a.e. as $k \to +\infty$.

Proof of (A.6). Since $|f^s_{a,k}| = |1 - \psi(|\tilde{f}_k|)| \leq \chi_{\{|\tilde{f}_k| < 1/2\}}$, we deduce from (A.4) that

$$\|f^s_{a,k}\|_{L^p(\mathbb{R}^2)} \leq o(\varepsilon_k^{1/p}) \quad \forall p < +\infty\,. \tag{A.8}$$

On the other hand, on $\mathbb{R}^2 \setminus \Sigma_k^a$ we have

$$|\nabla f^s_{a,k}| \leq C\frac{|\nabla \tilde{f}_k|}{|\tilde{f}_k - a|}(1 - \psi(|\tilde{f}_k|)) + |\psi'(|\tilde{f}_k|)|\,|\nabla \tilde{f}_k| \,\leq C|\nabla \tilde{f}_k|\chi_{\{|\tilde{f}_k| < 1/2\}}.$$

Now we use an averaging process due to Hardt, Kinderlehrer and Lin [28]. By Sard's theorem, the regular values of $\tilde{f}_k$ have full measure so that

$$\int_{B_{1/10}}\int_{\mathbb{R}^2} |\nabla f^s_{a,k}(x)|^p dx da \leq C_p \int_{\{|\tilde{f}_k| < 1/2\}} |\nabla \tilde{f}_k(x)|^p dx \quad \forall 1 < p < 2\,.$$

Next we fix $1 < p_0 < 2$. Then (A.2), (A.4) and Hölder inequality yield

$$\begin{aligned}\int_{B_{1/10}}\int_{\mathbb{R}^2} |\nabla f^s_{a,k}(x)|^{p_0} dx da &\leq C\|\nabla \tilde{f}_k\|^{p_0}_{L^2(\mathbb{R}^2)}\big|\{|\tilde{f}_k| < 1/2\}\big|^{1-\frac{p_0}{2}} \\ &\leq o(\varepsilon_k^{-\frac{p_0}{2}}\varepsilon_k^{1-\frac{p_0}{2}}) = o(\varepsilon_k^{1-p_0})\,.\end{aligned} \tag{A.9}$$

Observe that $f^s_{a,k}$ is compactly supported by Step 2. By the homogeneous Gagliardo-Nirenberg inequality (see e.g. [15]), we have

$$|f^s_{a,k}|^{2p_0}_{1/2} \leq C\|f^s_{a,k}\|^{p_0}_{L^{p_0'}(\mathbb{R}^2)}\|\nabla f^s_{a,k}\|^{p_0}_{L^{p_0}(\mathbb{R}^2)}\,. \tag{A.10}$$

Combining (A.8), (A.9) and (A.10), we obtain

$$\int_{B_{1/10}} |f^s_{a,k}|^{2p_0}_{1/2}\, da \le o(\varepsilon_k^{p_0-1}\varepsilon_k^{1-p_0}) = o(1).$$

Hence we may choose $a = a_k \in B_{1/10}$, a regular value of $\tilde{f}_k$ such that (A.6) holds.

Proof of (A.7). We consider $L_k : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L_k(\xi) = \pi_{a_k}(\xi)\psi(|\xi|)$. By the specific choice of the function ψ, L_k satisfies a uniform (with respect to k) Lipschitz condition. Observe that $L_k(f) = f$. Now we derive exactly as in the proof of statement (5.43) in [9] that $|f^r_{a_k,k} - f|_{1/2} = |L_k(\tilde{f}_k) - L_k(f)|_{1/2} \to 0$ as $n \to +\infty$ because of (A.3).

Step 4. According to Theorem 2.5, we have $f = g + f^\infty$ for some constant $f^\infty \in \mathbb{S}^1$. Then observe that $\tilde{f}_k = \tilde{g}_k + f^\infty$ with $\tilde{g}_k = \varrho_{\varepsilon_k} * g$. We claim that the set $\{|\tilde{g}_k| > 1/2\}$ is bounded. Indeed, one may argue as in Step 2 using that $|\nabla \tilde{g}_k| \le C_0\varepsilon_k^{-1}$ by (A.5) and $\|\tilde{g}_k\|_{L^4(\mathbb{R}^2)} < +\infty$ by Proposition 2.1. Hence there exists $r_k > 0$ such that

$$|\tilde{g}_k| < 1/2 \quad \text{in } \mathbb{R}^2 \setminus B_{r_k}. \tag{A.11}$$

In particular $|\tilde{f}_k| > 1/2$ in $\mathbb{R}^2 \setminus B_{r_k}$ so that $\hat{f}_k = L_k(\tilde{f}_k)$ in $\mathbb{R}^2 \setminus B_{r_k}$ and

$$|\hat{f}_k - f^\infty| = |L_k(\tilde{f}_k) - L_k(f^\infty)| \le C|\tilde{f}_k - f^\infty| = C|\tilde{g}_k| \quad \text{in } \mathbb{R}^2 \setminus B_{r_k}.$$

Therefore $\hat{f}_k - f^\infty \in L^4(\mathbb{R}^2)$ which clearly implies that $(\hat{f}_k)^\infty = f^\infty$ for every k. Then we write $\hat{f}_k = \hat{g}_k + f^\infty$. Introducing a cut-off function $\chi \in C^\infty_0(\mathbb{R},\mathbb{R})$, $0 \le \chi \le 1$, being supported in $[-2,2]$ with $\chi \equiv 1$ on $[-1,1]$, we consider for $r >> r_k$,

$$\hat{g}_{k,r}(x) = \chi\left(\frac{|x|}{r}\right)\hat{g}_k(x)\,.$$

We claim that $|\hat{g}_{k,r} - \hat{g}_k|_{1/2} \to 0$ as $r \to +\infty$. Setting $\hat{v}_k$ to be the (unique finite energy) harmonic extension of $\hat{g}_k$ to the half space and $\hat{v}_{k,r}(x) = \chi(r^{-1}|x|)\hat{v}_k(x)$, it is enough to show that $\|\nabla\big(\hat{v}_{k,r} - \hat{v}_k\big)\|_{L^2(\mathbb{R}^3_+)} \to 0$ as $r \to +\infty$. A straightforward computation yields

$$\int_{\mathbb{R}^3_+} |\nabla\big(\hat{v}_{k,r} - \hat{v}_k\big)|^2 dx \le 2\int_{\{|x|\ge r\}\cap\mathbb{R}^3_+} |\nabla \hat{v}_k|^2 dx + \frac{C}{r^2}\int_{\{r\le|x|\le 2r\}\cap\mathbb{R}^3_+} |\hat{v}_k|^2 dx =: I_r + II_r\,.$$

Clearly $I_r \to 0$ since $\nabla\hat{v}_k \in L^2(\mathbb{R}^3_+)$. Next we infer from Proposition 2.2, claim 2),

$$II_r \le C\int_{\{r\le|x|\le 2r\}\cap\mathbb{R}^3_+} \frac{|\hat{v}_k|^2}{|x|^2}\, dx \underset{r\to+\infty}{\longrightarrow} 0$$

and the claim is proved. Setting $\hat{f}_{k,r} = \hat{g}_{k,r} + f^\infty$, we conclude from Step 3 that

$$\lim_{k\to+\infty}\lim_{r\to+\infty} |\hat{f}_{k,r} - f|_{1/2} = 0 \tag{A.12}$$

Step 5. We observe that (A.11) implies that $\Sigma^{a_k}_k \subset B_{r_k}$ and $|\hat{g}_k| < 3/4$ in $\mathbb{R}^2 \setminus B_{r_k}$. In particular, $|\hat{f}_{k,r}| > 1/4$ in $B_{2r} \setminus B_r$ since $r >> r_k$ and $|\hat{f}_{k,r}| = 1$ elsewhere by construction. Therefore the map

$$f_{k,r} := \frac{\hat{f}_{k,r}}{|\hat{f}_{k,r}|}$$

is $\mathbb{S}^1$–valued, smooth outside the finite set $\Sigma^{a_k}_k$ and $f_{k,r} \equiv f^\infty$ outside B_{2r}. Moreover, arguing as in the proof of (A.7), (A.12) yields $\lim\limits_{k\to+\infty}\lim\limits_{r\to+\infty} |f_{k,r} - f|_{1/2} = 0$. Hence a suitable diagonal sequence $f_n := f_{k_n,r_n}$ satisfies the requirement. ■

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