1. MATRICES  $AB - BA$

Definition For a matrix $A = [a_{ij}]_{i,j \leq n}$ we define its trace as
$$\text{tr } A = \sum_{i=1}^{n} a_{ii}.$$ 

$$\text{tr } AB = \text{tr } BA,$$ or $$\text{tr } (AB - BA) = 0.$$

$\forall$ tr is a good invariant since plugging $B^{-1}A$ instead of $A$ we get
$$\text{tr } B^{-1}AB = \text{tr } A.$$

Question 1 Suppose $AB - BA = A$ for some $A, B \in M_{n \times n}(\mathbb{R})$. Prove that $\det A = 0$.

Solution Assume conversely that $A$ is invertible. Then by the assumption
$$B - A^{-1}BA = I.$$ 

Taking trace yields a contradiction. $\square$

Question 2 Let $A, B \in M_{2 \times 2}(\mathbb{R})$ satisfy $(AB - BA)^n = I$ for some natural number $n$. Prove that

(a) $2 | n$
(b) $(AB - BA)^4 = I$.

Solution Since $\text{tr } (AB - BA) = 0$ we can write
$$AB - BA = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = X.$$

Then
$$X^2 = \begin{bmatrix} a^2 + bc & 0 \\ 0 & a^2 + bc \end{bmatrix} = (a^2 + bc)I,$$
$$X^4 = (a^2 + bc)^2 I.$$
\[ X^3 = X^2 \cdot X = xI \cdot X = xX, \]
\[ X^4 = x \cdot x^2 = x^2 I, \]
\[ \vdots \]
\[ X^{2k+1} = x^k I, \]
\[ X^{2k} = x^k X. \]

We know that \( I = X^n = \begin{cases} 
  x^k I & \text{if } n = 2k \\
  x^k X & \text{if } n = 2k+1 
\end{cases} \).

Clearly the second case does not hold (take e.g. \( tr \)).

So \( n = 2k \), which proves (a).

For (b) notice that \( I = x^k I \) implies \( x^k = 1 \),
so \( x = \pm 1 \), hence \( x^2 = 1 \) and
\[ X^4 = x^2 I = I. \]

\[ \square \]

2. NILPOTENT MATRICES

Definition: A matrix \( A \in M_{n \times n}(\mathbb{R}) \) is called nilpotent if \( A^n = 0 \) for some natural \( n > 0 \).

Question 3: Let an \( n \times n \) matrix \( A \) be nilpotent. Show that \( A^n = 0 \).

Solution: Suppose \( A^n \neq 0 \). Then there exists a vector \( v \) such that \( A^n v \neq 0 \). Consider the vectors
\[ v, Av, ..., A^n v. \]
They are linearly independent which contradicts the fact that they are in \( \mathbb{R}^n \) and there are \( n+1 \) of them. Indeed,
\[ \alpha_0 v + \alpha_1 Av + ... + \alpha_n A^n v = 0. \]
Solution: Take the smallest $m$ such that $A^m = 0$ (if $A$ is nilpotent, so such $m$ exists). Since $A^{m-1} \neq 0$ there is a vector $v$ such that $A^{m-1}v \neq 0$. Consider the vectors $\mathbf{0}, Av, \ldots, A^{m-1}v$.

They are linearly independent. Indeed,

\[
\alpha_0 v + \alpha_1 Av + \ldots + \alpha_{m-1} A^{m-1} v = 0 \quad / \quad A^{m-1}
\]

\[
\alpha_0 A^{m-1} v = 0 \Rightarrow \alpha_0 = 0,
\]

so

\[
\alpha_1 A v + \ldots + \alpha_{m-1} A^{m-1} v = 0 \quad / \quad A^{m-1}
\]

\[
\alpha_1 A^{m-1} v = 0 \Rightarrow \alpha_1 = 0,
\]

and so on.

Therefore $m \leq \dim V = n$. □

**Question 4** Let $n \times n$ matrices $A, B$ be such that $A + tB$ is nilpotent for some $n+1$ distinct reals $t_1, \ldots, t_{n+1}$. Show that $A$ and $B$ are nilpotent too.

**Solution** Consider the matrix

\[
C(t) = (A + tB)^n = A^n + \ldots + t^n B^n.
\]

We know that $C(t; c) = 0$, $c = 1, \ldots, n+1$. At each entry, $C(t)$ is a polynomial of degree $n$ with respect to $t$. Since such a polynomial has got $n+1$ distinct roots, it must be the zero polynomial. In particular each entry of $A^n$ and $B^n$ equals 0 which means that $A^n = 0 = B^n$. □

**Question 5** $A \in M_{n \times n}(\mathbb{C})$ is nilpotent iff $\text{tr} A^k = 0$, for $k = 1, \ldots$.
3. RANK

**Definition** The rank of a matrix $A \in \mathbb{M}_{m \times n}(\mathbb{R})$ is the dimension of the subspace spanned by its columns or, equivalently, rows,

$$\text{rank } A = \dim \text{ span } \lbrace \text{column of } A \rbrace = \dim \text{ span } \lbrace \text{rows of } A \rbrace \Rightarrow \text{rank } A = \text{rank } A^T.$$ 

In other words

$$\text{rank } A = \dim \text{ im } A,$$

when we consider $A$ as a linear mapping $\mathbb{R}^n \to \mathbb{R}^m$.

**Question 6** Prove that for any $A, B \in \mathbb{M}_{m \times n}(\mathbb{R})$ we have

(a) $\text{rank } AB \leq \text{rank } A \cdot \text{rank } B$

(b) $\text{rank } AB \geq \text{rank } A + \text{rank } B - n$.

**Solution** (a) Consider $A, B$ as linear maps $f, g: \mathbb{R}^n \to \mathbb{R}^m$.

Clearly

$$\text{im } fog \subset \text{im } f,$$

so

$$\text{rank } AB \leq \text{rank } A.$$ 

Moreover

$$\text{rank } (AB)^T = \text{rank } B^TA^T \leq \text{rank } B^T = \text{rank } B,$$

which proves (a).

(b)  

\[ \text{dim} \text{ im } fog \geq \text{dim} \text{ im } g \]

\[ = \text{dim im g} + \text{dim ker f} - n \]

\[ \geq \text{dim im g} + \text{dim im f} - n \]

As a corollary we find

**\( \triangle \)** rank is a good invariant too

$$\text{rank } A = \text{rank } B^{-1}AB.$$
Question 7 Let \( A_1, \ldots, A_k \) be \( n \times n \) matrices of rank \( n-1 \). Prove that if \( k < n \) then \( A_1, \ldots, A_k \neq 0 \).

Solution By Question 6 (b) we get
\[
\text{rank } (A_1, \ldots, A_k) \geq \sum_{i=1}^{k} \text{rank } A_i - (k-1) \cdot n
\]
\[
= k \cdot (n-1) - (k-1) \cdot n = n-k > 0. \square
\]

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Question 8 Let \( A, B \in M_{n \times n} (\mathbb{R}) \) satisfy \( AB - BA = 0 \cdot A \).

Prove that

(a) \( A^k B - BA^k = k \cdot A^k \) for all integers \( k \geq 0 \),

(b) \( A \) is nilpotent.

Solution (a) easy induction.

(b) Consider the linear map \( L : M_{n \times n} (\mathbb{R}) \to M_{n \times n} (\mathbb{R}) \)
\[
L(X) = XB - BX.
\]

(a) says that \( A^k \) is an eigenvector of \( L \) with eigenvalue \( k \). Since \( L \) must have infinitely many distinct eigenvalues, there must be \( A^k = 0 \) for \( k \geq m \) for certain \( m \). □

Question 9 There are \( n \geq 2 \) people sitting at the round table. Each person has got a prime number written on a sticky note. At each minute a certain person modifies her or his number multiplying it by a number of a neighbour. Is it possible that at some point two people have got the same number?

Solution At time \( t \), the \( i \)-th person has got a number \( p_{1}^{x_{1,i}^{(t)}} \cdots p_{m}^{x_{m,i}^{(t)}} \), where \( p_j \) is the prime attached at the very beginning to the \( j \)-th person. Consider the matrix
\[ A_t = [ x_{i,j}^{(t)} ] : i,j \leq n. \]

Clearly \( A_0 = I \). Moreover, \( A_{t+1} \) is obtained from \( A_t \) by adding to the \( k \)-th column the \( k-1 \) or \( k+1 \)-th one. Therefore

\[ \text{rank } A_t = \text{const } = n. \]

Yet, if there were two people at time \( t^* \) with the same number, two columns of \( A_{t^*} \) would be the same. That would contradict the above. \( \square \)

5. Homework

**Question 1** Suppose that 2x2 real matrices \( A \) and \( B \) satisfy \( AB - BA = B^2 \). Prove that \( AB = BA \).

**Question 2** Let \( X \) be an \( n \times n \) real invertible matrix with columns \( x_1, x_2, \ldots, x_n \). Let \( Y \) be the matrix with columns \( X_2, X_3, \ldots, X_n, 0 \). Prove that \( YX^{-1} \) is a rank \( n-1 \) matrix.