1. Let $X$ be a nonnegative random variable. Show that for $p > 0$ we have
   \[ \mathbb{E}X^p = \int_0^{\infty} pt^{p-1} \mathbb{P}(X > t) \, dt. \]

2. Let $X$ be a random variable such that $\mathbb{E}|X|^p < \infty$ for some $p > 0$. Show that
   \[ \lim_{t \to \infty} t^p \mathbb{P}(|X| > t) = 0. \]

3. Show that the probability that in $n$ throws of a fair die the number of sixes lies between
   \[ \frac{1}{6}n - \sqrt{n} \text{ and } \frac{1}{6}n + \sqrt{n} \]
   is at least $\frac{31}{36}$.

4. Let $X$ be a random variable with values in an interval $[0, a]$. Show that for every $t$ in this interval we have
   \[ \mathbb{P}(X \geq t) \geq \frac{\mathbb{E}X - t}{a - t}. \]

5. Prove the Paley-Zygmund inequality: for a nonnegative random variable $X$ and every $\theta \in [0, 1]$ we have
   \[ \mathbb{P}(X > \theta \mathbb{E}X) \geq (1 - \theta)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}. \]

6. Let $\varepsilon_1, \ldots, \varepsilon_n$ be independent random signs. Prove that there is a positive constant $c$ such that for every $n \geq 1$ and real numbers $a_1, \ldots, a_n$ we have
   \[ \mathbb{P} \left( \left| \sum_{i=1}^{n} a_i \varepsilon_i \right| > \frac{1}{2} \sqrt{\sum_{i=1}^{n} a_i^2} \right) \geq c. \]
   
   \textit{Hint. Use the Paley-Zygmund inequality and Q4 HW5.}

7. Prove that for nonnegative random variables $X$ and $Y$ we have
   \[ \mathbb{E} \frac{X}{Y} \geq \left( \frac{\mathbb{E}\sqrt{X}}{\mathbb{E}Y} \right)^2. \]

8. Suppose that $X = (X_1, \ldots, X_n)$ is a random vector uniformly distributed on the cube $[-\sqrt{3}, \sqrt{3}]^n$. Show that $X_1, \ldots, X_n$ are independent. Find $\mathbb{E}X_1$, $\mathbb{E}X_1^2$ and $\text{Var}(X_1^2)$. Let $\|X\| = X_1^2 + \ldots + X_n^2$ denote the distance from the point $X$ to the origin. Show
   \[ \mathbb{E}\|X\| - \sqrt{n} \|^2 < 1. \]
   
   Conclude that for $t > 0$,
   \[ \mathbb{P} \left( \|X\| - \sqrt{n} > t \right) < \frac{1}{t^2}. \]
   
   In particular, $\mathbb{P} \left( \|X\| - \sqrt{n} > 10 \right) \leq 1/100$, which implies that a random point $X$ lands in the thin shell of width 20 around the sphere with radius $\sqrt{n}$ with probability greater than 99/100 (think of $n$ as being large).