Reviews in Mathematical Physics
Vol. 18, No. 2 (2006) 119–162

ENERGY EXPANSION AND VORTEX LOCATION FOR A TWO-DIMENSIONAL ROTATING BOSE–EINSTEIN CONDENSATE

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Received 26 June 2005
Revised 16 February 2006

We continue the analysis started in [14] on a model describing a two-dimensional rotating Bose–Einstein condensate. This model consists in minimizing under the unit mass constraint, a Gross–Pitaevskii energy defined in $\mathbb{R}^2$. In this contribution, we estimate the critical rotational speeds Ω_d for having exactly d vortices in the bulk of the condensate and we determine their topological charge and their precise location. Our approach relies on asymptotic energy expansion techniques developed by Serfaty [20–22] for the Ginzburg–Landau energy of superconductivity in the high κ limit.

Keywords: Bose–Einstein condensate; renormalized energy; vortices.

Mathematics Subject Classification 2000: 35A15, 35Q55

1. Introduction

Since its first experimental achievement in dilute alkali gases, the phenomenon of the Bose–Einstein condensation has given rise to a very active area of research in condensed matter physics. A Bose–Einstein condensate (BEC) is a quantum object in which every atom is in the lowest quantum state, so that it can be described by a single wave function. One of the most interesting feature of these systems is their superfluid behavior (see [10]): above some critical velocity, a BEC rotates through the existence of vortices, i.e. zeroes of the wave function around which there is a circulation of phase. When the angular speed gets larger, the number of vortices increases and they arrange themselves in a regular pattern around the center of the condensate. This has been observed experimentally by the ENS group [16, 17] and by the MIT group [1].

We consider here a two-dimensional model describing a condensate placed in a trap that strongly confines the atoms in the direction of the rotation axis (see [10, 11]). In the non-dimensionalized form (see [2, 14]), the wave function minimizes the Gross–Pitaevskii (GP) energy

$$F_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}[(|u|^2 - a(x))^2 - (a^-(x))^2] - \Omega x^\perp \cdot (iu, \nabla u) \right\} dx \tag{1.1}$$

under the constraint

$$\int_{\mathbb{R}^2} |u|^2 = 1, \tag{1.2}$$

where $\varepsilon > 0$ is small and describes the ratio of two characteristic lengths and $\Omega = \Omega(\varepsilon) \geq 0$ is the angular velocity. The function $a(x)$ in (1.1) comes from the existence of a potential trapping the atoms, and is normalized such that $\int_{\mathbb{R}^2} a^+(x) = 1$. We will restrict our attention to the specific case of a harmonic trapping, that is $a(x) = a_0 - x_1^2 - \Lambda^2 x_2^2$ with $a_0 = \sqrt{2\Lambda/\pi}$ for some constant $\Lambda \in (0, 1]$, which corresponds to actual experiments (see [16, 17]).

Our goal is to compute an asymptotic expansion of the energy $F_\varepsilon(u_\varepsilon)$ and to determine the number and the location of vortices according to the value of the angular speed $\Omega(\varepsilon)$ in the limit $\varepsilon \to 0$. More precisely, we want to estimate the critical velocity Ω_d for which the dth vortex becomes energetically favorable and to derive a reduced energy governing the location of the vortices (the so-called "*renormalized energy*" by analogy with [8, 20, 21]).

We have started in [14] the analysis of minimizers u_ε of the functional F_ε under the constraint (1.2) and we have already determined the critical rotational speed $\Omega_1 = \frac{\sqrt{\pi}(1+\Lambda^2)}{\sqrt{2\Lambda}}|\ln \varepsilon|$ of nucleation of the first vortex inside the domain

$$\mathcal{D} = \{x \in \mathbb{R}^2 \, : \, a(x) > 0\}.$$

In the physical context, the set $\mathcal{D}$ represents the region occupied by the condensate since in the limit $\varepsilon \to 0$, the minimization of F_ε forces $|u_\varepsilon|^2$ to be close to the function $a^+(x)$($F_\varepsilon(u_\varepsilon)$ remaining small in front of $1/\varepsilon^2$). We proved that for subcritical velocities $\Omega \leq \Omega_1 - \delta \ln|\ln \varepsilon|$ with $-\delta < \omega_1^\star < 0$ for some constant $\omega_1^\star$, there is no vortices in the region $\mathcal{D}$ and u_ε behaves as the *vortex-free* profile $\tilde{\eta}_\varepsilon e^{i\Omega S}$ where the phase function $S : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$S(x) = \frac{\Lambda^2 - 1}{\Lambda^2 + 1} x_1 x_2 \tag{1.3}$$

and $\tilde{\eta}_\varepsilon$ is the (unique) positive solution of the minimization problem

$$\text{Min}\big\{E_\varepsilon(u) \, : \, u \in \mathcal{H}, \, \|u\|_{L^2(\mathbb{R}^2)} = 1\big\} \tag{1.4}$$

with

$$E_\varepsilon(u) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}[(|u|^2 - a(x))^2 - (a^-(x))^2] \quad \text{and}$$

$$\mathcal{H} = \left\{ u \in H^1(\mathbb{R}^2, \mathbb{C}) \, : \, \int_{\mathbb{R}^2} |x|^2|u|^2 < \infty \right\}.$$

In this contribution which constitutes the sequel of [14], we push forward the study of minimizers u_ε. First, we prove the following estimate on the critical speed Ω_d for any integer $d \geq 1$ in the asymptotic $\varepsilon \to 0$,

$$\Omega_d = \frac{1+\Lambda^2}{a_0}(|\ln\varepsilon| + (d-1)\ln|\ln\varepsilon|) = \frac{\sqrt{\pi}(1+\Lambda^2)}{\sqrt{2\Lambda}}(|\ln\varepsilon| + (d-1)\ln|\ln\varepsilon|)\,.$$

Then, we show that for velocities ranged between Ω_d and Ω_{d+1}, any minimizer has exactly d vortices of degree $+1$ inside $\mathcal{D}$. Establishing an asymptotic expansion of $F_\varepsilon(u_\varepsilon)$ as $\varepsilon \to 0$, we derive the distribution of vortices within $\mathcal{D}$ as a minimizing configuration of the reduced energy given by (1.5) below. We also improve the result stated in [14] for the non-existence of vortices in the subcritical case by showing that the best constant is $\omega_1^\star = 0$, that is subcritical velocities go up to $\Omega_1 - \delta\ln|\ln\varepsilon|$ for any $\delta > 0$.

Our main theorem can be stated as follows:

Theorem 1.1. *Let u_ε be any minimizer of F_ε in $\mathcal{H}$ under the constraint* (1.2) *and let $0 < \delta \ll 1$ be any small constant.*

(i) *If $\Omega \leq \Omega_1 - \delta\ln|\ln\varepsilon|$, then for any $R_0 < \sqrt{a_0}$, there exists $\varepsilon_0 = \varepsilon_0(R_0,\delta) > 0$ such that for any $\varepsilon < \varepsilon_0$, u_ε is vortex free in $B_{R_0}^\Lambda = \{x \in \mathbb{R}^2 \, : \, |x|_\Lambda^2 = x_1^2 + \Lambda^2 x_2^2 < R_0^2\}$, i.e. u_ε does not vanish in $B_{R_0}^\Lambda$. In addition,*

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + o(1).$$

(ii) *If $\Omega_d + \delta\ln|\ln\varepsilon| \leq \Omega \leq \Omega_{d+1} - \delta\ln|\ln\varepsilon|$ for some integer $d \geq 1$, then for any $R_0 < \sqrt{a_0}$, there exists $\varepsilon_1 = \varepsilon_1(R_0,d,\delta) > 0$ such that for any $\varepsilon < \varepsilon_1$, u_ε has exactly d vortices $x_1^\varepsilon,\dots,x_d^\varepsilon$ of degree one in $B_{R_0}^\Lambda$. Moreover,*

$$|x_j^\varepsilon| \leq C\,\Omega^{-1/2} \quad \textit{for any } j = 1,\dots,d, \quad \textit{and}$$
$$|x_i^\varepsilon - x_j^\varepsilon| \geq C\,\Omega^{-1/2} \quad \textit{for any } i \neq j,$$

where $C > 0$ denotes a constant independent of ε. Setting $\tilde{x}_j^\varepsilon = \sqrt{\Omega}\,x_j^\varepsilon$, the configuration $(\tilde{x}_1^\varepsilon,\dots,\tilde{x}_d^\varepsilon)$ tends to minimize, as $\varepsilon \to 0$, the renormalized energy

$$w(b_1,\dots,b_d) = -\pi a_0 \sum_{i\neq j}\ln|b_i - b_j| + \frac{\pi a_0}{1+\Lambda^2}\sum_{j=1}^{d}|b_j|_\Lambda^2. \qquad (1.5)$$

In addition,

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\tilde\eta_\varepsilon e^{i\Omega S}) - \frac{\pi a_0^2 d}{1+\Lambda^2}(\Omega - \Omega_1) + \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| + \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{d,\Lambda} + o(1), \tag{1.6}$$

where $Q_{d,\Lambda}$ is a constant depending only on d and Λ.

These results are in agreement with the study made by Castin and Dum [11] who have looked for minimizers in a reduced class of functions. More precisely, we find the same critical angular velocities Ω_d as well as a distribution of vortices around the origin at a scale $\Omega^{-1/2}$. The minimizing configurations for the renormalized energy $w(\cdot)$ have been studied in the radial case $\Lambda = 1$ by Gueron and Shafrir in [12]. They prove that for $d \leq 6$, regular polygons centered at the origin and *stars* are local minimizers. For larger d, they numerically found minimizers with a shape of concentric polygons and then, triangular lattices as d increases. These figures are exactly the ones observed in physical experiments (see [16, 17]).

Our approach, suggested in [2] by Aftalion and Du, strongly relies on techniques developed by Serfaty [20–22] for the Ginzburg–Landau (GL) energy of superconductivity in the high κ limit. We point out that Serfaty has already applied the method to a simplified GP energy (the study is made in a ball instead of $\mathbb{R}^2$ with $a(x) \equiv 1$ and the minimization is performed without mass constraint) and has obtained in [23] a result analogue to Theorem 1.1 which shows that the simple model captures the main features of the full model concerning vortices. We emphasize once more that we treat here the exact physical model without any simplifying assumptions. The outline of our proof follows Serfaty's method but many technical difficulties arise from the specificities of the problem such as the unit mass constraint or the degenerate behavior of the function $a(x)$ near the boundary of $\mathcal{D}$. As we shall see, a very delicate analysis is required so that we prefer sometimes to write all the details even if some proofs follow closely to other authors. More precisely, we also make use of the following results on the GL functional [3–5, 9, 15, 18, 19, 24], starting from the pioneering work of Bethuel, Brezis and Hélein [8]. We finally refer to our first part [14] for additional references on mathematical studies of vortices in BECs.

For the convenience of the reader, we recall now some results already established in [14]. First, we have proved the existence and smoothness of any minimizer u_ε of F_ε under the constraint (1.2) in the regime

$$\Omega \leq \frac{1+\Lambda^2}{a_0}(|\ln\varepsilon| + \omega_1 \ln|\ln\varepsilon|) \tag{1.7}$$

for a constant $\omega_1 \in \mathbb{R}$, as well as some qualitative properties: $E_\varepsilon(u_\varepsilon) \leq C|\ln\varepsilon|^2$, $|u_\varepsilon| \lesssim \sqrt{a^+}$ in any compact $K \subset \mathcal{D}$ and $|u_\varepsilon|$ decreases exponentially fast to 0 outside $\mathcal{D}$. We have also showed the existence and uniqueness of the positive minimizer $\tilde\eta_\varepsilon$ of E_ε under the mass constraint (1.2) for every $\varepsilon > 0$. Concerning the Lagrange multiplier $k_\varepsilon \in \mathbb{R}$ associated to $\tilde{\eta}_\varepsilon$ and the qualitative properties of $\tilde{\eta}_\varepsilon$, we have obtained:

$$|k_\varepsilon| \le C|\ln \varepsilon|\,, \tag{1.8}$$

$E_\varepsilon(\tilde{\eta}_\varepsilon) \le C|\ln \varepsilon|$ for ε small and $\tilde{\eta}_\varepsilon \to \sqrt{a^+}$ in $L^\infty(\mathbb{R}^2) \cap C^1_{\text{loc}}(\mathcal{D})$ as $\varepsilon \to 0$. Using a splitting technique introduced by Lassoued and Mironescu [15], we were able to decouple into two independent parts the energy $F_\varepsilon(u)$ for any $u \in \mathcal{H}$. The first part corresponds to the energy of the vortex-free profile $\tilde{\eta}_\varepsilon e^{i\Omega S}$ and the second part to a reduced energy of $v = u/(\tilde{\eta}_\varepsilon e^{i\Omega S})$, i.e.

$$F_\varepsilon(u) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(v) + \tilde{\mathcal{T}}_\varepsilon(v), \tag{1.9}$$

where the functionals $\tilde{\mathcal{F}}_\varepsilon$ and $\tilde{\mathcal{T}}_\varepsilon$ are defined by

$$\tilde{\mathcal{F}}_\varepsilon(v) = \tilde{\mathcal{E}}_\varepsilon(v) + \tilde{\mathcal{R}}_\varepsilon(v)\,, \tag{1.10}$$

$$\begin{aligned} \tilde{\mathcal{E}}_\varepsilon(v) &= \int_{\mathbb{R}^2} \frac{\tilde{\eta}_\varepsilon^2}{2}|\nabla v|^2 + \frac{\tilde{\eta}_\varepsilon^4}{4\varepsilon^2}(|v|^2-1)^2, \\ \tilde{\mathcal{R}}_\varepsilon(v) &= \frac{\Omega}{1+\Lambda^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 \nabla^\perp a \cdot (iv, \nabla v), \end{aligned} \tag{1.11}$$

$$\tilde{\mathcal{T}}_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^2} \big(\Omega^2|\nabla S|^2 - 2\Omega^2 x^\perp \cdot \nabla S + k_\varepsilon\big)\tilde{\eta}_\varepsilon^2(|v|^2-1). \tag{1.12}$$

Since the function $\tilde{\eta}_\varepsilon$ does not vanish, the vortex structure of any minimizer u_ε can be studied via the map

$$v_\varepsilon = u_\varepsilon/(\tilde{\eta}_\varepsilon e^{i\Omega S}),$$

applying the Ginzburg–Landau techniques to the weighted energy $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon)$. It is intuitively clear that difficulties will arise in the region where $\tilde{\eta}_\varepsilon$ is small and we will require the following properties of v_ε inherited from u_ε and $\tilde{\eta}_\varepsilon$: $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \le C|\ln \varepsilon|^2$, $\big|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)\big| \le o(1)$, $|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon)| \le C|\ln \varepsilon|^2$, $|\nabla v_\varepsilon| \le C_K \varepsilon^{-1}$ and $|v_\varepsilon| \lesssim 1$ in any compact $K \subset \mathcal{D}$. In the sequel, it will be more convenient to replace in the different functionals the function $\tilde{\eta}_\varepsilon^2$ by its limit $a^+(x)$. We denote by $\mathcal{F}_\varepsilon$, $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ the corresponding functionals (see notations below). In the regime (1.7), we have computed in [14] some fundamental bounds for the energy of v_ε in a domain slightly smaller than $\mathcal{D}$:

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le o(1), \tag{1.13}$$

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le C_{\omega_1}|\ln \varepsilon|, \tag{1.14}$$

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \backslash \{|x|_\Lambda < 2|\ln \varepsilon|^{-1/6}\}) \le C_{\omega_1} \ln|\ln \varepsilon|, \tag{1.15}$$

where

$$\mathcal{D}_\varepsilon = \{x \in \mathcal{D}\,:\, a(x) > \nu_\varepsilon |\ln \varepsilon|^{-3/2}\} \tag{1.16}$$

and ν_ε is a chosen parameter in the interval $(1,2)$ (see Proposition 2.13). These estimates represent the starting point of our analysis here.

The plan of the paper is as follows. In Sec. 2, we prove that the subset of $\mathcal{D}$ where $|v_\varepsilon|$ is smaller than $1/2$ can be covered by a family of disjoint discs such that each radius vanishes as $\varepsilon \to 0$, the cardinal of this family is uniformly bounded with respect to ε and v_ε has a non-vanishing degree around each disc of the family. We will call such a collection of discs a *fine structure of vortices* and a *vortex* one of these discs (identified with their center). In Sec. 3, we establish various lower energy estimates namely inside a vortex and away from the vortices. In Sec. 4, we prove Theorem 1.1 matching the lower energy estimates with upper estimates coming from the construction of trial functions. These constructions are presented in Sec. 5 which can be read independently from the rest of the paper. Finally, we prove in the Appendix, an auxiliary result that we shall use in the proof of Theorem 1.1.

Notations. Throughout the paper, we denote by C a positive constant independent of ε and we use the subscript to point out a possible dependence on the argument. For $x = (x_1, x_2) \in \mathbb{R}^2$, we write

$$|x|_\Lambda = \sqrt{x_1^2 + \Lambda^2 x_2^2} \quad \text{and} \quad B_R^\Lambda = \{x \in \mathbb{R}^2,\ |x|_\Lambda < R\}$$

and for $\mathcal{A} \subset \mathbb{R}^2$,

$$\begin{aligned}
\tilde{\mathcal{E}}_\varepsilon(v, \mathcal{A}) &= \int_{\mathcal{A}} \frac{1}{2}\, \tilde{\eta}_\varepsilon^2 |\nabla v|^2 + \frac{\tilde{\eta}_\varepsilon^4}{4\varepsilon^2}(1 - |v|^2)^2 \,,\\
\mathcal{E}_\varepsilon(v, \mathcal{A}) &= \int_{\mathcal{A}} \frac{1}{2}\, a|\nabla v|^2 + \frac{a^2}{4\varepsilon^2}(1 - |v|^2)^2,\\
\tilde{\mathcal{R}}_\varepsilon(v, \mathcal{A}) &= \frac{\Omega}{1+\Lambda^2} \int_{\mathcal{A}} \tilde{\eta}_\varepsilon^2 \nabla^\perp a \cdot (iv, \nabla v)\,,\\
\mathcal{R}_\varepsilon(v, \mathcal{A}) &= \frac{\Omega}{1+\Lambda^2} \int_{\mathcal{A}} a\nabla^\perp a \cdot (iv, \nabla v),\\
\tilde{\mathcal{F}}_\varepsilon(v, \mathcal{A}) &= \tilde{\mathcal{E}}_\varepsilon(v, \mathcal{A}) + \tilde{\mathcal{R}}_\varepsilon(v, \mathcal{A})\,,\\
\mathcal{F}_\varepsilon(v, \mathcal{A}) &= \mathcal{E}_\varepsilon(v, \mathcal{A}) + \mathcal{R}_\varepsilon(v, \mathcal{A}).
\end{aligned} \tag{1.17}$$

We do not write the dependence on $\mathcal{A}$ when $\mathcal{A} = \mathbb{R}^2$.

2. Fine Structure of Vortices

The main goal of this section is to construct a fine structure of vortices away from the boundary of $\mathcal{D}$. The analysis here follows the ideas in [8, 9]. The main difficulty in our situation is due to the presence in the energy of the weight function $a(x)$ which vanishes on $\partial\mathcal{D}$ and it does not allow us to construct the structure up to the boundary because of the resulting degeneracy in the energy estimates. Throughout this paper, we assume that Ω satisfies (1.7), so that (1.13)–(1.15) hold. We will prove the following results for the map $v_\varepsilon = u_\varepsilon/(\tilde{\eta}_\varepsilon e^{i\Omega S})$:

Theorem 2.1. (1) *For any* $R \in \left(\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\right)$, *there exists* $\varepsilon_R > 0$ *such that for any* $\varepsilon < \varepsilon_R$,

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } B_R^\Lambda \backslash B^\Lambda_{\frac{\sqrt{a_0}}{2}}.$$

(2) *There exist some constants* $N \in \mathbb{N}$, $\lambda_0 > 0$ *and* $\varepsilon_0 > 0$ (*which only depend on* ω_1) *such that for any* $\varepsilon < \varepsilon_0$, *one can find a finite collection of points* $\{x_j^\varepsilon\}_{j\in J_\varepsilon} \subset B^\Lambda_{\frac{\sqrt{a_0}}{4}}$ *such that* $\mathrm{Card}(J_\varepsilon) \leq N$ *and*

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } \bar{B}^\Lambda_{\frac{\sqrt{a_0}}{2}} \Bigg\backslash \left(\bigcup_{j\in J_\varepsilon} B(x_j^\varepsilon, \lambda_0\varepsilon)\right).$$

Remark 2.2. The statement of Theorem 2.1 also holds if the radius $\frac{\sqrt{a_0}}{2}$ is replaced by an arbitrary $r \in (0, R)$ but then the constants in Theorem 2.1 depend on r. For the sake of simplicity, we prefer to fix $r = \frac{\sqrt{a_0}}{2}$.

In the next proposition, we replace as in [20] the discs $\{B(x_j^\varepsilon, \lambda_0\varepsilon)\}_{j\in J_\varepsilon}$ obtained in Theorem 2.1 by slightly larger discs $B(x_j^\varepsilon, \rho)$ (deleting some of the points x_j^ε, if necessary), in order to get a precise information on the behavior of v_ε on $\partial B(x_j^\varepsilon, \rho)$. The resulting family of discs will represent the vortices of the map v_ε (and hence, the vortices of u_ε also).

Proposition 2.3. *Let* $0 < \beta < \mu < 1$ *be given constants such that* $\bar{\mu} := \mu^{N+1} > \beta$ *and let* $\{x_j^\varepsilon\}_{j\in J_\varepsilon}$ *be the collection of points given by* (2) *in Theorem* 2.1. *There exists* $0 < \varepsilon_1 < \varepsilon_0$ *such that for any* $\varepsilon < \varepsilon_1$, *we can find* $\tilde{J}_\varepsilon \subset J_\varepsilon$ *and* $\rho > 0$ *verifying*

(i) $\lambda_0\varepsilon \leq \varepsilon^\mu \leq \rho \leq \varepsilon^{\bar{\mu}} < \varepsilon^\beta$,

(ii) $|v_\varepsilon| \geq \frac{1}{2}$ *in* $\bar{B}^\Lambda_{\frac{\sqrt{a_0}}{2}} \backslash \cup_{j\in\tilde{J}_\varepsilon} B(x_j^\varepsilon, \rho)$,

(iii) $|v_\varepsilon| \geq 1 - \frac{2}{|\ln\varepsilon|^2}$ *on* $\partial B(x_j^\varepsilon, \rho)$ *for every* $j \in \tilde{J}_\varepsilon$,

(iv) $\displaystyle\int_{\partial B(x_j^\varepsilon,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \leq \frac{C(\beta,\mu)}{\rho}$ *for every* $j \in \tilde{J}_\varepsilon$,

(v) $|x_i^\varepsilon - x_j^\varepsilon| \geq 8\rho$ *for every* $i, j \in \tilde{J}_\varepsilon$ *with* $i \neq j$.

Moreover, for each $j \in \tilde{J}_\varepsilon$, *we have*

$$D_j := \deg\left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(x_j^\varepsilon, \rho)\right) \neq 0 \quad \text{and} \quad |D_j| \leq C \tag{2.1}$$

for a constant C *independent of* ε.

Remark 2.4. We point out that for every $j \in \tilde{J}_\varepsilon$, the disc $B(x_j^\varepsilon, \rho)$ carries at least one zero of v_ε since the degree $D_j \neq 0$.

2.1. *Some local estimates*

We start with a fundamental lemma. It strongly relies on Pohozaev's identity and it will play a similar role as in [8, Theorem III.2]. In our situation, we only derive local estimates as in [3, 9, 24]. Some of the arguments used in the proof are taken from [3, 9].

Lemma 2.5. *For any* $0 < R < \sqrt{a_0}$ *and* $\frac{2}{3} < \alpha < 1$, *there exists a positive constant* $C_{R,\alpha}$ *such that*

$$\frac{1}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(1-|v_\varepsilon|^2)^2 \le C_{R,\alpha} \quad \text{for any } x_0 \in B_R^\Lambda.$$

Proof. Step 1. Set $\tilde{u}_\varepsilon = u_\varepsilon e^{-i\Omega S}$. We claim that

$$E_\varepsilon(\tilde{u}_\varepsilon, \mathcal{D}_\varepsilon) \le C|\ln\varepsilon|, \tag{2.2}$$

where $\mathcal{D}_\varepsilon$ is defined in (1.16). Indeed, since $\tilde{u}_\varepsilon = \tilde{\eta}_\varepsilon v_\varepsilon$, we get that

$$|\nabla\tilde{u}_\varepsilon|^2 \le C\big(\tilde{\eta}_\varepsilon^2|\nabla v_\varepsilon|^2 + |v_\varepsilon|^2|\nabla\tilde{\eta}_\varepsilon|^2\big).$$

By [14, Propositions 2.2 and 3.3], $|v_\varepsilon| \le C$ in $\mathcal{D}_\varepsilon$, $\tilde{\eta}_\varepsilon^2 \le Ca$ in $\mathcal{D}_\varepsilon$ and $E_\varepsilon(\tilde{\eta}_\varepsilon) \le C|\ln\varepsilon|$ and consequently,

$$\int_{\mathcal{D}_\varepsilon}|\nabla\tilde{u}_\varepsilon|^2 \le C\bigg(\int_{\mathcal{D}_\varepsilon} a(x)|\nabla v_\varepsilon|^2 + \int_{\mathcal{D}_\varepsilon}|\nabla\tilde{\eta}_\varepsilon|^2\bigg) \le C|\ln\varepsilon|$$

by (1.14). On the other hand, we also have

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\mathcal{D}_\varepsilon}(a(x)-|\tilde{u}_\varepsilon|^2)^2 &\le \frac{C}{\varepsilon^2}\int_{\mathcal{D}_\varepsilon}\big[(a(x)-\tilde{\eta}_\varepsilon^2)^2 + \tilde{\eta}_\varepsilon^4(1-|v_\varepsilon|^2)^2\big]\\ &\le \frac{C}{\varepsilon^2}\bigg(\int_{\mathcal{D}_\varepsilon}(a(x)-\tilde{\eta}_\varepsilon^2)^2 + \int_{\mathcal{D}_\varepsilon}a^2(x)(1-|v_\varepsilon|^2)^2\bigg) \le C|\ln\varepsilon|\end{aligned}$$

and therefore (2.2) follows.

Step 2. We are going to show that one can find a constant $C_{R,\alpha} > 0$, independent of ε, such that for any $x_0 \in B_R^\Lambda$, there is some $r_0 \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3})$ satisfying

$$E_\varepsilon(\tilde{u}_\varepsilon, \partial B(x_0, r_0)) \le \frac{C_{R,\alpha}}{r_0}.$$

We proceed by contradiction. Assume that for all $M > 0$, there is $x_M \in B_R^\Lambda$ such that

$$E_\varepsilon(\tilde{u}_\varepsilon, \partial B(x_M, r)) \ge \frac{M}{r}, \quad \text{for any } r \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3}). \tag{2.3}$$

Obviously, for ε small, $B(x_M, \varepsilon^{\alpha/2+1/3}) \subset \mathcal{D}_\varepsilon$. Integrating (2.3) for $r \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3})$, we derive that

$$E_\varepsilon(\tilde{u}_\varepsilon, \mathcal{D}_\varepsilon) \ge M\int_{\varepsilon^\alpha}^{\varepsilon^{\alpha/2+1/3}}\frac{dr}{r} = M(\alpha/2 - 1/3)|\ln\varepsilon|$$

which contradicts Step 1 for M large enough.

Step 3. Fix $x_0 \in B_R^\Lambda$ and let $r_0 \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3})$ be given by Step 2. We recall that any minimizer u_ε of F_ε in $\{u \in \mathcal{H},\ \|u\|_{L^2(\mathbb{R}^2)} = 1\}$ satisfies

$$-\Delta u_\varepsilon + 2i\Omega x^\perp \cdot \nabla u_\varepsilon = \frac{1}{\varepsilon^2}(a(x) - |u_\varepsilon|^2)u_\varepsilon + \ell_\varepsilon u_\varepsilon \quad \text{in } \mathbb{R}^2,$$

where ℓ_ε denotes the Lagrange multiplier. Therefore, we have

$$\begin{aligned} -\Delta \tilde{u}_\varepsilon = {} & \frac{1}{\varepsilon^2}(a(x_0) - |\tilde{u}_\varepsilon|^2)\tilde{u}_\varepsilon + \frac{1}{\varepsilon^2}(a(x) - a(x_0))\tilde{u}_\varepsilon + 2i\Omega(\nabla S - x^\perp) \cdot \nabla \tilde{u}_\varepsilon \\ & + (\ell_\varepsilon + 2\Omega^2 x^\perp \cdot \nabla S - \Omega^2 |\nabla S|^2)\tilde{u}_\varepsilon \quad \text{in } B(x_0, r_0). \end{aligned} \tag{2.4}$$

As in the proof of the Pohozaev identity, we multiply (2.4) by $(x - x_0) \cdot \nabla \tilde{u}_\varepsilon$ and we integrate by parts in $B(x_0, r_0)$. We have

$$\int_{B(x_0,r_0)} -\Delta \tilde{u}_\varepsilon \cdot [(x - x_0) \cdot \nabla \tilde{u}_\varepsilon] = \frac{r_0}{2} \int_{\partial B(x_0,r_0)} |\nabla \tilde{u}_\varepsilon|^2 - r_0 \int_{\partial B(x_0,r_0)} \left| \frac{\partial \tilde{u}_\varepsilon}{\partial \nu} \right|^2 \tag{2.5}$$

and

$$\begin{aligned} & \frac{1}{\varepsilon^2} \int_{B(x_0,r_0)} (a(x_0) - |\tilde{u}_\varepsilon|^2)\tilde{u}_\varepsilon \cdot [(x - x_0) \cdot \nabla \tilde{u}_\varepsilon] \\ & \quad = \frac{1}{2\varepsilon^2} \int_{B(x_0,r_0)} (a(x_0) - |\tilde{u}_\varepsilon|^2)^2 - \frac{r_0}{4\varepsilon^2} \int_{\partial B(x_0,r_0)} (a(x_0) - |\tilde{u}_\varepsilon|^2)^2 \end{aligned} \tag{2.6}$$

(where ν is the outer normal vector to $\partial B(x_0, r_0)$). From (2.4)–(2.6), we derive that

$$\begin{aligned} & \frac{1}{\varepsilon^2} \int_{B(x_0,r_0)} (a(x_0) - |\tilde{u}_\varepsilon|^2)^2 \\ & \quad \le C\Bigg(r_0 \int_{\partial B(x_0,r_0)} |\nabla \tilde{u}_\varepsilon|^2 + r_0 \varepsilon^{-2} \int_{\partial B(x_0,r_0)} (a(x_0) - |\tilde{u}_\varepsilon|^2)^2 \\ & \qquad + r_0 \varepsilon^{-2} \int_{B(x_0,r_0)} |a(x) - a(x_0)||\tilde{u}_\varepsilon||\nabla \tilde{u}_\varepsilon| + \Omega r_0 \int_{B(x_0,r_0)} |\nabla \tilde{u}_\varepsilon|^2 \\ & \qquad + (\Omega^2 + |\ell_\varepsilon|) r_0 \int_{B(x_0,r_0)} |\tilde{u}_\varepsilon||\nabla \tilde{u}_\varepsilon| \Bigg). \end{aligned}$$

Then, we estimate each integral term in the right-hand side of the previous inequality. By [14, Proposition 3.2], we have $|\ell_\varepsilon| \le C\varepsilon^{-1}|\ln \varepsilon|$ and $|\tilde{u}_\varepsilon| \le C$ in $\mathbb{R}^2$. According to (2.2), we obtain

$$\begin{aligned} \varepsilon^{-2} \int_{\partial B(x_0,r_0)} (a(x_0) - |\tilde{u}_\varepsilon|^2)^2 & \le C\varepsilon^{-2} \int_{\partial B(x_0,r_0)} \left[(a(x_0) - a(x))^2 + (a(x) - |\tilde{u}_\varepsilon|^2)^2 \right] \\ & \le C\varepsilon^{-2} \int_{\partial B(x_0,r_0)} (a(x) - |\tilde{u}_\varepsilon|^2)^2 + C_R \varepsilon^{\frac{3}{2}\alpha - 1}, \end{aligned}$$

and

$$\Omega r_0 \int_{B(x_0,r_0)} |\nabla \tilde{u}_\varepsilon|^2 \le 2\Omega r_0 E_\varepsilon(\tilde{u}_\varepsilon, \mathcal{D}_\varepsilon) \le C_R \varepsilon^{\alpha/2+1/3} |\ln \varepsilon|^2,$$

and

$$r_0\varepsilon^{-2}\int_{B(x_0,r_0)}|a(x)-a(x_0)||\tilde{u}_\varepsilon||\nabla\tilde{u}_\varepsilon| \le C_R r_0^2\varepsilon^{-2}\int_{B(x_0,r_0)}|\nabla\tilde{u}_\varepsilon|$$
$$\le C_R r_0^3\varepsilon^{-2}[E_\varepsilon(\tilde{u}_\varepsilon,\mathcal{D}_\varepsilon)]^{1/2}$$
$$\le C_R\varepsilon^{\frac{3}{2}\alpha-1}|\ln\varepsilon|^{1/2},$$

and

$$(\Omega^2+|\ell_\varepsilon|)r_0\int_{B(x_0,r_0)}|\tilde{u}_\varepsilon||\nabla\tilde{u}_\varepsilon| \le C_R\varepsilon^{-1}|\ln\varepsilon|r_0^2\,[E_\varepsilon(\tilde{u}_\varepsilon,\mathcal{D}_\varepsilon)]^{1/2}$$
$$\le C_R\,\varepsilon^{\alpha-\frac{1}{3}}|\ln\varepsilon|^{3/2}$$

(here we use that $|a(x)-a(x_0)|\le C_R\,r_0$ for any $x\in B(x_0,r_0)$). We finally get that

$$\frac{1}{\varepsilon^2}\int_{B(x_0,r_0)}(a(x_0)-|\tilde{u}_\varepsilon|^2)^2 \le C_{R,\alpha}\big(1+r_0E_\varepsilon\big(\tilde{u}_\varepsilon,\partial B(x_0,r_0)\big)\big)$$

for some constant $C_{R,\alpha}$ independent of ε. By Step 2, we conclude that

$$\frac{1}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(a(x_0)-|\tilde{u}_\varepsilon|^2)^2\le C_{R,\alpha}. \tag{2.7}$$

Since $\|\tilde{\eta}_\varepsilon-\sqrt{a}\|_{C^1(B_R^\Lambda)}\le C_R\varepsilon^2|\ln\varepsilon|$ by [14, Proposition 2.2], we have

$$\frac{1}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(1-|v_\varepsilon|^2)^2 \le \frac{C_R}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(\tilde{\eta}_\varepsilon^2-|\tilde{u}_\varepsilon|^2)^2$$
$$\le \frac{C_R}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(a(x)-|\tilde{u}_\varepsilon|^2)^2+o(1)$$
$$\le \frac{C_R}{\varepsilon^2}\int_{B(x_0,\varepsilon^\alpha)}(a(x_0)-|\tilde{u}_\varepsilon|^2)^2+o(1)\le C_{R,\alpha}$$

and we conclude with (2.7). □

The next result will allow us to define the notion of a bad disc as in [8].

Proposition 2.6. *For any* $0<R<\sqrt{a_0}$, *there exist two positive constants* λ_R *and* μ_R *such that if*

$$\frac{1}{\varepsilon^2}\int_{B(x_0,2l)}(1-|v_\varepsilon|^2)^2\le\mu_R \quad \textit{with } x_0\in B_R^\Lambda,\quad \frac{l}{\varepsilon}\ge\lambda_R \ \textit{ and } l\le\frac{\sqrt{a_0}-R}{2},$$

then $|v_\varepsilon|\ge 1/2$ *in* $B(x_0,l)$.

Proof. In [14, Proposition 3.3], we proved the existence of a constant $C_R>0$ independent of ε such that

$$|\nabla v_\varepsilon|\le\frac{C_R}{\varepsilon}\quad \text{in } B^\Lambda_{\frac{\sqrt{a_0}+R}{2}}.$$

Then, the result follows as in [8, Theorem III.3]. □

Definition 2.7. For $0 < R < \sqrt{a_0}$ and $x \in B_R^\Lambda$, we say that $B(x, \lambda_R\varepsilon)$ is a **bad disc** if

$$\frac{1}{\varepsilon^2}\int_{B(x,2\lambda_R\varepsilon)} (1-|v_\varepsilon|^2)^2 \geq \mu_R.$$

Now we can give a local version of Theorem 2.1. We will see that Lemma 2.5 plays a crucial role in the proof.

Proposition 2.8. *For any $0 < R < \sqrt{a_0}$ and $\frac{2}{3} < \alpha < 1$, there exist positive constants $N_{R,\alpha}$ and $\varepsilon_{R,\alpha}$ such that for every $\varepsilon < \varepsilon_{R,\alpha}$ and $x_0 \in B_R^\Lambda$, one can find $x_1, \ldots, x_{N_\varepsilon} \in B(x_0, \varepsilon^\alpha)$ with $N_\varepsilon \leq N_{R,\alpha}$ verifying*

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } B(x_0,\varepsilon^\alpha) \Big\backslash \left(\bigcup_{k=1}^{N_\varepsilon} B(x_k,\lambda_R\varepsilon)\right).$$

Proof. We follow the ideas in [8, Chapter IV]. Consider a family of discs $\{B(x_i,\lambda_R\varepsilon)\}_{i\in\mathcal{F}}$ such that

$$x_i \in B(x_0,\varepsilon^\alpha), \tag{2.8}$$

$$B\left(x_i, \frac{\lambda_R\varepsilon}{4}\right) \cap B\left(x_j, \frac{\lambda_R\varepsilon}{4}\right) = \emptyset \quad \text{for } i \neq j, \tag{2.9}$$

$$B(x_0,\varepsilon^\alpha) \subset \bigcup_{i\in\mathcal{F}} B(x_i,\lambda_R\varepsilon).$$

Obviously, the discs $\{B(x_i,2\lambda_R\varepsilon)\}_{i\in\mathcal{F}}$ cannot intersect more that C times (where C is a universal constant) and

$$\bigcup_{i\in\mathcal{F}} B(x_i,2\lambda_R\varepsilon) \subset B(x_0,\varepsilon^{\alpha'})$$

with $\alpha' = \frac{1}{2}(\alpha + \frac{2}{3})$. We denote by $\mathcal{F}'$ the set of indices $i \in \mathcal{F}$ such that $B(x_i,\lambda_R\varepsilon)$ is a bad disc. We derive from Definition 2.7 that

$$\mu_R \operatorname{Card}(\mathcal{F}') \leq \sum_{i\in\mathcal{F}} \frac{1}{\varepsilon^2}\int_{B(x_i,2\lambda_R\varepsilon)} (1-|v_\varepsilon|^2)^2 \leq \frac{C}{\varepsilon^2}\int_{B(x_0,\varepsilon^{\alpha'})} (1-|v_\varepsilon|^2)^2.$$

The conclusion now follows by Lemma 2.5 and Proposition 2.6. □

Remark 2.9. By the proof of Proposition 2.8, it follows that any family of discs $\{B(x_i,\lambda_R\varepsilon)\}_{i\in\mathcal{F}}$ satisfying (2.8) and (2.9) cannot contain more than $N_{R,\alpha}$ bad discs.

In the sequel, we will require the following crucial lemma to prove that vortices of degree zero do not occur. This result has its source in [3, 9] and the proof is based on the construction of a suitable test function. Hence, the main difference and difficulty in our case come from the mass constraint we have to take into account in the construction of test functions.

Lemma 2.10. *Let $D > 0$, $0 < \beta < 1$ and $\gamma > 1$ be given constants such that $\gamma\beta < 1$. Let $0 < R < \sqrt{a_0}$ and $0 < \rho < \varepsilon^\beta$ be such that $\rho^\gamma > \lambda_R \varepsilon$. We assume that for $x_0 \in B_R^\Lambda$,*

(i) $\displaystyle\int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 < \frac{D}{\rho}$,

(ii) $|v_\varepsilon| \geq \dfrac{1}{2}$ *on* $\partial B(x_0,\rho)$,

(iii) $\deg\left(\dfrac{v_\varepsilon}{|v_\varepsilon|}, \partial B(x_0,\rho)\right) = 0$.

Then, we have

$$|v_\varepsilon| \geq \frac{1}{2} \quad \textit{in } B(x_0, \rho^\gamma).$$

Proof of Lemma 2.10. We are going to construct a comparison function as in [3] or [9] to obtain the following estimate:

$$\int_{B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq C_{\beta,R}. \tag{2.10}$$

Since the degree of v_ε restricted to $\partial B(x_0,\rho)$ is zero, we may write on $\partial B(x_0,\rho)$

$$v_\varepsilon = |v_\varepsilon| e^{i\phi_\varepsilon},$$

where ϕ_ε is a smooth map from $\partial B(x_0,\rho)$ into $\mathbb{R}$. Then, we define $\hat{v}_\varepsilon : \mathbb{R}^2 \to \mathbb{C}$ by

$$\begin{cases} \hat{v}_\varepsilon = \chi_\varepsilon e^{i\psi_\varepsilon} & \text{in } B(x_0,\rho), \\ \hat{v}_\varepsilon = v_\varepsilon & \text{in } \mathbb{R}^2 \setminus B(x_0,\rho), \end{cases}$$

where ψ_ε is the solution of

$$\begin{cases} \Delta\psi_\varepsilon = 0 & \text{in } B(x_0,\rho), \\ \psi_\varepsilon = \phi_\varepsilon & \text{on } \partial B(x_0,\rho), \end{cases}$$

and χ_ε has the form, written in polar coordinates centered at x_0,

$$\chi_\varepsilon(r,\theta) = (|v_\varepsilon(\rho e^{i\theta})| - 1)\xi(r) + 1$$

and ξ is a smooth function taking values in $[0,1]$ with small support near ρ with $\xi(\rho) = 1$. By [14, Proposition 3.3], we know that $|v_\varepsilon(x)| \leq 1 + C\varepsilon^{1/3}$ for $x \in \mathcal{D}$ with $|x|_\Lambda \geq \sqrt{a_0} - \varepsilon^{1/8}$ and we deduce that $0 \leq \chi_\varepsilon \leq 1 + C\varepsilon^{1/3}$. Arguing as in [7, proof of Theorem 2], we may prove that

$$\int_{B(x_0,\rho)} |\nabla\psi_\varepsilon|^2 \leq C\rho \int_{\partial B(x_0,\rho)} \left|\frac{\partial\phi_\varepsilon}{\partial\tau}\right|^2 \leq C\rho \int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 \tag{2.11}$$

and

$$\int_{B(x_0,\rho)} |\nabla\chi_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-\chi_\varepsilon^2)^2 \leq C\rho \int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 + O(\rho). \tag{2.12}$$

From (2.11), (2.12) and assumption (i), we infer that

$$\int_{B(x_0,\rho)} |\nabla \hat{v}_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2 \le C. \tag{2.13}$$

We set $\tilde{v}_\varepsilon = m_\varepsilon^{-1}\hat{v}_\varepsilon$ with $m_\varepsilon = \|\tilde{\eta}_\varepsilon \hat{v}_\varepsilon\|_{L^2(\mathbb{R}^2)}$. Clearly, $\tilde{\eta}_\varepsilon e^{i\Omega S}\tilde{v}_\varepsilon \in \mathcal{H}$ and $\|\tilde{\eta}_\varepsilon e^{i\Omega S}\tilde{v}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. Since $u_\varepsilon = \tilde{\eta}_\varepsilon e^{i\Omega S}v_\varepsilon$ minimizes the functional F_ε under the constraint (1.2), we have $F_\varepsilon(u_\varepsilon) \le F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}\tilde{v}_\varepsilon)$ and by (1.9), it yields

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) \le \tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon). \tag{2.14}$$

We claim that

$$\tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) \le \tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon) + C\rho|\ln\varepsilon|^2 \quad \text{and} \quad \big|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) - \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\big| = O\big(\rho^2|\ln\varepsilon|^2\big). \tag{2.15}$$

Indeed, we have already established in the proof of [14, Proposition 3.3] that

$$\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \le C|\ln\varepsilon|^2 \quad \text{and} \quad \big|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon)\big| \le C|\ln\varepsilon|^2 \tag{2.16}$$

so that, using (2.13), $\|\tilde{\eta}_\varepsilon v_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, $\hat{v}_\varepsilon = v_\varepsilon$ in $\mathbb{R}^2\backslash B(x_0,\rho)$ and (2.16), we obtain

$$\begin{aligned} m_\varepsilon^2 &= 1 + \int_{B(x_0,\rho)} \tilde{\eta}_\varepsilon^2\big(|\hat{v}_\varepsilon|^2 - 1\big) + \int_{B(x_0,\rho)} \tilde{\eta}_\varepsilon^2\big(1-|v_\varepsilon|^2\big) \\ &= 1 + O(\rho\,\varepsilon|\ln\varepsilon|). \end{aligned} \tag{2.17}$$

From (2.13), (2.16) and (2.17), we derive

$$\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla\tilde{v}_\varepsilon|^2 = m_\varepsilon^{-2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla\hat{v}_\varepsilon|^2 = \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla\hat{v}_\varepsilon|^2 + O(\rho\varepsilon|\ln\varepsilon|^3) \tag{2.18}$$

and

$$\tilde{\mathcal{R}}_\varepsilon(\tilde{v}_\varepsilon) = m_\varepsilon^{-2}\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) = \tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) + O(\rho\varepsilon|\ln\varepsilon|^3). \tag{2.19}$$

Since u_ε remains bounded in $\mathbb{R}^2$ and $E_\varepsilon(u_\varepsilon) \le C|\ln\varepsilon|^2$ by [14, Proposition 3.3], we infer from (2.16),

$$\begin{aligned} \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1-|\tilde{v}_\varepsilon|^2)^2 &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1-|\hat{v}_\varepsilon|^2)^2 + \frac{2(1-m_\varepsilon^{-2})}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2(1-|\hat{v}_\varepsilon|^2)|\tilde{\eta}_\varepsilon\hat{v}_\varepsilon|^2 \\ &\quad + \frac{(1-m_\varepsilon^{-2})^2}{\varepsilon^2}\int_{\mathbb{R}^2} |\tilde{\eta}_\varepsilon\hat{v}_\varepsilon|^4 \\ &\le \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1-|\hat{v}_\varepsilon|^2)^2 \\ &\quad + C\rho|\ln\varepsilon|\left(\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2\backslash B(x_0,\rho)} \tilde{\eta}_\varepsilon^4(1-|v_\varepsilon|^2)^2\right)^{1/2} \\ &\quad \times\left(\int_{\mathbb{R}^2\backslash B(x_0,\rho)} |u_\varepsilon|^4\right)^{1/2} + C\rho^2|\ln\varepsilon|^2 \\ &\le \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4(1-|\hat{v}_\varepsilon|^2)^2 + C\rho|\ln\varepsilon|^2. \end{aligned} \tag{2.20}$$

Finally, we obtain in the same way,

$$\big|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) - \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\big| \le \big|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) - \tilde{\mathcal{T}}_\varepsilon(\hat{v}_\varepsilon)\big| + \big|\tilde{\mathcal{T}}_\varepsilon(\hat{v}_\varepsilon) - \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\big| \tag{2.21}$$

$$\le C|\ln\varepsilon|^2\bigg(\int_{B(x_0,\rho)}(1+|x|^2)\tilde{\eta}_\varepsilon^2 + |1-m_\varepsilon^{-2}|\int_{\mathbb{R}^2}(1+|x|^2)\tilde{\eta}_\varepsilon^2|\hat{v}_\varepsilon|^2\bigg)$$

$$\le C\rho^2|\ln\varepsilon|^2. \tag{2.22}$$

From (2.18)–(2.21), we conclude that (2.15) holds.

Since $\hat{v}_\varepsilon = v_\varepsilon$ in $\mathbb{R}^2\backslash B(x_0,\rho)$, we get from (2.14) and (2.15) that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, B(x_0,\rho)) \le \tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon, B(x_0,\rho)) + C\rho|\ln\varepsilon|^2.$$

By (2.13), we have $\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon, B(x_0,\rho)) \le C$ and therefore,

$$\big|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon, B(x_0,\rho))\big| \le C\Omega\int_{B(x_0,\rho)}|\nabla\hat{v}_\varepsilon| \le C\Omega\rho\|\nabla\hat{v}_\varepsilon\|_{L^2(B(x_0,\rho))} = O(\rho|\ln\varepsilon|). \tag{2.23}$$

Hence, $\tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon, B(x_0,\rho)) \le C$ and we conclude that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, B(x_0,\rho)) \le C_\beta.$$

As for (2.23), using (2.16), we easily derive that $|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon, B(x_0,\rho))| = O(\rho|\ln\varepsilon|^2)$ and we finally get that $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon, B(x_0,\rho)) \le C_\beta$ which clearly implies (2.10) since $\tilde{\eta}_\varepsilon^2 \to a^+$ uniformly as $\varepsilon \to 0$ (see [14, Proposition 2.2]).

We deduce from (2.10) that

$$\int_{2\rho^\gamma}^{\rho}\bigg(\int_{\partial B(x_0,s)}|\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\bigg)\,ds \le C_{\beta,R}.$$

Since $\int_{2\rho^\gamma}^{\rho}\frac{ds}{s|\ln s|^{1/2}} \ge C_\gamma|\ln\varepsilon|^{1/2}$, we derive that for small ε there exists $s_0 \in [2\rho^\gamma,\rho]$ such that

$$\int_{\partial B(x_0,s_0)}|\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \le \frac{C_{\beta,R}}{s_0|\ln s_0|^{1/2}}.$$

Repeating the arguments used to prove (2.10), we find that

$$\int_{B(x_0,s_0)}|\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \le \frac{C_{\beta,R}}{|\ln s_0|^{1/2}}.$$

In particular, we have

$$\frac{1}{\varepsilon^2}\int_{B(x_0,2\rho^\gamma)}(1-|v_\varepsilon|^2)^2 = o(1)$$

and the conclusion follows by Proposition 2.6. □

We obtain as in [9, Proposition IV.3] the following result which gives us an estimate of the contribution in the energy of any vortex. We reproduce here the proof for completeness.

Proposition 2.11. *Let $0 < R < \sqrt{a_0}$ and $\frac{2}{3} < \alpha < 1$. Let $x_0 \in B_R^\Lambda$ and assume that $|v_\varepsilon(x_0)| < \frac{1}{2}$. Then there exists a positive constant $C_{R,\alpha}$ (which only depends on R, α and ω_1) such that*

$$\int_{B(x_0,\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \geq C_{R,\alpha}|\ln\varepsilon|.$$

Proof. Let $N_{R,\alpha}$ and $x_1, \ldots, x_{N_\varepsilon} \in B(x_0, \varepsilon^\alpha)$ be as in Proposition 2.8. We set

$$\delta_\alpha = \frac{\alpha^{1/2} - \alpha}{3(N_{R,\alpha} + 1)}$$

and for $k = 0, \ldots, 3N_{R,\alpha} + 2$, we consider

$$\alpha_k = \alpha^{1/2} - k\delta_\alpha, \quad \mathcal{I}_k = [\varepsilon^{\alpha_k}, \varepsilon^{\alpha_{k+1}}] \quad \text{and} \quad \mathcal{C}_k = B(x_0, \varepsilon^{\alpha_{k+1}}) \backslash B(x_0, \varepsilon^{\alpha_k}).$$

Then, there is some $k_0 \in \{1, \ldots, 3N_{R,\alpha} + 1\}$ such that

$$\mathcal{C}_{k_0} \cap \left(\bigcup_{j=1}^{N_\varepsilon} B(x_j, \lambda_R \varepsilon) \right) = \emptyset. \tag{2.24}$$

Indeed, since $N_\varepsilon \leq N_{R,\alpha}$ and $2\lambda_R\varepsilon < |\mathcal{I}_k|$ for small ε, the union of N_ε intervals of length $2\lambda_R\varepsilon$

$$\bigcup_{j=1}^{N_\varepsilon} \big(|x_i - x_0| - \lambda_R\varepsilon, |x_i - x_0| + \lambda_R\varepsilon\big)$$

cannot intersect all the intervals $\mathcal{I}_k$ of disjoint interior, for $1 \leq k \leq 3N_{R,\alpha} + 1$. From (2.24), we deduce that

$$|v_\varepsilon(x)| \geq \frac{1}{2} \quad \text{for any } x \in \mathcal{C}_{k_0}.$$

Therefore, for every $\rho \in \mathcal{I}_{k_0}$,

$$d_{k_0} = \deg\left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(x_0, \rho) \right)$$

is well defined and does not depend on ρ. We claim that

$$d_{k_0} \neq 0. \tag{2.25}$$

By contradiction, we suppose that $d_{k_0} = 0$. According to (1.14), it results that

$$\int_{B^\Lambda_{\frac{\sqrt{a_0}+R}{2}}} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \leq C_R|\ln\varepsilon|.$$

Using the same argument as in Step 2 of the proof of Lemma 2.5, there is a constant $C_{R,\alpha}$ such that

$$\int_{\partial B(x_0,\rho_0)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \leq \frac{C_{R,\alpha}}{\rho_0} \quad \text{for some } \rho_0 \in \mathcal{I}_{k_0}.$$

According to Lemma 2.10 (with $\beta = \alpha_{k_0+1}$ and $\gamma = \frac{\alpha_{k_0-1}}{\alpha_{k_0}}$), we should have $|v_\varepsilon(x_0)| \geq \frac{1}{2}$ which is a contradiction.

By (2.25), we obtain for every $\rho \in \mathcal{I}_{k_0}$,

$$1 \leq |d_{k_0}| = \frac{1}{2\pi}\left|\int_{\partial B(x_0,\rho)} \frac{1}{|v_\varepsilon|^2}\left(v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial \tau}\right)\right| \leq C\int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|$$

(we use that $|v_\varepsilon| \geq \frac{1}{2}$ in $\mathcal{C}_{k_0}$). Then, the Cauchy–Schwarz inequality yields

$$\int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 \geq \frac{C}{\rho} \quad \text{for any } \rho \in \mathcal{I}_{k_0}$$

and the conclusion follows integrating on $\mathcal{I}_{k_0}$. □

2.2. *Proofs of Theorem 2.1 and Proposition 2.1*

The part (1) in Theorem 2.1 follows directly from Lemma 2.12 below.

Lemma 2.12. *There exists a constant* $\varepsilon_R > 0$ *such that for any* $0 < \varepsilon < \varepsilon_R$,

$$|v_\varepsilon| \geq \frac{1}{2} \quad \textit{in } B_R^\Lambda \backslash B^\Lambda_{\frac{\sqrt{a_0}}{5}}.$$

Proof. First, we fix some $\alpha \in (\frac{2}{3}, 1)$. We proceed by contradiction. Suppose that there is some $x_0 \in B_R^\Lambda \backslash B^\Lambda_{\frac{\sqrt{a_0}}{5}}$ such that $|v_\varepsilon(x_0)| < 1/2$. Then, for any ε sufficiently small, we have $B(x_0, \varepsilon^\alpha) \subset \mathcal{D}_\varepsilon \backslash \{|x|_\Lambda < 2|\ln\varepsilon|^{-1/6}\}$ and therefore, by (1.15), we get that

$$\int_{B(x_0,\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \leq C_R\, \mathcal{E}_\varepsilon\big(v_\varepsilon, \mathcal{D}_\varepsilon \backslash \big\{|x|_\Lambda < 2|\ln\varepsilon|^{-1/6}\big\}\big) \leq C_R \ln|\ln\varepsilon|$$

which contradicts Proposition 2.11 for ε small enough. □

Proof of (2) in Theorem 2.1. We fix some $\frac{2}{3} < \alpha < 1$. As in the proof of Proposition 2.8, we consider a finite family of points $\{x_j\}_{j\in\mathcal{J}}$ satisfying

$$x_j \in B^\Lambda_{\frac{\sqrt{a_0}}{2}}$$
$$B\left(x_i, \frac{\lambda_0\varepsilon}{4}\right) \cap B\left(x_j, \frac{\lambda_0\varepsilon}{4}\right) = \emptyset \quad \text{for } i \neq j,$$
$$B^\Lambda_{\frac{\sqrt{a_0}}{2}} \subset \bigcup_{j\in\mathcal{J}} B(x_j, \lambda_0\varepsilon),$$

where $\lambda_0 := \lambda_{\frac{\sqrt{a_0}}{2}}$ (defined in Proposition 2.6 with $R = \frac{\sqrt{a_0}}{2}$) and we denote by J_ε the set of indices $j \in \mathcal{J}$ such that $B(x_j, \lambda_0\varepsilon)$ contains at least one point y_j verifying

$$|v_\varepsilon(y_j)| < \frac{1}{2}. \tag{2.26}$$

Obviously, $B(x_j, \lambda_0\varepsilon)$ is a bad disc for every $j \in J_\varepsilon$. Applying Lemma 2.12 (with $R = \frac{3\sqrt{a_0}}{4}$), we infer that there exists ε_0 such that for any $0 < \varepsilon < \varepsilon_0$,

$$B(x_j, \lambda_0\varepsilon) \subset B^\Lambda_{\frac{\sqrt{a_0}}{4}} \quad \text{for any } j \in J_\varepsilon. \tag{2.27}$$

Then, it remains to prove that $\mathrm{Card}(J_\varepsilon)$ is bounded independently of ε. Using Proposition 2.11 (with $R = \frac{\sqrt{a_0}}{2}$), we derive that for any $j \in J_\varepsilon$ and any point y_j satisfying (2.26) in the ball $B(x_j, \lambda_0\varepsilon)$,

$$\int_{B(x_j, 2\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \geq \int_{B(y_j, \varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \geq C_\alpha |\ln \varepsilon| \tag{2.28}$$

for some positive constant C_α which only depends on α. We set for ε small enough,

$$W = \bigcup_{j \in J_\varepsilon} B(x_j, 2\varepsilon^\alpha) \subset B^\Lambda_{\frac{\sqrt{a_0}}{3}}.$$

We claim that there is a positive integer M_α independent of ε such that any $y \in W$ belongs to at most M_α balls in the collection $\{B(x_j, 2\varepsilon^\alpha)\}_{j \in J_\varepsilon}$. Indeed, for each $y \in W$, consider the subset $K_y \subset J_\varepsilon$ defined by

$$K_y = \big\{ j \in J_\varepsilon \ : \ y \in B(x_j, 2\varepsilon^\alpha) \big\}.$$

We have for every $j \in K_y$,

$$x_j \in B(y, 2\varepsilon^\alpha) \subset B(y, \varepsilon^{\alpha'}) \subset B^\Lambda_{\frac{\sqrt{a_0}}{2}} \quad \text{with } \alpha' = \frac{1}{2}\left(\alpha + \frac{2}{3}\right). \tag{2.29}$$

Since the family of discs $\{B(x_j, \lambda_0\varepsilon)\}_{j \in K_y}$ is a subcover of $B(y, \varepsilon^{\alpha'})$ satisfying (2.8) and (2.9), we conclude from Remark 2.9 that

$$\mathrm{Card}(K_y) \leq M_\alpha$$

with $M_\alpha = N_{\frac{\sqrt{a_0}}{2}, \alpha'}$. From (2.28), we infer that

$$\int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}}} |\nabla v_\varepsilon|^2 \geq \int_W |\nabla v_\varepsilon|^2 \geq \frac{1}{M_\alpha} \sum_{j \in J_\varepsilon} \int_{B(x_j, 2\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \geq C_\alpha \ \mathrm{Card}(J_\varepsilon) |\ln \varepsilon|. \tag{2.30}$$

On the other hand, we know by (1.14),

$$\int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}}} |\nabla v_\varepsilon|^2 \leq C \int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}}} a(x) |\nabla v_\varepsilon|^2 \leq C |\ln \varepsilon| \tag{2.31}$$

for a constant C independent of ε. Matching (2.30) and (2.31), we conclude that $\mathrm{Card}(J_\varepsilon)$ is uniformly bounded. □

In the following, we will prove Proposition 2.3. We proceed exactly as in [20, Theorem 2.1] and an adaptation of [3, Theorem V.1]. Before starting our proof, we recall, for the convenience of the reader, a result obtained in [14, Proposition 4.1], by a method due to Sandier [18] and Sandier–Serfaty [19]:

Proposition 2.13 [14]. *There exists a positive constant $\mathcal{K}_0$ such that for ε sufficiently small, there exist $\nu_\varepsilon \in (1,2)$ and a finite collection of disjoint balls $\{B_i\}_{i\in I_\varepsilon} := \{B(p_i, r_i)\}_{i\in I_\varepsilon}$ satisfying*:

(i) *for every $i \in I_\varepsilon, B_i \subset\subset \mathcal{D}_\varepsilon = \{x \in \mathbb{R}^2,\, a(x) > \nu_\varepsilon |\ln\varepsilon|^{-3/2}\}$,*

(ii) $\{x \in \mathcal{D}_\varepsilon,\, |v_\varepsilon(x)| < 1 - |\ln\varepsilon|^{-5}\} \subset \cup_{i\in I_\varepsilon} B_i$,

(iii) $\displaystyle\sum_{i\in I_\varepsilon} r_i \le |\ln\varepsilon|^{-10}$,

(iv) $\displaystyle\frac{1}{2}\int_{B_i} a(x)|\nabla v_\varepsilon|^2 \ge \pi a(p_i)|d_i|\big(|\ln\varepsilon| - \mathcal{K}_0 \ln|\ln\varepsilon|\big)$,

where $d_i = \deg\left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B_i\right)$ for every $i \in I_\varepsilon$.

Proof of Proposition 2.3. By Theorem 2.1, we have for ε small enough,

$$\bigcup_{j\in J_\varepsilon} B(x_j^\varepsilon, \lambda_0\varepsilon) \subset B^\Lambda_{\frac{\sqrt{a_0}}{3}}.$$

From (iii) in Proposition 2.13, there exists a radius $r_\varepsilon \in (\frac{\sqrt{a_0}}{3}, \frac{\sqrt{a_0}}{2}]$ such that

$$\bar{B}_i \cap \partial B^\Lambda_{r_\varepsilon} = \emptyset \quad \text{for every } i \in I_\varepsilon. \tag{2.32}$$

Hence, we have

$$|v_\varepsilon| \ge 1 - |\ln\varepsilon|^{-5} \quad \text{on } \partial B^\Lambda_{r_\varepsilon}.$$

The existence of a subset $\tilde{J}_\varepsilon \subset J_\varepsilon$ satisfying (i)–(v) can now be proved identically as in [20, Proposition 3.2] and it remains to prove (2.1). From the proof of Theorem 2.1, we know (by construction) that each disc $B(x_k^\varepsilon, \lambda_0\varepsilon)$, $k \in J_\varepsilon$, contains at least one point y_k such that $|v_\varepsilon(y_k)| < \frac{1}{2}$. Therefore, each disc $B(x_j^\varepsilon, \rho)$, $j \in \tilde{J}_\varepsilon$, contains at least one of the y_k's with $|x_j^\varepsilon - y_k| < \lambda_0\varepsilon$. Assume now that $D_j = 0$. By Lemma 2.10 with $\gamma = \mu^{-1/2}$, it would lead to $|v_\varepsilon| \ge \frac{1}{2}$ in $B(x_j^\varepsilon, \rho^\gamma)$ and then $|v_\varepsilon(y_k)| \ge \frac{1}{2}$ for ε small enough, contradiction. We also find a bound on the degrees D_j:

$$|D_j| = \frac{1}{2\pi}\left|\int_{\partial B(x_j^\varepsilon,\rho)} \frac{1}{|v_\varepsilon|^2}\left(v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial\tau}\right)\right| \le C\|\nabla v_\varepsilon\|_{L^2(\partial B(x_j^\varepsilon,\rho))}\sqrt{\rho} \le C$$

by (iv) in Proposition 2.3. □

3. Some Lower Energy Estimates

In this section, we obtain various lower energy estimates for v_ε in terms of the vortex structure defined in Sec. 2, Proposition 2.3. We start by proving a lower bound on the kinetic energy away from the vortices which brings out the interaction between vortices. The method that we use is based on the techniques developed in [3, 8, 20, 21]. As in the previous section, the main difficulty is due to the degenerate behavior near the boundary of $\mathcal{D}$ of the function $a(x)$ since the method involves in our case the operator $-\mathrm{div}(a^{-1}\nabla)$ which is not uniformly elliptic in $\mathcal{D}$. To avoid this problem, we shall establish our estimates in B_R^Λ for an arbitrary radius $R \in [\sqrt{a_0}/2, \sqrt{a_0}\,)$. The underlying idea here is to let $R \to \sqrt{a_0}$ at the end of the analysis. To emphasize the possible dependence on R in the "error term", we will denote by $O_R(1)$ (respectively, $o_R(1)$) any quantity which remains uniformly bounded in ε for fixed R (respectively, any quantity which tends to 0 as $\varepsilon \to 0$ for fixed R). In the sequel, we will also write $\tilde{J}_\varepsilon = \{1, \ldots, n_\varepsilon\}$.

Proposition 3.1. *For any* $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$, *let* $\Theta_\rho = B_R^\Lambda \backslash \cup_{j=1}^{n_\varepsilon} B(x_j^\varepsilon, \rho)$. *We have*

$$\frac{1}{2}\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \geq \pi \sum_{j=1}^{n_\varepsilon} D_j^2\, a(x_j^\varepsilon)|\ln\rho| \\ + W_{R,\varepsilon}\big((x_1^\varepsilon, D_1), \ldots, (x_{n_\varepsilon}^\varepsilon, D_{n_\varepsilon})\big) + O_R(1), \tag{3.1}$$

where

$$W_{R,\varepsilon}\big((x_1^\varepsilon, D_1), \ldots, (x_{n_\varepsilon}^\varepsilon, D_{n_\varepsilon})\big) = -\pi \sum_{i\neq j} D_i D_j a(x_j^\varepsilon)\ln|x_i^\varepsilon - x_j^\varepsilon| - \pi \sum_{j=1}^{n_\varepsilon} D_j \Psi_{R,\varepsilon}(x_j^\varepsilon)$$

and $\Psi_{R,\varepsilon}$ *is the unique solution of*

$$\begin{cases} \mathrm{div}\left(\dfrac{1}{a}\nabla\Psi_{R,\varepsilon}\right) = -\displaystyle\sum_{j=1}^{n_\varepsilon} D_j a(x_j^\varepsilon)\,\nabla\left(\dfrac{1}{a}\right)\cdot\nabla\big(\ln|x - x_j^\varepsilon|\big) & \text{in } B_R^\Lambda, \\ \Psi_{R,\varepsilon} = -\displaystyle\sum_{j=1}^{n_\varepsilon} D_j a(x_j^\varepsilon)\ln|x - x_j^\varepsilon| & \text{on } \partial B_R^\Lambda. \end{cases} \tag{3.2}$$

Moreover, if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ *as* $\varepsilon \to 0$ *for any* $i \neq j$, *then the term* $O_R(1)$ *in* (3.1) *is in fact* $o_R(1)$.

Remark 3.2. We point out that the dependence on R in the interaction term $W_{R,\varepsilon}$ only appears in the function $\Psi_{R,\varepsilon}$. Moreover, for $\Psi_{R,\varepsilon}$ to be well defined, $1/a(x)$ has to be bounded inside B_R^Λ so that we cannot pass to the limit $R \to \sqrt{a_0}$ in (3.1) without an *a priori* deterioration of the error term.

Proof of Proposition 3.1. We consider the solution Φ_ρ of the linear problem

$$\begin{cases} \mathrm{div}\left(\dfrac{1}{a}\nabla\Phi_\rho\right) = 0 & \text{in } \Theta_\rho, \\ \Phi_\rho = 0 & \text{on } \partial B_R^\Lambda, \\ \Phi_\rho = \text{const.} & \text{on } \partial B(x_j^\varepsilon, \rho), \\ \displaystyle\int_{\partial B(x_j^\varepsilon,\rho)} \dfrac{1}{a}\dfrac{\partial\Phi_\rho}{\partial\nu} = 2\pi D_j & \text{for } j = 1, \ldots, n_\varepsilon, \end{cases}$$

and $\Phi_{R,\varepsilon}$ the solution of

$$\begin{cases} \operatorname{div}\left(\frac{1}{a}\nabla\Phi_{R,\varepsilon}\right) = 2\pi\sum_{j=1}^{n_\varepsilon} D_j\,\delta_{x_j^\varepsilon} & \text{in } B_R^\Lambda, \\ \Phi_{R,\varepsilon} = 0 & \text{on } \partial B_R^\Lambda. \end{cases} \tag{3.3}$$

For $x \in \Theta_\rho$, we set $w_\varepsilon(x) = \frac{v_\varepsilon(x)}{|v_\varepsilon(x)|}$ and

$$\mathcal{S} = \left(-w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial x_2} + \frac{1}{a}\frac{\partial \Phi_\rho}{\partial x_1}, w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial x_1} + \frac{1}{a}\frac{\partial \Phi_\rho}{\partial x_2}\right).$$

We easily check that div $\mathcal{S} = 0$ in Θ_ρ and $\int_{\partial B_R^\Lambda} \mathcal{S}\cdot\nu = \int_{\partial B(x_j^\varepsilon,\rho)} \mathcal{S}\cdot\nu = 0$. By [8, Lemma I.1], there exists $H \in C^1(\bar{\Theta}_\rho)$ such that $\mathcal{S} = \nabla^\perp H$ and hence, we can write the Hodge–de Rham type decomposition

$$w_\varepsilon \wedge \nabla w_\varepsilon = \frac{1}{a}\,\nabla^\perp\Phi_\rho + \nabla H.$$

Consequently,

$$\begin{aligned} \int_{\Theta_\rho} a(x)|\nabla w_\varepsilon|^2 &= \int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 + 2\int_{\Theta_\rho} \nabla^\perp\Phi_\rho\cdot\nabla H + \int_{\Theta_\rho} a(x)|\nabla H|^2 \\ &\geq \int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 + 2\int_{\Theta_\rho} \nabla^\perp\Phi_\rho\cdot\nabla H. \end{aligned}$$

We observe that the last term is in fact equal to zero since it is the integral of a Jacobian and Φ_ρ is constant on $\partial\Theta_\rho$. Hence,

$$\int_{\Theta_\rho} a(x)|\nabla w_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2.$$

Since $|\nabla v_\varepsilon|^2 \geq |v_\varepsilon|^2|\nabla w_\varepsilon|^2$ in Θ_ρ, we derive that

$$\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 + T_1 + 2T_2$$

with

$$T_1 = \int_{\Theta_\rho} \big(|v_\varepsilon|^2 - 1\big)\,\frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 \quad \text{and} \quad T_2 = \int_{\Theta_\rho} \big(|v_\varepsilon|^2 - 1\big)\nabla\Phi_\rho^\perp\cdot\nabla H.$$

Arguing as in [3] (see Step 4 in the proof of Theorem 6), it turns out that $T_1 = o_R(1)$ and $T_2 = o_R(1)$ and therefore,

$$\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 + o_R(1). \tag{3.4}$$

On the other hand, by integrating by parts, we obtain

$$\int_{\Theta_\rho} \frac{1}{a(x)}\,|\nabla\Phi_\rho|^2 = \int_{\partial\Theta_\rho} \frac{1}{a(x)}\frac{\partial\Phi_\rho}{\partial\nu}\Phi_\rho = -2\pi\sum_{j=1}^{n_\varepsilon} D_j\Phi_\rho(z_j)$$

for any point $z_j \in \partial B(x_j^\varepsilon, \rho)$. Since n_ε and each D_j remain uniformly bounded in ε by Proposition 2.3, we may rewrite this equality as

$$\int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 = -2\pi \sum_{j=1}^{n_\varepsilon} D_j \Phi_{R,\varepsilon}(z_j) + O\big(\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)}\big). \tag{3.5}$$

Using an adaptation of [8, Lemma I.4] (see, e.g., [6, Lemma 3.5]), we derive that

$$\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)} \leq \sum_{j=1}^{n_\varepsilon} \left(\sup_{\partial B(x_j^\varepsilon, \rho)} \Phi_{R,\varepsilon} - \inf_{\partial B(x_j^\varepsilon,\rho)} \Phi_{R,\varepsilon} \right). \tag{3.6}$$

To estimate the right-hand side term in (3.6), we introduce for $x \in B_R^\Lambda$,

$$\Psi_{R,\varepsilon}(x) = \Phi_{R,\varepsilon}(x) - \sum_{j=1}^{n_\varepsilon} D_j\, a(x_j^\varepsilon) \ln|x - x_j^\varepsilon|.$$

Since $\Phi_{R,\varepsilon}$ solves (3.3), we deduce that $\Psi_{R,\varepsilon}$ may be characterized as the solution of Eq. (3.2). By elliptic regularity, we infer that $\|\Psi_{R,\varepsilon}\|_{W^{2,p}(B_R^\Lambda)} \leq C_{R,p}$ for any $1 \leq p < 2$ (here we used that $\{x_j^\varepsilon\}_{j=1}^{n_\varepsilon} \subset B^\Lambda_{\frac{\sqrt{a_0}}{4}}$ by Theorem 2.1). In particular, $\Psi_{R,\varepsilon}$ is uniformly bounded with respect to ε in $C^{0,1/2}(B_R^\Lambda)$ and hence,

$$\sup_{\partial B(x_j^\varepsilon,\rho)} \Psi_{R,\varepsilon} - \inf_{\partial B(x_j^\varepsilon,\rho)} \Psi_{R,\varepsilon} \leq C_R\sqrt{\rho} = o_R(1).$$

Since $|x_j^\varepsilon - x_i^\varepsilon| \geq 8\rho$, we derive from (2.1),

$$\begin{aligned} &\sup_{\partial B(x_j^\varepsilon,\rho)} \left(\sum_{i=1}^{n_\varepsilon} D_i a(x_i^\varepsilon) \ln|x - x_i^\varepsilon| \right) - \inf_{\partial B(x_j^\varepsilon,\rho)} \left(\sum_{i=1}^{n_\varepsilon} D_i a(x_i^\varepsilon) \ln|x - x_i^\varepsilon| \right) \\ &\quad \leq \rho \sum_{i=1,\, i\neq j}^{n_\varepsilon} a(x_i^\varepsilon) \sup_{\partial B(x_j^\varepsilon,\rho)} \frac{|D_i|}{|x - x_i^\varepsilon|} \leq O(1), \end{aligned}$$

(respectively, $\leq o(1)$ if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \neq j$). Coming back to (3.6), we obtain that $\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)} \leq O_R(1)$ (respectively, $\leq o_R(1)$ if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \neq j$). Inserting this estimate in (3.5), we get that

$$\begin{aligned} \int_{\Theta_\rho} \frac{1}{a(x)} |\nabla \Phi_\rho|^2 &= -2\pi \sum_{j=1}^{n_\varepsilon} D_j \Phi_{R,\varepsilon}(z_j) + O_R(1) \\ &= -2\pi \sum_{j=1}^{n_\varepsilon} D_j \Psi_{R,\varepsilon}(z_j) - 2\pi \sum_{i\neq j} D_i D_j a(x_i^\varepsilon) \ln|z_j - x_i^\varepsilon| \\ &\quad + 2\pi \sum_{j=1}^{n_\varepsilon} D_j^2\, a(x_j^\varepsilon) |\ln \rho| + O_R(1) \end{aligned} \tag{3.7}$$

(respectively, $+\,o_R(1)$ as $\varepsilon \to 0$). Since $\Psi_{R,\varepsilon}$ is uniformly bounded with respect to ε in $C^{0,1/2}(B_R^\Lambda)$, we have $|\Psi_{R,\varepsilon}(z_j) - \Psi_{R,\varepsilon}(x_j^\varepsilon)| \leq C_R\sqrt{\rho} = o_R(1)$. Moreover, using

(2.1) and $|x_j^\varepsilon - x_i^\varepsilon| \geq 8\rho$, we derive that

$$\left|\sum_{i\neq j} D_i D_j a(x_i^\varepsilon)(\ln|z_j - x_i^\varepsilon| - \ln|x_j^\varepsilon - x_i^\varepsilon|)\right| \leq \sum_{i\neq j} |D_i||D_j| \ln\left|1 + \frac{z_j - x_j^\varepsilon}{x_j^\varepsilon - x_i^\varepsilon}\right|$$

$$\leq \sum_{i\neq j} |D_i||D_j| \frac{\rho}{|x_j^\varepsilon - x_i^\varepsilon|} \leq O(1)$$

(respectively, $\leq o(1)$ as $\varepsilon \to 0$). Hence, (3.7) yields

$$\int_{\Theta_\rho} \frac{1}{a(x)} |\nabla\Phi_\rho|^2 = -2\pi \sum_{j=1}^{n_\varepsilon} D_j \Psi_{R,\varepsilon}(x_j^\varepsilon) - 2\pi \sum_{i\neq j} D_i D_j a(x_i^\varepsilon) \ln|x_j^\varepsilon - x_i^\varepsilon|$$

$$+ 2\pi \sum_{j=1}^{n_\varepsilon} D_j^2 a(x_j^\varepsilon)|\ln\rho| + O_R(1)$$

(respectively, $+\, o_R(1)$ as $\varepsilon \to 0$). Combining this estimate with (3.4), we obtain the announced result. □

Arguing as in [20, 21], we estimate the contribution in the energy of each vortex which yields the following lower bounds for $\mathcal{E}_\varepsilon(v_\varepsilon)$:

Lemma 3.3. *For any* $R \in \left[\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\right)$, *we have*

$$\mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) \geq \pi \sum_{j=1}^{n_\varepsilon} D_j^2 a(x_j^\varepsilon)|\ln\rho| + \pi \sum_{j=1}^{n_\varepsilon} |D_j| a(x_j^\varepsilon) \ln\frac{\rho}{\varepsilon} + W_{R,\varepsilon} + O_R(1) \qquad (3.8)$$

and

$$\mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) \geq \pi \sum_{j=1}^{n_\varepsilon} |D_j| a(x_j^\varepsilon) \ln\frac{\rho}{\varepsilon} + O(1). \qquad (3.9)$$

Proof. In view of Proposition 3.1, it suffices to show that

$$\mathcal{E}_\varepsilon(v_\varepsilon, B(x_j^\varepsilon, \rho)) \geq \pi|D_j| a(x_j^\varepsilon) \ln\frac{\rho}{\varepsilon} + O(1) \quad \text{for } j = 1, \ldots, n_\varepsilon,$$

which is equivalent to

$$\frac{1}{2}\int_{B(x_j^\varepsilon,\rho)} |\nabla v_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \geq \pi|D_j| \ln\frac{\rho}{\varepsilon} + O(1) \quad \text{for } j = 1, \ldots, n_\varepsilon \qquad (3.10)$$

(we used that $|a(x) - a(x_j^\varepsilon)| \leq C\rho$ for $x \in B(x_j^\varepsilon, \rho)$ and $\mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) \leq C_R|\ln\varepsilon|$). Setting

$$\hat{v}(y) = v_\varepsilon(\rho y + x_j^\varepsilon) \quad \text{for } y \in B(0,1) \quad \text{and} \quad \hat{\varepsilon} = \frac{\varepsilon}{\rho\sqrt{a(x_j^\varepsilon)}},$$

we infer from Proposition 2.3 that $|\hat{v}| \geq 1 - \frac{2}{|\ln \varepsilon|^2}$ on $\partial B(0,1)$,

$$\frac{1}{2}\int_{\partial B(0,1)} |\nabla \hat{v}|^2 + \frac{1}{2\hat{\varepsilon}^2}(1-|\hat{v}|^2)^2 = \frac{\rho}{2}\int_{\partial B(x_j^\varepsilon,\rho)} |\nabla v_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq C \tag{3.11}$$

and

$$\frac{1}{2}\int_{B(0,1)} |\nabla \hat{v}|^2 + \frac{1}{2\hat{\varepsilon}^2}(1-|\hat{v}|^2)^2 = \frac{1}{2}\int_{B(x_j^\varepsilon,\rho)} |\nabla v_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2.$$

As in the proof of [3, Lemma VI.1], (3.11) yields for ε small enough,

$$\frac{1}{2}\int_{B(0,1)} |\nabla \hat{v}|^2 + \frac{1}{2\hat{\varepsilon}^2}(1-|\hat{v}|^2)^2 \geq \pi|D_j|\,|\ln \hat{\varepsilon}| + O(1) = \pi|D_j| \ln\frac{\rho}{\varepsilon} + O(1)$$

and hence, (3.10) holds. □

As in [14, Proposition 4.2], we may compute an asymptotic expansion of $\mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon)$ in terms of vortices which leads, in view of Lemma 3.3, to lower expansions of $\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon)$:

Lemma 3.4. *For any $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0})$, we have*

$$\begin{aligned}\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi \sum_{j=1}^{n_\varepsilon} D_j^2\, a(x_j^\varepsilon)|\ln \rho| + \pi \sum_{j=1}^{n_\varepsilon} |D_j| a(x_j^\varepsilon) \ln\frac{\rho}{\varepsilon} \\ - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) D_j + W_{R,\varepsilon} + O_R(1)\end{aligned} \tag{3.12}$$

and

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi \sum_{j=1}^{n_\varepsilon} |D_j| a(x_j^\varepsilon) \ln\frac{\rho}{\varepsilon} - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) D_j + O(1). \tag{3.13}$$

Proof. We consider the family of balls $\{B_i\}_{i\in I_\varepsilon}$ given in Proposition 2.13. As in the proof of Proposition 2.3, we can find $r_\varepsilon \in [R, (R+\sqrt{a_0})/2]$ such that (2.32) holds. Setting

$$I_R^+ = \{i \in I_\varepsilon, |p_i|_\Lambda > r_\varepsilon \text{ and } d_i \geq 0\}$$

and

$$I_R^- = \{i \in I_\varepsilon,\, |p_i|_\Lambda > r_\varepsilon \text{ and } d_i < 0\}, \tag{3.14}$$

we have $\bar{B}_i \subset \mathcal{D}_\varepsilon \setminus \bar{B}_{r_\varepsilon}^\Lambda$ for any $i \in I_R^+ \cup I_R^-$. By Theorem 2.1, Propositions 2.3 and 2.13, we infer that for ε small enough,

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } \Xi_\varepsilon := \mathcal{D}_\varepsilon \Bigg\backslash \left(\bigcup_{i\in I_R^+\cup I_R^-} B_i \cup \bigcup_{j=1}^{n_\varepsilon} B(x_j^\varepsilon,\rho)\right).$$

Arguing exactly as in [14, Proposition 4.2], we obtain that

$$\mathcal{R}_\varepsilon(v_\varepsilon,\Xi_\varepsilon) = \frac{-\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon)D_j - \frac{\pi\Omega}{1+\Lambda^2}\sum_{i\in I_R^+\cup I_R^-}\big(a^2(p_i)-\nu_\varepsilon^2|\ln\varepsilon|^{-3}\big)d_i + o_R(1). \tag{3.15}$$

We recall that we have showed in the proof of [14, Proposition 4.2] that $\mathcal{R}_\varepsilon(v_\varepsilon,\cup_{i\in I_R^+\cup I_R^-}B_i) = o(1)$. In the same way, we may prove that $\mathcal{R}_\varepsilon(v_\varepsilon,\cup_{j=1}^{n_\varepsilon}B(x_j^\varepsilon,\rho)) = o(1)$. From (iv) in Proposition 2.13 and (3.15), we deduce that

$$\begin{aligned}\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) &\geq \mathcal{E}_\varepsilon\Bigg(v_\varepsilon,\mathcal{D}_\varepsilon\Bigg\backslash\bigcup_{i\in I_R^+\cup I_R^-}B_i\Bigg) \\ &\quad + \sum_{i\in I_R^+\cup I_R^-}\frac{1}{2}\int_{B_i}a(x)|\nabla v_\varepsilon|^2 + \mathcal{R}_\varepsilon(v_\varepsilon,\Xi_\varepsilon) + o_R(1) \\ &\geq \mathcal{E}_\varepsilon(v_\varepsilon,B_R^\Lambda) - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}a^2(x_j^\varepsilon)D_j \\ &\quad + \pi\sum_{i\in I_R^+\cup I_R^-}a(p_i)|d_i|\big(|\ln\varepsilon|-\mathcal{K}_0\ln|\ln\varepsilon|\big) \\ &\quad - \frac{\pi\Omega}{1+\Lambda^2}\sum_{i\in I_R^+\cup I_R^-}\big(a^2(p_i)-\nu_\varepsilon^2|\ln\varepsilon|^{-3}\big)d_i + o_R(1).\end{aligned} \tag{3.16}$$

Since $p_i \notin \bar{B}_{r_\varepsilon}^\Lambda$ for $i\in I_R^+\cup I_R^-$, we have $a(p_i)\ll a_0$ and we deduce that for ε small enough,

$$\pi\sum_{i\in I_R^+\cup I_R^-}a(p_i)|d_i|\big(|\ln\varepsilon|-\mathcal{K}_0\ln|\ln\varepsilon|\big) - \frac{\pi\Omega}{1+\Lambda^2}\sum_{i\in I_R^+\cup I_R^-}\big(a^2(p_i)-\nu_\varepsilon^2|\ln\varepsilon|^{-3}\big)d_i \geq 0$$

which leads to

$$\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) \geq \mathcal{E}_\varepsilon(v_\varepsilon,B_R^\Lambda) - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}a^2(x_j^\varepsilon)D_j + o_R(1). \tag{3.17}$$

Combining (3.8) and (3.17), we obtain (3.12). Similarly, the inequality (3.17) applied with $R=\sqrt{a_0}/2$, and (3.9) yield (3.13). □

4. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1 in terms of the map v_ε. We start by showing that vortices must be of degree one. This yields a fundamental improvement of the estimates obtained in the previous section. Then, we treat separately the points (i) and (ii) of Theorem 1.1.

4.1. *Vortices have degree one*

Lemma 4.1. *Whenever ε is small enough, $D_j = +1$ for $j = 1, \ldots, n_\varepsilon$.*

Proof. By [14, Proposition 3.5], we have $\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le o(1)$. According to (3.13), it yields

$$\pi \sum_{j=1}^{n_\varepsilon} |D_j| a(x_j^\varepsilon) \ln \frac{\rho}{\varepsilon} - \frac{\pi a_0 \Omega}{1+\Lambda^2} \sum_{D_j>0} a(x_j^\varepsilon) D_j \le \pi \sum_{j=1}^{n_\varepsilon} |D_j| a(x_j^\varepsilon) \ln \frac{\rho}{\varepsilon} - \frac{\pi\Omega}{1+\Lambda^2} \sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) D_j \le O(1).$$

From (1.7), we derive that

$$\sum_{j=1}^{n_\varepsilon} |D_j| a(x_j^\varepsilon) \ln \frac{\rho}{\varepsilon} \le \sum_{D_j>0} D_j a(x_j^\varepsilon) |\ln \varepsilon| + o(|\ln \varepsilon|).$$

Since $\rho \ge \varepsilon^\mu$, it leads to (we recall that $D_j \ne 0$)

$$(1-\mu) \sum_{D_j<0} |D_j| a(x_j^\varepsilon) |\ln \varepsilon| \le \mu \sum_{D_j>0} |D_j| a(x_j^\varepsilon) |\ln \varepsilon| + o(|\ln \varepsilon|).$$

By Theorem 2.1, $a(x_j^\varepsilon) \ge a_0/2$ and consequently,

$$\sum_{D_j<0} |D_j| \le \frac{2\mu}{1-\mu} \sum_{D_j>0} |D_j| + o(1) \le \frac{C\mu}{1-\mu} + o(1).$$

Choosing μ sufficiently small, it yields $D_j > 0$ for $j = 1, \ldots, n_\varepsilon$ whenever ε is small enough. Since $|x_j^\varepsilon| \le C$ and $D_j > 0$, we may now assert that

$$-\pi \sum_{i \ne j} D_i D_j a(x_j^\varepsilon) \ln |x_i^\varepsilon - x_j^\varepsilon| \ge O(1)$$

and thus, $W_{\frac{\sqrt{a_0}}{2},\varepsilon} \ge -\pi \sum_{j=1}^{n_\varepsilon} D_j \Psi_{\frac{\sqrt{a_0}}{2},\varepsilon}(x_j^\varepsilon) = O(1)$. Hence, the inequality (3.12) (applied with $R = \sqrt{a_0}/2$) together with $\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le o(1)$ leads us to

$$\pi \sum_{j=1}^{n_\varepsilon} D_j^2 a(x_j^\varepsilon) |\ln \rho| + \pi \sum_{j=1}^{n_\varepsilon} D_j a(x_j^\varepsilon) \ln \frac{\rho}{\varepsilon} - \frac{\pi\Omega}{1+\Lambda^2} \sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) D_j \le O(1).$$

As previously, we derive from (1.7), $\sum_{j=1}^{n_\varepsilon} (D_j^2 - D_j) a(x_j^\varepsilon) |\ln \rho| \le o(|\ln \varepsilon|)$. Since $\rho \le \varepsilon^{\bar\mu}$ and $a(x_j^\varepsilon) \ge a_0/2$, we conclude that

$$\frac{\bar\mu\, a_0}{2} \sum_{j=1}^{n_\varepsilon} (D_j^2 - D_j) \le o(1)$$

which yields $D_j = +1$ whenever ε is small enough. □

As a direct consequence of Lemma 4.1, we obtain the following improvement of Lemma 3.4:

Corollary 4.2. *For any $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$, we have*

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) + W_{R,\varepsilon}\big((x_1^\varepsilon,+1),\ldots,(x_{n_\varepsilon}^\varepsilon,+1)\big) + O_R(1).$$

Proof. It follows directly from (3.12) and Lemma 4.1 that for any $R \in \big[\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,\big)$,

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) + W_{R,\varepsilon}\big((x_1^\varepsilon,+1),\ldots,(x_{n_\varepsilon}^\varepsilon,+1)\big) + O_R(1).$$

On the other hand, we have proved in the proofs of [14, Propositions 3.4 and 3.5], that $|\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) - \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon)| = o(1)$ and $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, \mathbb{R}^2\setminus\mathcal{D}_\varepsilon) \geq o(1)$. Hence, we have $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq \mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) + o(1)$ and the conclusion follows. □

4.2. *The subcritical case*

We are now able to prove (i) in Theorem 1.1. Following the proof of [14, Theorem 1.1], it suffices to show Proposition 4.3 below.

Proposition 4.3. *Assume that* (1.7) *holds with* $\omega_1 < 0$. *Then, for* ε *sufficiently small, we have that*

$$|v_\varepsilon| \to 1 \quad \text{in } L^\infty_{\text{loc}}(\mathcal{D}) \text{ as } \varepsilon \to 0. \tag{4.1}$$

Moreover,

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = o(1) \quad \text{and} \quad \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) = o(1). \tag{4.2}$$

Proof. We fix some $\frac{\sqrt{a_0}}{2} < R_0 < \sqrt{a_0}$. In the proof of [14, Proposition 3.4], we have proved that $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \leq o(1)$ so that Corollary 4.2 applied with $R = \frac{\sqrt{a_0}}{2}$ leads to

$$\pi\sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi a_0\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon) \leq \pi\sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)|\ln\varepsilon| - \frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) \leq O(1).$$

Since $a(x_j^\varepsilon) \geq a_0/2$ and $\omega_1 < 0$, we deduce that

$$\frac{a_0|\omega_1|\,n_\varepsilon}{2}\ln|\ln\varepsilon| \leq -\omega_1\sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)\ln|\ln\varepsilon| \leq O(1)$$

and then $n_\varepsilon \le o(1)$ which implies that $n_\varepsilon \equiv 0$ whenever ε is small enough. Using the notation (3.14), we derive from (3.16) that

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \ge \pi \sum_{i \in I^+_{R_0} \cup I^-_{R_0}} a(p_i)|d_i|\big(|\ln\varepsilon| - \mathcal{K}_0 \ln|\ln\varepsilon|\big) - \frac{\pi\Omega}{1+\Lambda^2} \sum_{i \in I^+_{R_0} \cup I^-_{R_0}} \big(a^2(p_i) - \nu_\varepsilon^2|\ln\varepsilon|^{-3}\big)d_i.$$

By [14, Proposition 3.5], we have $\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le O(|\ln\varepsilon|^{-1})$. Since $a(p_i) \ll a_0$ for $i \in I^+_{R_0} \cup I^-_{R_0}$, we infer that exists $c > 0$ independent of ε such that

$$\begin{aligned} c \sum_{i \in I^+_{R_0} \cup I^-_{R_0}} a(p_i)|d_i||\ln\varepsilon| &\le \pi \sum_{i \in I^+_{R_0} \cup I^-_{R_0}} a(p_i)|d_i|\big(|\ln\varepsilon| - \mathcal{K}_0 \ln|\ln\varepsilon|\big) \\ &\quad - \frac{\pi\Omega}{1+\Lambda^2} \sum_{i \in I^+_{R_0} \cup I^-_{R_0}} \big(a^2(p_i) - \nu_\varepsilon^2|\ln\varepsilon|^{-3}\big)d_i \\ &\le O(|\ln\varepsilon|^{-1}). \end{aligned}$$

Since $a(x) \ge |\ln\varepsilon|^{-3/2}$ in $\mathcal{D}_\varepsilon$, we finally obtain

$$\sum_{i \in I^+_{R_0} \cup I^-_{R_0}} |d_i| \le O(|\ln\varepsilon|^{-1/2}).$$

Hence, $\sum_{i \in I^+_{R_0} \cup I^-_{R_0}} |d_i| = 0$ for ε sufficiently small and we conclude from (3.15),

$$\mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \backslash \cup_{i \in I^+_{R_0} \cup I^-_{R_0}} B_i) = o(1).$$

By the proof of [14, Proposition 4.2], we also have $\mathcal{R}_\varepsilon(v_\varepsilon, \cup_{i \in I^+_{R_0} \cup I^-_{R_0}} B_i) = o(1)$ so that $\mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = o(1)$. Consequently,

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = \mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) + o(1) \le o(1).$$

Then the rest of the proof follows as in [14, Proposition 4.3]. □

4.3. *The supercritical case*

In this section, we will prove (ii) in Theorem 1.1. Writing

$$\Omega = \frac{1+\Lambda^2}{a_0}\big(|\ln\varepsilon| + \omega(\varepsilon)\ln|\ln\varepsilon|\big), \tag{4.3}$$

we assume that

$$(d-1) + \delta \le \omega(\varepsilon) \le d - \delta \tag{4.4}$$

for some integer $d \ge 1$ and some positive number $\delta \ll 1$ independent of ε. We start by proving that, in this regime, v_ε has vortices whenever ε is small enough:

Proposition 4.4. *Assume that* (4.4) *holds. Then, for ε sufficiently small, v_ε has exactly d vortices of degree one, i.e. $n_\varepsilon \equiv d$, and*

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = -\pi a_0\ d\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| + O(1). \tag{4.5}$$

Proof. Step 1. We start by proving that $n_\varepsilon \geq 1$ for ε sufficiently small. By Theorem 5.1 in Sec. 5 (with $d = 1$), there exists $\tilde{u}_\varepsilon \in \mathcal{H}$ such that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and

$$F_\varepsilon(\tilde{u}_\varepsilon) \leq F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) - \pi a_0 \omega(\varepsilon) \ln|\ln \varepsilon| + O(1).$$

By the minimizing property of u_ε and (1.9), we have

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\eta_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) \leq F_\varepsilon(\tilde{u}_\varepsilon)$$

and since $|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)| = o(1)$ (see [14, Proposition 3.3]), we deduce that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \leq -\pi a_0 \omega(\varepsilon) \ln|\ln \varepsilon| + O(1).$$

From here, it turns out by Corollary 4.2 applied with $R = \frac{\sqrt{a_0}}{2}$ (recall that $W_{\frac{\sqrt{a_0}}{2},\varepsilon} \geq O(1)$),

$$\begin{aligned}
-\pi a_0 \omega(\varepsilon) \ln|\ln \varepsilon| + O(1) \geq \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) &\geq \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon) |\ln \varepsilon| - \frac{\pi\Omega}{1+\Lambda^2} \sum_{j=1}^{n_\varepsilon} a^2(x_j^\varepsilon) + O(1) \\
&\geq \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon) \left(-\omega(\varepsilon) \ln|\ln \varepsilon| + \frac{\Omega |x_j^\varepsilon|_\Lambda^2}{1+\Lambda^2} \right) + O(1) \\
&\geq -\pi a_0 \omega(\varepsilon) n_\varepsilon \ln|\ln \varepsilon| + O(1).
\end{aligned}$$

Hence, $n_\varepsilon \geq 1 + o(1)$ and the conclusion follows.

Step 2. Now, we show that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq -\pi a_0 n_\varepsilon \omega(\varepsilon) \ln|\ln \varepsilon| + \frac{\pi a_0}{2} (n_\varepsilon^2 - n_\varepsilon) \ln|\ln \varepsilon| + O(1). \tag{4.6}$$

In the case $n_\varepsilon = 1$, we have already proved the result in the previous step. Then, we may assume that $n_\varepsilon \geq 2$. Since $\|\Psi_{\frac{\sqrt{a_0}}{2},\varepsilon}\|_\infty = O(1)$, we get from Corollary 4.2 applied with $R = \frac{\sqrt{a_0}}{2}$,

$$\begin{aligned}
\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) &\geq \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon) \left(|\ln \varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^{n_\varepsilon} \ln|x_i^\varepsilon - x_j^\varepsilon| - \frac{\Omega a(x_j^\varepsilon)}{1+\Lambda^2} \right) + O(1) \\
&\geq \pi \sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon) \left(-\omega(\varepsilon) \ln|\ln \varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^{n_\varepsilon} \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega |x_j^\varepsilon|_\Lambda^2}{1+\Lambda^2} \right) + O(1). \qquad (4.7)
\end{aligned}$$

Since $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \le o(1)$, we derive that

$$-\sum_{i\neq j} \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon} |x_j^\varepsilon|_\Lambda^2 \le C\ln|\ln\varepsilon|.$$

On the other hand, $-\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| \ge O(1)$ so that $|x_j^\varepsilon|^2 \le C\big(\ln|\ln\varepsilon|\big)|\ln\varepsilon|^{-1}$ and hence,

$$\begin{aligned}
&\pi\sum_{j=1}^{n_\varepsilon} a(x_j^\varepsilon)\left(-\omega(\varepsilon)\ln|\ln\varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^{n_\varepsilon}\ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega|x_j^\varepsilon|_\Lambda^2}{1+\Lambda^2}\right)\\
&\quad = -\pi a_0 n_\varepsilon\omega(\varepsilon)\ln|\ln\varepsilon| - \pi a_0\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\pi a_0\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}|x_j^\varepsilon|_\Lambda^2 + o(1). \qquad (4.8)
\end{aligned}$$

Setting $r = \max_j |x_j^\varepsilon|$, we remark that

$$\begin{aligned}
-\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega}{1+\Lambda^2}\sum_{j=1}^{n_\varepsilon}|x_j^\varepsilon|_\Lambda^2 &\ge -(n_\varepsilon^2 - n_\varepsilon)\ln 2r\\
&\quad + \frac{\Omega\Lambda^2 r^2}{1+\Lambda^2} \ge \frac{n_\varepsilon^2 - n_\varepsilon}{2}\ln|\ln\varepsilon| + O(1). \qquad (4.9)
\end{aligned}$$

Combining (4.7)–(4.9), we obtain (4.6).

Step 3. We start by proving that $n_\varepsilon \ge d$. The case $d = 1$ is proved in Step 1 so that we may assume that $d \ge 2$. By Theorem 5.1 in Sec. 5, there exists for ε small enough, $\tilde{u}_\varepsilon \in \mathcal{H}$ such that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and

$$F_\varepsilon(\tilde{u}_\varepsilon) \le F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) - \pi a_0\, d\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| + O(1).$$

As in Step 1, $F_\varepsilon(u_\varepsilon) \le F_\varepsilon(\tilde{u}_\varepsilon)$ yields

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \le -\pi a_0\, d\omega(\varepsilon)\ln|\ln\varepsilon| + \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| + O(1). \qquad (4.10)$$

Matching (4.6) with (4.10), we deduce that

$$-\omega(\varepsilon)n_\varepsilon + \frac{n_\varepsilon^2 - n_\varepsilon}{2} \le -\omega(\varepsilon)d + \frac{d^2 - d}{2} + o(1)$$

and it yields

$$\omega(\varepsilon)(d - n_\varepsilon) \le \frac{(d - n_\varepsilon)(d + n_\varepsilon - 1)}{2} + o(1). \qquad (4.11)$$

If assume that $n_\varepsilon \le d - 1$, it would lead to

$$(d-1) + \delta \le \frac{d + n_\varepsilon - 1}{2} + o(1) \le d - 1 + o(1)$$

which is impossible for ε small enough.

Assume now that $n_\varepsilon \geq d+1$. As previously, we infer that (4.11) holds and therefore,

$$d-\delta \geq \frac{d+n_\varepsilon-1}{2}+o(1) \geq d+o(1)$$

which is also impossible for ε small. Hence, $n_\varepsilon \equiv d$ whenever ε is small enough which leads to (4.5) by (4.6) and (4.10). □

By Proposition 4.4, we may now assume that v_ε has exactly d vortices. We move on a first information on their location:

Lemma 4.5. *We have*

$$\begin{aligned} |x_j^\varepsilon| &\leq C|\ln\varepsilon|^{-1/2} \quad \textit{for } j=1,\dots,d \quad \textit{and} \quad \textit{if } d\geq 2,\\ |x_i^\varepsilon - x_j^\varepsilon| &\geq C|\ln\varepsilon|^{-1/2} \quad \textit{for } i\neq j. \end{aligned}$$

Proof. Matching (4.5) with (4.7) and (4.8) and using that $n_\varepsilon = d$, we deduce that

$$-\pi a_0 \sum_{i\neq j} \ln|x_i^\varepsilon - x_j^\varepsilon| + \frac{\pi a_0 \Omega}{1+\Lambda^2}\sum_{j=1}^{d}|x_j^\varepsilon|_\Lambda^2 \leq \pi a_0(d^2-d)\ln\big(|\ln\varepsilon|^{1/2}\big)+O(1).$$

Hence,

$$\sum_{j=1}^{d}\left(-\sum_{i\neq j}\ln\Big(\sqrt{|\ln\varepsilon|}\,|x_i^\varepsilon - x_j^\varepsilon|\Big)+\frac{\Omega|x_j^\varepsilon|^2}{2}\right)\leq O(1)$$

and the conclusion follows. □

Since $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} = o(1)$ by Lemma 4.5, we may now improve the lower estimates obtained in Lemma 3.3 following the method of the proof of Proposition 5.2 in [20, 21].

Lemma 4.6. *For any* $R\in\big[\frac{\sqrt{a_0}}{2},\sqrt{a_0}\big)$, *we have*

$$\mathcal{E}_\varepsilon(v_\varepsilon, B_R^\Lambda) \geq \pi a_0 \sum_{j=1}^{d} a(x_j^\varepsilon)|\ln\varepsilon| + W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon)+\frac{\pi a_0 d}{2}\ln a_0 + a_0\, d\gamma_0 + o_R(1),$$

where γ_0 *is an absolute constant.*

Proof. Since $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} = o(1)$ and $D_j = 1$, Proposition 3.1 yields

$$\frac{1}{2}\int_{\Theta_\rho} a(x)|\nabla v_\varepsilon|^2 \geq \pi\sum_{j=1}^{d} a(x_j^\varepsilon)|\ln\rho| + W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon)+o_R(1) \tag{4.12}$$

and it remains to estimate $\mathcal{E}_\varepsilon(v_\varepsilon, B(x_j^\varepsilon,\rho))$ for $j=1,\dots,d$. We proceed as follows. Since $D_j = 1$, we may write on $\partial B(x_j^\varepsilon,\rho)$ in polar coordinates with center x_j^ε,

$$v_\varepsilon(x) = |v_\varepsilon(x)|e^{i(\theta+\psi_j(\theta))}, \quad \theta\in[0,2\pi],$$

where $\psi_j \in H^1([0,2\pi],\mathbb{R})$ and $\psi_j(0) = \psi_j(2\pi) = 0$. Then, in each disc $B(x_j^\varepsilon, 2\rho)$, we consider the map $\hat{v}_\varepsilon$ defined by

$$\hat{v}_\varepsilon(x) = \begin{cases} v_\varepsilon(x) & \text{if } x \in B(x_j^\varepsilon, \rho), \\ \left(\dfrac{r-\rho}{\rho} + \dfrac{2\rho - r}{\rho}|v_\varepsilon(x_j^\varepsilon + \rho\, e^{i\theta})|\right) & \\ \quad \times \exp\, i\left(\theta + \psi_j(\theta)\dfrac{2\rho - r}{\rho} + \psi_j(0)\dfrac{\rho - r}{\rho}\right) & \text{if } x \in B(x_j^\varepsilon, 2\rho)\backslash B(x_j^\varepsilon, \rho). \end{cases}$$

Then, $\hat{v}_\varepsilon = \exp\, i(\theta + \psi_j(0))$ on $\partial B(x_j^\varepsilon, 2\rho)$. Exactly as in the proof of Proposition 5.2 in [20, 21], we prove that

$$\left|\mathcal{E}_\varepsilon(\hat{v}_\varepsilon, B(x_j^\varepsilon, 2\rho)\backslash B(x_j^\varepsilon, \rho)) - \pi a(x_j^\varepsilon)\ln 2\right| = o(1). \tag{4.13}$$

Since $|a(x) - a(x_j^\varepsilon)| = O(\rho)$ in $B(x_j^\varepsilon, 2\rho)$, we may write

$$\mathcal{E}_\varepsilon(\hat{v}_\varepsilon, B(x_j^\varepsilon, 2\rho)) = \frac{a(x_j^\varepsilon)}{2}\int_{B(x_j^\varepsilon, 2\rho)} |\nabla \hat{v}_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1 - |\hat{v}_\varepsilon|^2)^2 + o(1). \tag{4.14}$$

Now, we shall recall a result in [8]. For $\tilde{\varepsilon} > 0$, we consider

$$I(\tilde{\varepsilon}) = \underset{u \in \mathcal{C}}{\text{Min}}\ \frac{1}{2}\int_{B(0,1)} |\nabla u|^2 + \frac{1}{2\tilde{\varepsilon}^2}(1 - |u|^2)^2,$$

where

$$\mathcal{C} = \left\{u \in H^1(B(0,1), \mathbb{C}),\ u(x) = \frac{x}{|x|} \text{ on } \partial B(0,1)\right\}.$$

Then, we have

$$\lim_{\tilde{\varepsilon} \to 0} \left(I(\tilde{\varepsilon}) + \pi \ln \tilde{\varepsilon}\right) = \gamma_0. \tag{4.15}$$

Since $\hat{v}_\varepsilon(x) = \frac{x - x_j^\varepsilon}{|x - x_j^\varepsilon|}\, e^{i\psi_j(0)}$ on $\partial B(x_j^\varepsilon, 2\rho)$, we obtain by scaling

$$\begin{aligned} &\frac{1}{2}\int_{B(x_j^\varepsilon, 2\rho)} |\nabla \hat{v}_\varepsilon|^2 + \frac{a(x_j^\varepsilon)}{2\varepsilon^2}(1 - |\hat{v}_\varepsilon|^2)^2 \\ &\qquad \geq I\left(\frac{\varepsilon}{2\rho\sqrt{a(x_j^\varepsilon)}}\right) = \pi \ln \frac{\rho}{\varepsilon} + \pi \ln 2 + \frac{\pi}{2}\ln a(x_j^\varepsilon) + \gamma_0 + o(1). \end{aligned}$$

With (4.13) and (4.14), we derive that for $j = 1, \ldots, d$,

$$\begin{aligned} \mathcal{E}_\varepsilon(v_\varepsilon, B(x_j^\varepsilon, \rho)) &\geq \pi a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} + \frac{\pi a(x_j^\varepsilon)}{2}\ln a(x_j^\varepsilon) + a(x_j^\varepsilon)\gamma_0 + o(1) \\ &\geq \pi a(x_j^\varepsilon)\ln\frac{\rho}{\varepsilon} + \frac{\pi a_0}{2}\ln a_0 + a_0\gamma_0 + o(1). \end{aligned}$$

Combining this estimate with (4.12), we get the result. □

We are now able to give the asymptotic expansion of $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon)$ which will allow us to locate precisely the vortices. This concludes the proof of Theorem 1.1.

Proposition 4.7. *Setting* $\tilde{x}_j^\varepsilon = \sqrt{\Omega}\, x_j^\varepsilon$ *for* $j = 1, \ldots, d$, *as* $\varepsilon \to 0$ *the* $\tilde{x}_j^\varepsilon$*'s tend to minimize the renormalized energy* $w : \mathbb{R}^{2d} \to \mathbb{R}$ *given by*

$$w(b_1, \ldots, b_d) = -\pi a_0 \sum_{i \neq j} \ln|b_i - b_j| + \frac{\pi a_0}{1+\Lambda^2} \sum_{j=1}^{d} |b_j|_\Lambda^2 .$$

Moreover, we have

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = -\pi a_0\, d\, \omega(\varepsilon) \ln|\ln \varepsilon| + \frac{\pi a_0}{2}(d^2 - d) \ln|\ln \varepsilon| + \operatorname*{Min}_{b \in \mathbb{R}^{2d}} w(b) + Q_{\Lambda,d} + o(1) \tag{4.16}$$

where $Q_{\Lambda,d} = \frac{\pi a_0}{2}(d^2 - d)\ln(1+\Lambda^2) + \pi a_0 d \ln a_0 - \frac{\pi a_0 d^2}{2} \ln a_0 + a_0\, d\gamma_0 - \pi a_0 d^2 \ell(\Lambda)$ *and* $\ell(\Lambda)$ *is given by* (A.2).

Proof. From Lemma 4.6 and (3.17), we infer that for any $R \in [\frac{\sqrt{a_0}}{2}, \sqrt{a_0}\,)$,

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi \sum_{j=1}^{d} a(x_j^\varepsilon)|\ln \varepsilon| - \frac{\pi\Omega}{1+\Lambda^2} \sum_{j=1}^{d} a^2(x_j^\varepsilon) + W_{R,\varepsilon} + \frac{\pi a_0 d}{2} \ln a_0 + a_0\, d\gamma_0 + o_R(1).$$

As in the proof of Corollary 4.2, this estimate implies

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq \pi \sum_{j=1}^{d} a(x_j^\varepsilon)|\ln \varepsilon| - \frac{\pi\Omega}{1+\Lambda^2} \sum_{j=1}^{d} a^2(x_j^\varepsilon) + W_{R,\varepsilon} + \frac{\pi a_0 d}{2} \ln a_0 + a_0\, d\gamma_0 + o_R(1).$$

Expanding Ω and $a(x_j^\varepsilon)$, we derive that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq \pi \sum_{j=1}^{d} a(x_j^\varepsilon)\left(-\omega(\varepsilon)\ln|\ln \varepsilon| + \frac{\Omega |x_j^\varepsilon|_\Lambda^2}{1+\Lambda^2} \right) + W_{R,\varepsilon} + \frac{\pi a_0 d}{2} \ln a_0 + a_0\, d\gamma_0 + o_R(1)$$

and by Lemma 4.5, it yields

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \geq -\pi a_0\, d\omega(\varepsilon)\ln|\ln \varepsilon| + \frac{\pi a_0}{1+\Lambda^2} \sum_{j=1}^{d} \Omega |x_j^\varepsilon|_\Lambda^2 + W_{R,\varepsilon} + \frac{\pi a_0 d}{2} \ln a_0 + a_0\, d\gamma_0 + o_R(1). \tag{4.17}$$

By Lemma 4.5, we also have

$$W_{R,\varepsilon} = -\pi a_0 \sum_{i \neq j} \ln|x_i^\varepsilon - x_j^\varepsilon| - \pi \sum_{j=1}^{d} \Psi_{R,\varepsilon}(x_j^\varepsilon) + o(1). \tag{4.18}$$

Since $D_j = 1$ for all j, the function $\Psi_{R,\varepsilon}$ satisfies the equation

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{a}\nabla \Psi_{R,\varepsilon}\right) = -\displaystyle\sum_{j=1}^{d} a(x_j^\varepsilon)\nabla\left(\frac{1}{a}\right) \cdot \nabla(\ln|x - x_j^\varepsilon|) & \text{in } B_R^\Lambda, \\ \Psi_{R,\varepsilon} = -\displaystyle\sum_{j=1}^{d} a(x_j^\varepsilon)\ln|x - x_j^\varepsilon| & \text{on } \partial B_R^\Lambda. \end{cases} \tag{4.19}$$

We infer from Lemma 4.5 that for $j = 1, \ldots, d$,

$$a(x_j^\varepsilon)\nabla\left(\frac{1}{a}\right)\cdot\nabla(\ln|x - x_j^\varepsilon|) = \frac{-2a_0|x|_\Lambda^2}{a^2(x)|x|^2} + f_\varepsilon^j(x),$$

where f_ε^j satisfies

$$\|f_\varepsilon^j\|_{L^p(B_R^\Lambda)} = o_R(1) \quad \text{for any } p \in [1, 2)$$

and

$$\big\|a_0\ln|x| - a(x_j^\varepsilon)\ln|x - x_j^\varepsilon|\,\big\|_{C^1(\partial B_R^\Lambda)} = o(1).$$

Letting Ψ_R to be the solution of the equation

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{a}\nabla\Psi_R\right) = \dfrac{-2|x|_\Lambda^2}{a^2(x)|x|^2} & \text{in } B_R^\Lambda, \\ \Psi_R = -\ln|x| & \text{on } \partial B_R^\Lambda, \end{cases} \tag{4.20}$$

it follows by classical results that $\|\Psi_{R,\varepsilon} - a_0 d\Psi_R\|_{L^\infty(B_R^\Lambda)} = o_R(1)$. Hence, we obtain from (4.18),

$$\lim_{\varepsilon\to 0}\left\{W_{R,\varepsilon}(x_1^\varepsilon, \ldots, x_d^\varepsilon) + \pi a_0\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon|\right\} = -\pi a_0 d^2\Psi_R(0). \tag{4.21}$$

Combining (4.17) and (4.21), we are led to

$$\liminf_{\varepsilon\to 0}\left\{\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0\, d\omega(\varepsilon)\ln|\ln\varepsilon| + \pi a_0\sum_{i\neq j}\ln|x_i^\varepsilon - x_j^\varepsilon| - \frac{\pi a_0}{1+\Lambda^2}\sum_{j=1}^d \Omega|x_j^\varepsilon|_\Lambda^2\right\}$$
$$\geq \frac{\pi a_0 d}{2}\ln a_0 + a_0\, d\gamma_0 - \pi a_0 d^2\Psi_R(0).$$

Setting $\tilde{x}_j^\varepsilon = \sqrt{\Omega}\,x_j^\varepsilon$, it yields

$$\liminf_{\varepsilon\to 0}\left\{\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0\, d\omega(\varepsilon)\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| - w(\tilde{x}_1^\varepsilon, \ldots, \tilde{x}_d^\varepsilon)\right\}$$
$$\geq \frac{\pi a_0}{2}(d^2 - d)\ln(1+\Lambda^2) + \pi a_0 d\ln a_0 - \frac{\pi a_0 d^2}{2}\ln a_0 + a_0\, d\gamma_0 - \pi a_0 d^2\Psi_R(0).$$

Since $\Psi_R(0) \to \ell(\Lambda)$ as $R \to \sqrt{a_0}$ by Lemma A.1 in Appendix A, we conclude that

$$\liminf_{\varepsilon\to 0} \left\{ \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0 \omega(\varepsilon) d \ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| - w(\tilde{x}_1^\varepsilon, \ldots, \tilde{x}_d^\varepsilon) \right\} \geq Q_{\Lambda,d} \tag{4.22}$$

and hence,

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \left\{ \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0 \omega(\varepsilon) d \ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} \\ &\quad \geq \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d}. \end{aligned} \tag{4.23}$$

By Theorem 5.1 in Sec. 5, for any $\delta' > 0$, there exists $\tilde{u}_\varepsilon \in \mathcal{H}$ such that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \left\{ F_\varepsilon(\tilde{u}_\varepsilon) - F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \pi a_0\, d\omega(\varepsilon)\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} \\ &\quad \leq \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d} + \delta'. \end{aligned}$$

As in the proof of Proposition 4.4, $F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(\tilde{u}_\varepsilon)$ implies

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \left\{ \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0\, d\omega(\varepsilon)\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} \\ &\quad \leq \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d} + \delta'. \end{aligned} \tag{4.24}$$

Matching (4.23) with (4.24), we conclude that

$$\lim_{\varepsilon\to 0} \left\{ \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \pi a_0\, d\omega(\varepsilon)\ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon| \right\} = \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d}$$

since δ' is arbitrarily small. Coming back to (4.22), we are led to

$$\operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d} - \limsup_{\varepsilon\to 0}\, w(x_1^\varepsilon, \ldots, x_d^\varepsilon) \geq Q_{\Lambda,d}$$

and therefore, $\lim_{\varepsilon\to 0} w(\tilde{x}_1^\varepsilon, \ldots, \tilde{x}_d^\varepsilon) = \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b)$ which ends the proof. □

Remark 4.8. In the case $d = 1$, the expansion of the energy takes the simpler form

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = -\pi a_0 \omega(\varepsilon)\ln|\ln\varepsilon| + Q_{\Lambda,1} + o(1)$$

and the renormalized energy $w(\cdot)$ reduces to $w(b) = (\pi a_0 |b|_\Lambda^2)/(1+\Lambda^2)$. In particular, if x^ε denotes the single vortex of v_ε, we have $\sqrt{\Omega}\, x^\varepsilon \to 0$ as ε goes to 0.

5. Upper Bound of the Energy

Here, we give the construction of the test functions used in the previous sections. The difficulties are twofold: the mass constraint we have to take into account and the vanishing property of the function $a(x)$ on the boundary of $\mathcal{D}$. Hence, the classical methods cannot be applied directly. Concerning the mass constraint, we simply renormalize a suitable trial function. This procedure requires a high precision in the energy estimates and an almost optimal choice of the preliminary trial function. To overcome the degeneracy problem induced by the function $a(x)$, we proceed by upper approximation of $a(x)$. In the sequel, we assume that (1.7) holds. Using notation (4.3), the result can be stated as follows:

Theorem 5.1. *Let $d \geq 1$ be an integer. For any $\delta > 0$, there exists $(\tilde{u}_\varepsilon)_{\varepsilon>0} \subset \mathcal{H}$ verifying $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$ and*

$$\limsup_{\varepsilon\to 0} \left\{ F_\varepsilon(\tilde{u}_\varepsilon) - F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \pi a_0 \omega(\varepsilon) d \ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| \right\}$$
$$\leq \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d} + \delta,$$

where the constant $Q_{\Lambda,d}$ is defined in Proposition 4.7.

As mentioned above, the proof of Theorem 5.1 is based on a first construction which is given by the following proposition. Here, some of the main ingredients are taken from a previous construction due to André and Shafrir [5].

Proposition 5.2. *Let $d \geq 1$ be an integer. For any $\delta > 0$, there exists $(\hat{v}_\varepsilon)_{\varepsilon>0}$ such that $\tilde{\eta}_\varepsilon \hat{v}_\varepsilon \in \mathcal{H}$ and*

$$\limsup_{\varepsilon\to 0} \left\{ \tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon) + \pi a_0 \omega(\varepsilon) d \ln|\ln\varepsilon| - \frac{\pi a_0}{2}(d^2 - d)\ln|\ln\varepsilon| \right\}$$
$$\leq \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) + Q_{\Lambda,d} + \delta.$$

Proof. Step 1. Let $\sigma > 0$ and $\kappa > 0$ be two small parameters that we will choose later. We consider the function $a_\sigma : \bar{\mathcal{D}} \to \mathbb{R}$ given by

$$a_\sigma(x) = \begin{cases} a(x) & \text{if } |x|_\Lambda \leq \sqrt{a_0 - \sigma}, \\ -2\sqrt{a_0-\sigma}\,|x|_\Lambda + 2a_0 - \sigma & \text{otherwise.} \end{cases}$$

It turns out that $a_\sigma \in C^1(\bar{\mathcal{D}})$, $a_\sigma \geq a$ and $a_\sigma \geq C\sigma^2$ in $\bar{\mathcal{D}}$ for some positive constant C. Since a_σ does not vanish in $\bar{\mathcal{D}}$, we may define $\Phi_\sigma : \mathcal{D} \to \mathbb{R}$ the solution of the equation

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{a_\sigma}\nabla\Phi_\sigma\right) = 2\pi\, d\delta_0 & \text{in } \mathcal{D}, \\ \Phi_\sigma = 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{5.1}$$

By the results in [8, Chap. I], we may find a map $v_0^\sigma \in C^2(\bar{\mathcal{D}}\setminus\{0\}, S^1)$ satisfying

$$v_0^\sigma \wedge \nabla v_0^\sigma = \frac{1}{a_\sigma}\nabla^\perp \Phi_\sigma \quad \text{in } \mathcal{D}\setminus\{0\}. \tag{5.2}$$

Set $\Theta_{\kappa,\varepsilon} = \mathcal{D}\setminus B(0,\kappa^{-1}\Omega^{-1/2})$. By (5.1) and (5.2), we have for ε small enough,

$$\begin{aligned}\int_{\Theta_{\kappa,\varepsilon}} a_\sigma|\nabla v_0^\sigma|^2 = \int_{\Theta_{\kappa,\varepsilon}} \frac{1}{a_\sigma}|\nabla\Phi_\sigma|^2 &= -\int_{\partial B(0,\kappa^{-1}\Omega^{-1/2})} \frac{1}{a}\frac{\partial\Phi_\sigma}{\partial\nu}\Phi_\sigma \\ &= -\int_{\partial B(0,\kappa^{-1}\Omega^{-1/2})} \frac{a_0^2 d^2}{a}\left(\frac{\partial\Psi_\sigma}{\partial\nu} + \frac{1}{|x|}\right) \\ &\qquad \times \big(\Psi_\sigma + \ln|x|\big),\end{aligned} \tag{5.3}$$

where $\Psi_\sigma(x) = (a_0 d)^{-1}\Phi_\sigma(x) - \ln|x|$. Notice that $\Psi_\sigma \in C^{1,\alpha}(\bar{\mathcal{D}})$ for any $0 < \alpha < 1$, since it satisfies the equation

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{a_\sigma}\nabla\Psi_\sigma\right) = f_\sigma(x) & \text{in } \mathcal{D}, \\ \Psi_\sigma = -\ln|x| & \text{on } \partial\mathcal{D} \end{cases} \tag{5.4}$$

with

$$f_\sigma(x) = -\nabla\left(\frac{1}{a_\sigma(x)}\right)\cdot\frac{x}{|x|^2} = \begin{cases} \dfrac{-2|x|_\Lambda^2}{a_\sigma^2(x)|x|^2} & \text{if } |x| \le \sqrt{a_0-\sigma}, \\ \dfrac{-2\sqrt{a_0-\sigma}\,|x|_\Lambda}{a_\sigma^2(x)|x|^2} & \text{otherwise.} \end{cases}$$

From (5.3), we derive that

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{\Theta_{\kappa,\varepsilon}} a|\nabla v_0^\sigma|^2 - \pi a_0 d^2\ln(\kappa\Omega^{1/2})\right\} \\ &\quad\le \lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{\Theta_{\kappa,\varepsilon}} a_\sigma|\nabla v_0^\sigma|^2 - \pi a_0 d^2\ln(\kappa\Omega^{1/2})\right\} \\ &\quad\le -\pi a_0 d^2\Psi_\sigma(0).\end{aligned}$$

By Lemma A.1 in Appendix A, $\Psi_\sigma(0) \to \ell(\Lambda)$ as $\sigma \to 0$ where the constant $\ell(\Lambda)$ is defined in (A.2). Consequently, we may choose σ small such that

$$\limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{\Theta_{\kappa,\varepsilon}} a|\nabla v_0^\sigma|^2 - \pi a_0 d^2\ln(\kappa\Omega^{1/2})\right\} \le -\pi a_0 d^2\ell(\Lambda) + \frac{\delta}{2}. \tag{5.5}$$

In $\mathbb{R}^2\setminus B(0,\kappa^{-1}\Omega^{-1/2})$, we define

$$\hat{v}_\varepsilon(x) = \begin{cases} v_0^\sigma(x) & \text{if } x \in \Theta_\kappa, \\ v_0^\sigma\left(\dfrac{\sqrt{a_0}\,x}{|x|_\Lambda}\right) & \text{if } x \in \mathbb{R}^2\setminus\mathcal{D}. \end{cases}$$

By [14, Proposition 2.2], we have $\|\tilde{\eta}_\varepsilon^2\|_{L^\infty(\mathbb{R}^2\setminus\mathcal{D}_\varepsilon)} = o(1)$. Since $\hat{v}_\varepsilon$ does not depend on ε in $\mathbb{R}^2\setminus\mathcal{D}_\varepsilon$ and $|\hat{v}_\varepsilon| = 1$ in $\mathbb{R}^2\setminus\mathcal{D}_\varepsilon$, we derive that

$$\lim_{\varepsilon\to 0} \tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon, \mathbb{R}^2\setminus\mathcal{D}_\varepsilon) = 0. \tag{5.6}$$

From [14, Proposition 2.2], we also know that

$$\left\| \frac{a - \tilde{\eta}_\varepsilon^2}{\tilde{\eta}_\varepsilon^2} \right\|_{L^\infty(\mathcal{D}_\varepsilon)} \leq C\varepsilon^{1/3} \tag{5.7}$$

and hence, (5.5) remains valid if one replaces a by $\tilde{\eta}_\varepsilon^2$ in the left-hand side. Since v_0^σ is S^1-valued, we deduce that

$$\limsup_{\varepsilon\to 0} \left\{ \tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon, \mathbb{R}^2\setminus B(0,\kappa^{-1}\Omega^{-1/2})) - \pi a_0 d^2 \ln(\kappa\Omega^{1/2}) \right\} \leq -\pi a_0 d^2 \ell(\Lambda) + \frac{\delta}{2}. \tag{5.8}$$

Step 2. We are going to extend $\hat{v}_\varepsilon$ to $B(0,\kappa^{-1}\Omega^{-1/2})$. As in [8], we may write in a neighborhood of 0 (using polar coordinates),

$$v_0^\sigma(x) = \exp\big(i(d\theta + \psi_\sigma(x))\big),$$

where ψ_σ is a smooth function in that neighborhood. Let $(b_1, \ldots, b_d) \in \mathbb{R}^{2d}$ be a minimizing configuration for $w(\cdot)$, i.e.

$$w(b_1, \ldots, b_d) = \operatorname*{Min}_{b\in\mathbb{R}^{2d}} w(b) \tag{5.9}$$

(note that we necesarily have $b_i \neq b_j$ for $i \neq j$). We choose κ sufficiently small such that $\max|b_j| \leq 1/4\kappa$ and we set $b_j^{(\varepsilon)} = \Omega^{-1/2}\, b_j$. Following the proof of [5, Lemma 2.6], we write

$$e^{i\psi_\sigma(0)} \prod_{j=1}^{d} \frac{x - b_j^{(\varepsilon)}}{|x - b_j^{(\varepsilon)}|} = \exp\big(i(d\theta + \phi_\varepsilon(x))\big)$$
$$\text{for } x \in A_{\kappa,\varepsilon} = B(0,\kappa^{-1}\Omega^{-1/2})\setminus B(0,(2\kappa)^{-1}\Omega^{-1/2}),$$

where ϕ_ε is a smooth function satisfying $|\nabla\phi_\varepsilon(x)| \leq C_\sigma\kappa^2\Omega^{1/2})$ and $|\phi_\varepsilon(x) - \psi_\sigma(0)| = C_\sigma\,\kappa^2$ for $x \in A_{\kappa,\varepsilon}$. We define in $A_{\kappa,\varepsilon}$,

$$\hat{v}_\varepsilon(x) = \exp\big(i(d\theta + \hat{\psi}_\varepsilon(x))\big)$$

with

$$\hat{\psi}_\varepsilon(x) = \big(2 - 2\kappa\Omega^{1/2}|x|\big)\phi_\varepsilon(x) + \big(2\kappa\Omega^{1/2}|x| - 1\big)\psi_\sigma(x).$$

As in [5], we get that (using (5.7)),

$$\limsup_{\varepsilon\to 0}\left\{\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon, A_{\kappa,\varepsilon}) - \pi a_0 d^2 \ln 2\right\}$$
$$\le \limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{A_{\kappa,\varepsilon}} a_\sigma|\nabla\hat{v}_\varepsilon|^2 - \pi a_0 d^2\ln 2\right\} \le C_\sigma\kappa^2. \tag{5.10}$$

Next, we define $\hat{v}_\varepsilon$ in $\Xi_{\kappa,\varepsilon} = B(0,(2\kappa)^{-1}\Omega^{-1/2})\setminus\cup_{j=1}^d B(b_j^{(\varepsilon)}, 2\kappa\Omega^{-1/2})$ by

$$\hat{v}_\varepsilon(x) = e^{i\psi_\sigma(0)}\prod_{j=1}^d \frac{x-b_j^{(\varepsilon)}}{|x-b_j^{(\varepsilon)}|}.$$

Once more as in [5], we have (using (5.7)),

$$\limsup_{\varepsilon\to 0}\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon,\Xi_{\kappa,\varepsilon}) \le \limsup_{\varepsilon\to 0}\frac{1}{2}\int_{\Xi_{\kappa,\varepsilon}} a_\sigma|\nabla\hat{v}_\varepsilon|^2$$
$$\le \pi a_0(d^2+d)\ln\frac{1}{2\kappa} - \pi a_0\sum_{i\neq j}\ln|b_i-b_j| + C_\sigma\kappa. \tag{5.11}$$

Finally, in each $B_j^{(\varepsilon)} := B(b_j^{(\varepsilon)}, 2\kappa\Omega^{-1/2})$, we set

$$\hat{v}_\varepsilon(x) = e^{i\psi_\sigma(0)}\tilde{w}_\varepsilon^j\left(\frac{x-b_j^{(\varepsilon)}}{2\kappa\Omega^{-1/2}}\right), \tag{5.12}$$

where $\tilde{w}_\varepsilon^j$ realizes

$$\mathrm{Min}\left\{\frac{1}{2}\int_{B(0,1)}|\nabla v|^2 + \frac{1}{2\hat{\varepsilon}^2}(1-|v|^2)^2, v(y) = \prod_{i=1}^d\frac{2\kappa y + b_j - b_i}{|2\kappa y + b_j - b_i|} \text{ on } \partial B(0,1)\right\} \tag{5.13}$$

with

$$\hat{\varepsilon} = \frac{\varepsilon}{2\kappa\sqrt{a_0}\,\Omega^{-1/2}}.$$

As in the proof of [5, Lemma 2.3], we derive

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{B(0,1)}|\nabla\tilde{w}_\varepsilon^j|^2 + \frac{1}{2\hat{\varepsilon}^2}(1-|\tilde{w}_\varepsilon^j|^2)^2 - \pi|\ln\hat{\varepsilon}|\right\} = \gamma_0 + X(\kappa),$$

where γ_0 is defined in (4.15) and $X(\kappa)$ denotes a quantity satisfying $X(\kappa)\to 0$ as $\kappa\to 0$. By scaling, we obtain

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{B_j^{(\varepsilon)}}|\nabla\hat{v}_\varepsilon|^2 + \frac{a_0}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2 - \pi\ln\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\right\} = \frac{\pi}{2}\ln a_0 + \gamma_0 + X(\kappa).$$

Notice that in $B_j^{(\varepsilon)}$,

$$a_\sigma(x) = a(x) \le a_0 - (|\ln\varepsilon| + \omega_1\ln|\ln\varepsilon|)^{-1}\min_{y\in B(b_j,2\kappa)}\frac{a_0|y|_\Lambda^2}{1+\Lambda^2}$$

and consequently,

$$\limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{B_j^{(\varepsilon)}} a_\sigma|\nabla\hat{v}_\varepsilon|^2+\frac{a_0a_\sigma}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2-\pi a_0\ln\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\right\}$$
$$\leq\frac{\pi a_0}{2}\ln a_0+a_0\gamma_0-\frac{\pi a_0|b_j|_\Lambda^2}{1+\Lambda^2}+X(\kappa).$$

By (5.7), it yields

$$\limsup_{\varepsilon\to 0}\left\{\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon,B_j^{(\varepsilon)})-\pi a_0\ln\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\right\}$$
$$\leq\frac{\pi a_0}{2}\ln a_0+a_0\gamma_0-\frac{\pi a_0|b_j|_\Lambda^2}{1+\Lambda^2}+X(\kappa). \tag{5.14}$$

Combining (5.8), (5.10), (5.11) and (5.14), we conclude that for κ small enough,

$$\limsup_{\varepsilon\to 0}\left\{\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon)-\pi a_0 d|\ln\varepsilon|-\frac{\pi a_0}{2}(d^2-d)\ln|\ln\varepsilon|\right\}$$
$$\leq-\pi a_0\sum_{i\neq j}\ln|b_i-b_j|-\frac{\pi a_0}{1+\Lambda^2}\sum_{j=1}^{d}|b_j|_\Lambda^2+Q_{\Lambda,d}+\delta. \tag{5.15}$$

Step 3. Now, it remains to estimate $\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon)$. The Cauchy–Schwartz inequality yields

$$|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon,\mathbb{R}^2\backslash\mathcal{D}_\varepsilon)|\leq C\Omega\left(\int_{\mathbb{R}^2\backslash\mathcal{D}_\varepsilon}|x|^2\tilde{\eta}_\varepsilon^2\right)^{1/2}\left(\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon,\mathbb{R}^2\backslash\mathcal{D}_\varepsilon)\right)^{1/2}. \tag{5.16}$$

By [14, Proposition 2.2], $\Omega^2\int_{\mathbb{R}^2\backslash\mathcal{D}_\varepsilon}|x|^2\tilde{\eta}_\varepsilon^2\to 0$ as $\varepsilon\to 0$ and according to (5.6), it leads to

$$\lim_{\varepsilon\to 0}\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon)-\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon)\right|=0. \tag{5.17}$$

By the results in [8, Chap. IX], for $\hat{\varepsilon}$ sufficiently small and each $j=1,\ldots,d$, there exists exactly one disc $\hat{D}_\varepsilon^j\subset B(0,1)$ with $\operatorname{diam}(\hat{D}_\varepsilon^j)\leq C\hat{\varepsilon}$ such that $|\tilde{w}_\varepsilon^j|\geq 1/2$ in $B(0,1)\backslash\hat{D}_\varepsilon^j$. By scaling, we infer that exist exactly d discs $D_\varepsilon^1,\ldots,D_\varepsilon^d$ with $D_\varepsilon^j\subset B_j^{(\varepsilon)}$ and $\operatorname{diam}(D_\varepsilon^j)\leq C\varepsilon$ such that

$$|\hat{v}_\varepsilon|\geq\frac{1}{2}\quad\text{in }\mathcal{D}_\varepsilon\Big\backslash\bigcup_{j=1}^{d}D_\varepsilon^j.$$

We derive from (5.14) that

$$\left|\tilde{\mathcal{R}}_\varepsilon\left(\hat{v}_\varepsilon,\bigcup_{j=1}^{d}D_\varepsilon^j\right)\right|\leq C\Omega\varepsilon\sum_{j=1}^{d}\left(\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon,B_j^{(\varepsilon)})\right)^{1/2}\underset{\varepsilon\to 0}{\longrightarrow}0,$$

and by (5.17), it leads to $\lim_{\varepsilon\to 0}\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon)-\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\backslash\cup_{j=1}^{d}D_\varepsilon^j)\right|=0$. From (5.7), we infer that

$$\lim_{\varepsilon\to 0}\left|\tilde{\mathcal{R}}_\varepsilon\left(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\Big\backslash\bigcup_{j=1}^{d}D_\varepsilon^j\right)-\mathcal{R}_\varepsilon\left(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\Big\backslash\bigcup_{j=1}^{d}D_\varepsilon^j\right)\right|=0$$

and hence,

$$\lim_{\varepsilon\to 0}\left|\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon)-\mathcal{R}_\varepsilon\left(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\Big\backslash\bigcup_{j=1}^d D_\varepsilon^j\right)\right|=0. \tag{5.18}$$

To compute $\mathcal{R}_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}\backslash\cup_{j=1}^d D_\varepsilon^j)$, we proceed as in [14, Proposition 4.2] (here, we use that $\tilde{\mathcal{E}}_\varepsilon(\hat{v}_\varepsilon)\le C|\ln\varepsilon|$ by (5.15)). It yields

$$\lim_{\varepsilon\to 0}\left(\mathcal{R}_\varepsilon\left(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\Big\backslash\bigcup_{j=1}^d D_\varepsilon^j\right)+\frac{\pi\Omega}{1+\Lambda^2}\sum_{j=1}^d a^2(b_j^{(\varepsilon)})\right)=0$$

since $\deg(\hat{v}_\varepsilon/|\hat{v}_\varepsilon|,\partial D_\varepsilon^j)=+1$ for $j=1,\dots,d$. Expanding $a^2(b_j^{(\varepsilon)})$ and Ω, we deduce from (5.18) that

$$\lim_{\varepsilon\to 0}\left(\tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon)+\pi a_0 d\,|\ln\varepsilon|+\pi a_0\omega(\varepsilon)d\ln|\ln\varepsilon|\right)=\frac{2\pi a_0}{1+\Lambda^2}\sum_{j=1}^d|b_j|_\Lambda^2. \tag{5.19}$$

Combining (5.9), (5.15) and (5.19), we obtain the announced result. □

Proof of Theorem 5.1. We consider the map $\hat{v}_\varepsilon$ given in Proposition 5.2 and we set

$$\tilde{v}_\varepsilon=m_\varepsilon^{-1}\hat{v}_\varepsilon\quad\text{and}\quad\tilde{u}_\varepsilon=\tilde{\eta}_\varepsilon e^{i\Omega S}\tilde{v}_\varepsilon\quad\text{with}\quad m_\varepsilon=\|\tilde{\eta}_\varepsilon\hat{v}_\varepsilon\|_{L^2(\mathbb{R}^2)}.$$

We are going to prove that the map $\tilde{u}_\varepsilon$ satisfies the required property. By [14, Lemma 3.2], we have

$$F_\varepsilon(\tilde{u}_\varepsilon)=F(\tilde{\eta}_\varepsilon e^{i\Omega S})+\tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon)+\tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon).$$

In view of Proposition 5.2, it suffices to prove that $\left|\tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon)-\tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon)\right|\to 0$ and $\tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\to 0$ as $\varepsilon\to 0$. We first estimate m_ε. Since $|\hat{v}_\varepsilon|=1$ in $\mathbb{R}^2\setminus\cup_{j=1}^d B_j^{(\varepsilon)}$ and $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)}=1$, we have

$$m_\varepsilon^2=\int_{\mathbb{R}^2}\tilde{\eta}_\varepsilon^2+\int_{\cup_{j=1}^d B_j^{(\varepsilon)}}\tilde{\eta}_\varepsilon^2(|\hat{v}_\varepsilon|^2-1)=1+\int_{\cup_{j=1}^d B_j^{(\varepsilon)}}\tilde{\eta}_\varepsilon^2(|\hat{v}_\varepsilon|^2-1).$$

Using the Cauchy–Schwarz inequality, we derive from (5.12), (5.13) and [8, Theorem III.2] that

$$\begin{aligned}\left|\int_{\cup_{j=1}^d B_j^{(\varepsilon)}}\tilde{\eta}_\varepsilon^2(|\hat{v}_\varepsilon|^2-1)\right|&\le C|\ln\varepsilon|^{-1/2}\left(\int_{\cup_{j=1}^d B_j^{(\varepsilon)}}(|\hat{v}_\varepsilon|^2-1)^2\right)^{1/2}\\&\le C\varepsilon|\ln\varepsilon|^{-1/2}\end{aligned} \tag{5.20}$$

and thus

$$m_\varepsilon^2=1+O(\varepsilon|\ln\varepsilon|^{-1/2}). \tag{5.21}$$

Using $|\hat{v}_\varepsilon| = 1$ in $\mathbb{R}^2 \setminus \cup_{j=1}^d B_j^{(\varepsilon)}$, $|\nabla S| \le C|x|$, $|k_\varepsilon| \le C|\ln \varepsilon|$, (5.20) and (5.21), we derive that

$$\begin{aligned}\left|\tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\right| \le C|\ln \varepsilon|^2 \Bigg(&|1 - m_\varepsilon^{-2}| \int_{\mathbb{R}^2} (1 + |x|^2)\tilde{\eta}_\varepsilon^2 \\ &+ \int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde{\eta}_\varepsilon^2 \left(|1 - m_\varepsilon^{-2}||\hat{v}_\varepsilon|^2 + (1 - |\hat{v}_\varepsilon|^2)\right)\Bigg) \\ \le C\varepsilon|\ln \varepsilon|^{3/2}&.\end{aligned}$$

Now, we may estimate using (5.15), (5.19) and (5.21),

$$\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |\nabla \tilde{v}_\varepsilon|^2 = m_\varepsilon^{-2} \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |\nabla \hat{v}_\varepsilon|^2 = \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |\nabla \hat{v}_\varepsilon|^2 + O(\varepsilon|\ln \varepsilon|^{1/2}), \tag{5.22}$$

and

$$\tilde{\mathcal{R}}_\varepsilon(\tilde{v}_\varepsilon) = m_\varepsilon^{-2} \tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) = \tilde{\mathcal{R}}_\varepsilon(\hat{v}_\varepsilon) + O(\varepsilon|\ln \varepsilon|^{1/2}). \tag{5.23}$$

We write

$$\begin{aligned}\frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4 (1 - |\tilde{v}_\varepsilon|^2)^2 = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} &\tilde{\eta}_\varepsilon^4 (1 - |\hat{v}_\varepsilon|^2)^2 + \frac{2(1 - m_\varepsilon^{-2})}{\varepsilon^2} \\ &\times \int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde{\eta}_\varepsilon^4 (1 - |\hat{v}_\varepsilon|^2)|\hat{v}_\varepsilon|^2 \\ &+ \frac{(1 - m_\varepsilon^{-2})^2}{\varepsilon^2} \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4 |\hat{v}_\varepsilon|^4.\end{aligned} \tag{5.24}$$

We infer from (5.15) and (5.21) that

$$\frac{(1 - m_\varepsilon^{-2})^2}{\varepsilon^2} \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^4 |\hat{v}_\varepsilon|^4 \le C|\ln \varepsilon|^{-1}, \tag{5.25}$$

and from (5.20) and (5.21),

$$\frac{|1 - m_\varepsilon^{-2}|}{\varepsilon^2} \int_{\cup_{j=1}^d B_j^{(\varepsilon)}} \tilde{\eta}_\varepsilon^4 |\hat{v}_\varepsilon|^2 \left|1 - |\hat{v}_\varepsilon|^2\right| \le C|\ln \varepsilon|^{-1}. \tag{5.26}$$

Combining (5.22)–(5.26), we finally obtain that $\tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) = \tilde{\mathcal{F}}_\varepsilon(\hat{v}_\varepsilon) + o(1)$ and the proof is complete. □

Acknowledgments

We express our gratitude to A. Aftalion, who suggested this problem to us, for his very helpful suggestions and comments. We also thank E. Sandier and I. Shafrir for very interesting discussions, and H. Brezis for his hearty encouragement and constant support. The research of the authors was partially supported by the RTN Program "Fronts-Singularities" of European Commission, HPRN-CT-2002-00274.

Appendix A

In this appendix, we prove that the functions Ψ_R and Ψ_σ defined by (4.20) and, respectively, (5.4) converge to the same limiting function as $R \to \sqrt{a_0}$ and $\sigma \to 0$. The proof is based on the construction of suitable *barrier* functions.

Lemma A.1 *For any* $0 < R < \sqrt{a_0}$*, respectively, any* $\sigma > 0$*, let* Ψ_R *be the solution of Eq.* (4.20)*, respectively,* Ψ_σ *the solution of* (5.4)*. Then,* $\Psi_R \to \Psi_\star$ *as* $R \to \sqrt{a_0}$*, respectively,* $\Psi_\sigma \to \Psi_\star$ *as* $\sigma \to 0$*, in* $C^1_{\mathrm{loc}}(\mathcal{D})$ *where* $\Psi_\star$ *is the unique solution in* $C^0(\bar{\mathcal{D}})$ *of*

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{a}\nabla\Psi_\star\right) = \dfrac{-2|x|^2_\Lambda}{a^2(x)|x|^2} & \textit{in } \mathcal{D}, \\ \Psi_\star = -\ln|x| & \textit{on } \partial\mathcal{D}. \end{cases} \tag{A.1}$$

In particular,

$$\lim_{R\to\sqrt{a_0}} \Psi_R(0) = \lim_{\sigma\to 0} \Psi_\sigma(0) = \Psi_\star(0) =: \ell(\Lambda). \tag{A.2}$$

Proof. Step 1. *Uniqueness of* $\Psi_\star$. Assume that (A.1) admits two solutions $\Psi^1_\star$ and $\Psi^2_\star$ in $C^0(\bar{\mathcal{D}})$. Then, the difference $\Psi^1_\star - \Psi^2_\star$ satisfies $\operatorname{div}(\frac{1}{a}\nabla(\Psi^1_\star - \Psi^2_\star)) = 0$ in $\mathcal{D}$ and $\Psi^1_\star - \Psi^2_\star = 0$ on $\partial\mathcal{D}$. By elliptic regularity, we infer that $\Psi^1_\star - \Psi^2_\star \in C^2(\mathcal{D}) \cap C^0(\bar{\mathcal{D}})$. Hence, it follows $\Psi^1_\star - \Psi^2_\star \equiv 0$ by the classical maximum principle.

Step 2: *Existence of* $\Psi_\star$. We set for $y \in \mathcal{D}$,

$$\Upsilon_R(y) = \Psi_R\left(\frac{Ry}{\sqrt{a_0}}\right) - \zeta(y) + \ln(R/\sqrt{a_0}),$$

where ζ is the solution of

$$\begin{cases} \Delta\zeta = 0 & \text{in } \mathcal{D}, \\ \zeta = -\ln|y| & \text{on } \partial\mathcal{D}. \end{cases}$$

Since Ψ_R solves (4.20), we deduce that Υ_R is the unique solution of

$$\begin{cases} -\operatorname{div}\left(\dfrac{1}{a_R(y)}\nabla\Upsilon_R\right) = \dfrac{f(y)}{a_R^2(y)} & \text{in } \mathcal{D}, \\ \Upsilon_R = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{A.3}$$

where $a_R(y) = a_0^2/R^2 - |y|^2_\Lambda$ and

$$f(y) = \frac{2|y|^2_\Lambda}{|y|^2} + 2(y_1, \Lambda^2 y_2)\cdot\nabla\zeta(y).$$

We easily check that $y \mapsto K a_R(y)$, respectively, $y \mapsto -K a_R(y)$, defines a supersolution, respectively, a subsolution, of (A.3) whenever the constant K satisfies $K \geq \|f\|_{L^\infty(\mathcal{D})}/(\Lambda^2 a_0)$. Hence,

$$|\Upsilon_R| \leq C a_R \quad \text{in } \mathcal{D} \tag{A.4}$$

for a constant C independent of R. By elliptic regularity, we deduce that Υ_R remains bounded in $W^{2,p}_{\mathrm{loc}}(\mathcal{D})$ as $R \to \sqrt{a_0}$ for any $1 \leq p < \infty$. Therefore, from any sequence

$R_n \to \sqrt{a_0}$, we may extract a subsequence, still denoted by (R_n), such that $\Upsilon_{R_n} \to \Upsilon_\star$ in $C^1_{\text{loc}}(\mathcal{D})$ where $\Upsilon_\star$ satisfies

$$-\text{div}\left(\frac{1}{a(y)}\nabla\Upsilon_\star\right) = \frac{f}{a^2(y)} \quad \text{in } \mathcal{D}.$$

We infer from (A.4) that $|\Upsilon_\star(y)| \le Ca(y)$ for any $y \in \mathcal{D}$ and hence, $\Upsilon_\star \in C^0(\bar{\mathcal{D}})$ with $\Upsilon_{\star|\partial\mathcal{D}} = 0$. Consequently, the function $\Psi_\star := \Upsilon_\star + \zeta$ defines a solution of (A.1) which is continuous in $\bar{\mathcal{D}}$.

Step 3. By the uniqueness of $\Psi_\star$, we have that $\Upsilon_R \to \Psi_\star - \zeta$ in $C^1_{\text{loc}}(\mathcal{D})$ as $R \to \sqrt{a_0}$ which clearly implies $\Psi_R \to \Psi_\star$ in $C^1_{\text{loc}}(\mathcal{D})$ as $R \to \sqrt{a_0}$. To prove that $\Psi_\sigma \to \Psi_\star$ in $C^1_{\text{loc}}(\mathcal{D})$ as $\sigma \to 0$, we may proceed as in Step 2. Indeed, we may show as in Step 2, that $|\Psi_\sigma - \zeta| \le Ca_\sigma$ in $\mathcal{D}$ for a constant C independent of σ. □

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