1. For $\alpha \in \mathbb{R}$, consider the function

$$F_\alpha(t) = \begin{cases} 
0, & t < -1, \\
\alpha(t - 1) + \frac{1}{2}, & -1 \leq t < 1, \\
1, & t \geq 1.
\end{cases}$$

Find all $\alpha$ such that $F_\alpha$ is the distribution function of a random variable. For those $\alpha$, let $X_\alpha$ be a random variable with the distribution function $F_\alpha$. Find $\mathbb{P}(X_\alpha = -1)$, $\mathbb{P}(X_\alpha = 1)$ and $\mathbb{P}(X_\alpha > 0)$. Is $X_\alpha$ a continuous random variable? Find the distribution function of $Y = X_0^2$.

2. Let $X$ and $Y$ be independent random variables such that $X$ is uniformly distributed on $[-1, 1]$ and $Y$ has the exponential distribution with parameter 1. Find $\mathbb{E}[(X + Y)^2]$.

3. Let $X$ and $Y$ be independent standard Gaussian random variables. Let $Z = 2X - Y$. Is $Z$ a Gaussian random variable? Find the mean and variance of $Z$. Find the density of $Z$. Consider the random vector $V = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$. Is $V$ a Gaussian random vector? Find the density of $V$. Does $V$ have independent coordinates?

4. Let $G$ be a standard Gaussian vector in $\mathbb{R}^2$. Let $u$ and $v$ be unit vectors in $\mathbb{R}^2$. Show that

$$\mathbb{E}\left[ \langle u, G \rangle \langle v, G \rangle \right] = \langle u, v \rangle$$

and

$$\mathbb{E}\left[ \text{sgn}(\langle u, G \rangle) \text{sgn}(\langle v, G \rangle) \right] = \frac{2}{\pi} \arcsin(\langle u, v \rangle).$$

Here $\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \rangle = x_1y_1 + x_2y_2$ is the standard scalar product and $\text{sgn}(t) = \begin{cases} 1, & t > 0, \\
0, & t = 0, \\
-1, & t < 0. \end{cases}$