1. Let \((V, \| \cdot \|)\) be a normed vector space. Fix a vector \(v \in V\). For every \(x \in V\) define
\[
\|x\|' = \|x + \|x\|v\| + \|x - \|x\|v\|.
\]
Show that \(\| \cdot \|'\) is a norm on \(V\) which is equivalent to \(\| \cdot \|\).

*Hint*: The function \(\mathbb{R} \ni t \mapsto \|x + tv\| + \|x - tv\| \in [0, +\infty)\) is even and convex.

2. Is it true that for every vector space \(V\) there is a function \(N : V \rightarrow [0, +\infty)\) which is a norm on \(V\)? (In other words, is every vector space normable?)

3. We know that any two norms on a finite dimensional vector space are equivalent. Does there exist an infinite dimensional vector space \(V\) with the property that any two norms on \(V\) are equivalent?

4. Determine whether the following functional spaces
\[
C(\mathbb{R}) = \{ f : \mathbb{R} \rightarrow \mathbb{R}, \ f \text{ is continuous} \},
\]
\[
C_{\text{van}}(\mathbb{R}) = \{ f \in C(\mathbb{R}), \ \lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0 \},
\]
\[
C_{\text{bd}}(\mathbb{R}) = \{ f \in C(\mathbb{R}), \ f \text{ is bounded} \},
\]
\[
C_{0}(\mathbb{R}) = \{ f \in C(\mathbb{R}), \ \text{cl}\{x \in \mathbb{R}, \ f(x) \neq 0\} \text{ is compact} \},
\]
equipped with the supremum norm are Banach spaces.

5. Give an example of a Banach space which is not a Hilbert space, that is whose norm is not induced by any scalar product.

*Hint*: Parallelogram law.

6* Give an example of a nonseparable Hilbert space.

7. Does there exist a bounded linear operator on Hilbert space with empty point spectrum?

8. Solve the Sturm-Liouville problem for the Laplacian operator on the interval \((0, 1)\), that is find the \(\lambda\) for which the problem
\[
\begin{cases}
 u''(x) = \lambda u(x), & \text{on } (0, 1), \\
u(0) = u(1) = 0
\end{cases}
\]
has a nontrivial solution \(u\) and show that the corresponding solutions (eigen-vectors) form an orthonormal basis of \(L^2(0, 1)\).