A few baby inequalities

Question 1. Prove that for real numbers $x_1, \ldots, x_n$ we have

$$|x_1 + \ldots + x_n| \leq |x_1| + \ldots + |x_n|.$$ 

Question 2. Given a natural number $n \geq 1$ find minimum value of $\sum_{i=1}^{n} \sqrt{a_i^2 + (2i - 1)^2}$ subject to positive numbers $a_i$ satisfying $\sum_{i=1}^{n} a_i = n^2$.

Question 3. Prove that for $x \geq 1$ and $h > 0$ we have

$$\sqrt{x+h} - \sqrt{x} \leq \frac{h}{2}.$$ 

Does this hold for all $x > 0$?

Question 4. Prove that for positive numbers $x_1, \ldots, x_n$ the arithmetic mean is greater or equal that the geometric mean,

$$\frac{n}{\sqrt[n]{x_1 \cdots x_n}} \leq \frac{x_1 + \ldots + x_n}{n}.$$ 

Question 5. Prove that for numbers $x_1, \ldots, x_n > -1$ that have the same sign we have

$$(1 + x_1) \cdots (1 + x_n) \geq 1 + x_1 \ldots + x_n.$$ 

Question 6 (Bernoulli’s inequality). Prove that for a real number $x > -1$ and a natural number $n \geq 1$ we have

$$(1 + x)^n \geq 1 + nx.$$ 

Question 7 (The Cauchy-Schwarz inequality). Prove that for real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ we have

$$\sum_{i=1}^{n} x_i y_i \leq \sqrt{\sum_{i=1}^{n} x_i^2} \sqrt{\sum_{i=1}^{n} y_i^2}.$$ 

Question 8 (Hölder’s inequality). Prove that for real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ and $p, q \geq 1$ which satisfy $1/p + 1/q = 1$ we have

$$\sum_{i=1}^{n} x_i y_i \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |y_i|^q \right)^{1/q}.$$ 

Question 9. Prove that for $x \in (0, \pi/2)$ we have

$$\sin x < x < \tan x.$$
Question 10. Prove that for $x \in \mathbb{R}$ we have 

$$1 + x \leq e^x.$$ 

Question 11. Prove that for $x > 0$ we have 

$$\frac{x}{x + 1} \leq \ln(1 + x) \leq x.$$ 

A FEW ESTIMATES INVOLVING FACTORIALS

Question 12. Prove that 

$$\left(\frac{n}{e}\right)^n \leq n! \leq 3 \frac{n^{n+1}}{e^n}.$$ 

Conclude that $\sqrt[n]{n!/n} \to 1/e.$

Question 13. Prove that 

$$n^{n/2} \leq n! \leq \left(\frac{n+1}{2}\right)^n.$$ 

Question 14. Prove that 

$$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k.$$ 

Question 15. Prove that 

$$\frac{4^n}{2\sqrt{n}} \leq \binom{2n}{n} \leq 4^n.$$ 

Question 16. Give a natural number $n \geq 1$ prove that the sequence $a_k = \binom{n}{k}$ is log-concave, i.e. $a_k^2 \geq a_{k-1}a_{k+1}$ for $k = 2, \ldots, n-1.$

Question 17 (†). Prove that for a natural number $n \geq 1$ we have 

$$\frac{e}{2n+2} < e - \left(1 + \frac{1}{n}\right)^n < \frac{e}{2n+1}.$$ 

Conclude that 

$$\left(\frac{(1+1/n)^n}{e}\right)^n \to 1/\sqrt{e}. $$
**Analysis II, Term 2 2012/2013**

Tomasz Tkocz

Support class 2

**Question 1** (♥). Let \( f(x) = \sum_{k=-2013}^{2013} |x - k| \). Given \( c \in \mathbb{R} \) and \( \epsilon > 0 \) find \( \delta > 0 \) such that

\[
\forall x \in \mathbb{R} \quad |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.
\]

Hint: You may want to prove that for all \( x, y \in \mathbb{R} \)

\[
|f(x) - f(y)| \leq 4027|x - y|.
\]

**Question 2** (♥). Find all \( a \in \mathbb{R} \) such that the function \( f_a: \mathbb{R} \to \mathbb{R} \) given by

\[
f_a(x) = \begin{cases} 
\min\{1/|x|, a\}, & x \neq 0 \\
 a^2 - 1, & x = 0
\end{cases}
\]

is continuous.

**Question 3** (♥). Let \( f, g: \mathbb{R} \to \mathbb{R} \) be continuous functions. Prove that the functions \( M(x) = \max\{f(x), g(x)\} \) and \( m(x) = \min\{f(x), g(x)\} \) are also continuous.

**Question 4** (♥). Suppose that for some \( c \in \mathbb{R} \) a function \( f: \mathbb{R} \to \mathbb{R} \) satisfies the following property

\[
\forall (x_n)_{n=1}^{\infty} \ x_n \to c \implies f(x_n) \to f(c).
\]

Prove that \( f \) is continuous at \( c \).

**Remark.** Cf. Exercise 6 from Assignment 1.

**Question 5** (♠). Let \( f, g: \mathbb{R} \to \mathbb{R} \) be bounded below, say \( f(x), g(x) \geq 0 \) for all \( x \in \mathbb{R} \). Suppose that \( |g(x) - g(y)| \leq |x - y| \) for all \( x, y \in \mathbb{R} \). Prove that the function \( f \Box g \) defined by

\[
(f \Box g)(x) = \inf_{t \in \mathbb{R}} \{ f(x) + g(x - t) \}
\]

is continuous.
Question 1 (♥). Let \( f: [0, +\infty) \rightarrow [0, +\infty) \) satisfy for all \( x, y \geq 0 \)
\[
|f(x) - f(y)| \leq q|x - y|,
\]
with some constant \( q \in (0, 1) \). Fix \( x_0 \geq 0 \) and define recursively the sequence \((x_n)_{n\geq0}\) by \( x_{n+1} = f(x_n) \), \( n \geq 0 \). Prove that it converges. What can be said about the limit?

Question 2 (♥). Let \( f(x) = \sqrt{1 + x} \) for \( x \geq 0 \). Prove that for any \( x_0 \geq 0 \) the sequence \((x_n)_{n\geq0}\) defined recursively by \( x_{n+1} = f(x_n) \) for \( n \geq 0 \) converges and compute the limit.

Question 3 (♥). Define the function
\[
f(x) = \begin{cases} 
0 & \text{if } x \in \mathbb{Q} \\
|\sin x| & \text{if } x \notin \mathbb{Q}.
\end{cases}
\]
At which points is \( f \) continuous?

Definition. We say that a function \( f: (A,B) \rightarrow \mathbb{R} \) possesses the Intermediate Value Property if for any \( a < b \) in the domain such that \( f(a) \neq f(b) \), and any \( z \) between \( f(a) \) and \( f(b) \) there is some \( c \in (a,b) \) between \( a \) and \( b \) with \( f(c) = z \).

Question 4 (♠). Give an example of a function which is not continuous, and yet has got the Intermediate Value Property.

Question 5 (♥). Prove that the equation \((1-x)\cos x = \sin x\) has a solution in the interval \((0,1)\).

Question 6 (♥). Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be a \( T \) - periodic continuous function, i.e. \( f(x+T) = f(x) \) for all \( x \in \mathbb{R} \), where \( T > 0 \) is the period. Prove that there exists \( x_0 \) such that
\[
f(x_0 + T/2) = f(x_0).
\]

Question 7 (♥/♠). Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be additive, i.e. satisfy for all \( x, y \in \mathbb{R} \) Cauchy’s functional equation
\[
f(x + y) = f(x) + f(y).
\]
Prove that
(a) \( f(0) = 0 \)
(b) \( f(-x) = f(x) \)
(c) For any \( x_1, \ldots, x_n \in \mathbb{R} \) we have \( f(x_1 + \ldots + x_n) = f(x_1) + \ldots + f(x_n) \)
(d) For an integer \( k \) and a real number \( x \) we have \( f(kx) = kf(x) \)
(e) For a rational number $q$ we have $f(q) = qf(1)$

(f) In addition, if $f$ satisfies one of these assumptions:

   (i) $f$ is continuous
   (ii) $f$ is continuous at one point
   (iii) $f$ is monotone
   (iv) $f$ is bounded above/below

then $f(x) = xf(1)$ for all $x \in \mathbb{R}$. 
Question 1 (♥/♠). Compute the following limit (if it exists)
(a) \( \lim_{x \to 0} x \cos \frac{1}{x} \)
(b) \( \lim_{x \to +\infty} x \left( \sqrt{x^2 + 1} - \sqrt{x^4 + 1} \right) \)
(c) \( \lim_{x \to 0} \frac{\cos \left( \frac{x}{2} \cos x \right)}{\sin (\sin x)} \).

Question 2 (♥). Prove that for any \( x \in \mathbb{R} \) we have \( x - 1 < |x| \leq x \).

Question 3 (♥). Compute the following limit (if it exists)
(a) \( \lim_{x \to 0} x \left| \frac{1}{x} \right| \)
(b) \( \lim_{x \to 0} \frac{|x|}{x} \).
(c) \( \lim_{x \to 0} x^2 \left( 1 + 2 + \ldots + \left| \frac{1}{|x|} \right| \right) \).

Question 4 (♠). Let \( f: \mathbb{R} \longrightarrow \mathbb{R} \) be an increasing function such that \( \lim_{x \to \infty} \frac{f(2x)}{f(x)} = 1 \). Prove that \( \lim_{x \to \infty} \frac{f(cx)}{f(x)} = 1 \) for every \( c > 0 \).

Question 5 (★). Let \( f: [0, \infty) \longrightarrow \mathbb{R} \) possess the property: for every \( a \geq 0 \) the limit \( \lim_{n \to \infty} f(a + n) \) exists and equals 0. Does it imply that the limit \( \lim_{x \to \infty} f(x) \) exists?
Question 1 (♥). Prove that
(a) \( \lim_{x \to \infty} \frac{e^x}{x^n} = \infty, \ n \geq 0 \)
(b) \( \lim_{x \to \infty} x^n e^{-x} = 0, \ n \geq 0 \)
(c) \( \lim_{x \to 0^+} x^p \ln x = 0, \ p \in (0, 1) \)
(d) \( \lim_{x \to \infty} \frac{\ln^n x}{x} = 0, \ n \geq 0 \).

Question 2 (♠). Let \( f: \mathbb{R} \to \mathbb{R} \) be given by the formula
\[
f(x) = \begin{cases} 
  e^{-1/x^2}, & x > 0 \\
  0, & x \leq 0
\end{cases}
\]
Is \( f \) differentiable? What can you say about the second derivative? About the higher order derivatives?

Question 3 (♥). Let \( f: \mathbb{R} \to \mathbb{R} \) be a differentiable function with \( \sup_{x \in \mathbb{R}} |f'(x)| = L < \infty \). Prove that \( f \) is \( L \)-Lipschitz, i.e.
\[
|f(x) - f(y)| \leq L|x - y|, \quad \text{for all reals } x, y.
\]

Question 4 (★). Find all differentiable functions \( f: \mathbb{R} \to \mathbb{R} \) satisfying for all reals \( x \neq y \)
\[
\frac{f(y) - f(x)}{y - x} = f'\left(\frac{x + y}{2}\right).
\]
What is the geometric interpretation of this equation?
**Question 1 (♠).** Let \( f: (a,b) \to \mathbb{R} \) be differentiable. Prove that \( f' \) possesses the intermediate value property.

**Question 2 (♠).** Does there exist a differentiable function \( f: \mathbb{R} \to \mathbb{R} \) such that
\[
    f'(x) = \begin{cases} 
        -1, & x < 0 \\
        0, & x = 0 \\
        1, & x > 0 
    \end{cases}
\]

**Question 3 (♥).** Suppose that \( f(0) = 0 \) and that \( f'(0) \) exists. Given a positive integer \( k \) compute
\[
    \lim_{x \to 0} \frac{1}{x} \left( f(x) + f\left(\frac{x}{2}\right) + \ldots + f\left(\frac{x}{k}\right) \right).
\]

**Question 4 (♠).** Let \( f(x) = a_1 \sin x + a_2 \sin(2x) + \ldots + a_n \sin(nx) \), where \( a_1, a_2, \ldots, a_n \) are reals. Prove that if \( |f(x)| \leq |\sin x| \) for every \( x \in \mathbb{R} \) then \( |a_1 + 2a_2 + \ldots + na_n| \leq 1 \).
Question 1 (♥). Prove that if $|x| < 1/2$ then the approximate formula

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

gives the value of $\sqrt{1+x}$ with the error at most $\frac{1}{2}|x|^3$.

Observe that $\sqrt{1+1/8} = \frac{3\sqrt{2}}{4}$ and find an approximate value of $\sqrt{2}$. What is the error?

Question 2 (Bernoulli’s inequality in full glory ♥). Let $x > -1$, $x \neq 0$. Prove that

(a) $(1+x)^\alpha > 1 + \alpha x$, if $\alpha > 1$ or $\alpha < 0$

(b) $(1+x)^\alpha < 1 + \alpha x$, if $0 < \alpha < 1$.

Question 3 (♥). Prove that if $f''(x)$ exists then

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$ 

Question 4 (♠). Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable and

$$M_k = \sup \{|f^{(k)}(x)|; x \in \mathbb{R}\} < \infty, \quad k = 0, 1, 2.$$ 

Prove that

$$M_1 \leq \sqrt{2M_0M_2}.$$ 

Question 5 (♥). Find

$$\lim_{x \to 0} \frac{e^{x^2/2} - 1}{\cosh x - 1}.$$

Question 6 (♠). Prove that

(a) $\cosh x \leq e^{x^2/2}$ for $x \in \mathbb{R}$

(b) $\cos x \leq e^{-x^2/2}$ for $x \in [0, \pi/2]$. 


Questions difficulty legend: ♥ - easy ♠ - medium ★ - hard † - very hard

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Analysis II, Term 2 2012/2013
Tomasz Tkocz

Support class 8

Question 1 (♥♠). Evaluate

(a) \( \lim_{x \to 0} \frac{\ln(1+ex)}{x} \).

(b) \( \lim_{x \to 1} \frac{\arctan\left(\frac{x^2-1}{x^2+1}\right)}{x-1} \).

(c) \( \lim_{x \to \infty} x \left(\left(1 + \frac{1}{x}\right)^x - e\right) \).

(d) \( \lim_{x \to 0^+} \left(\frac{\sin x}{x}\right)^{1/x} \).

(e) \( \lim_{x \to 0^+} \left(\frac{\sin x}{x}\right)^{1/x^2} \).

Question 2 (♥). Determine the interval of convergence for the series

(a) \( \sum_{n=1}^{\infty} \frac{2^n x^n}{n!} \).

(b) \( \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n \left(-1\right)^n n^2 x^n \).

Question 3 (♥). Recall the definition of \( \lim_{n \to \infty} a_n \) and \( \overline{\lim}_{n \to \infty} a_n \). Prove that

(a) if \( a_n \leq b_n \) eventually, then \( \overline{\lim}_{n \to \infty} a_n \leq \overline{\lim}_{n \to \infty} b_n \).

(b) \( \overline{\lim}(a_n + b_n) \leq \overline{\lim} a_n + \overline{\lim} b_n \).

(c) \( \overline{\lim} |a_nb_n| \leq \overline{\lim} |a_n| \cdot \overline{\lim} |b_n| \). Show that the inequality can be strict.

(d) \( \lim\min\{a_n, b_n\} = \min\{\lim a_n, \lim b_n\} \).

Question 4 (♠). Suppose that \( f: (-1,1) \to \mathbb{R} \) is a function of class \( C^2 \) such that \( f(0) = 0 \). Compute

\[
\lim_{x \to 0^+} \sum_{k=1}^{\lfloor 1/\sqrt{x} \rfloor} f(kx).
\]
**Analysis II, Term 2 2012/2013**

Tomasz Tkocz

**Support class 9**

**Question 1** (♥). Evaluate

(a) \( \lim_{x \to 5} (6 - x)^{1/(x-5)} \).

(b) \( \lim_{x \to \infty} \frac{x - \sin x}{2x + \sin x} \).

**Question 2** (♥♠). Prove that for \( x \in (0, \pi/2) \) and a positive integer \( n \) we have

(a) \( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{x^{4n-3}}{(4n-3)!} - \frac{x^{4n-1}}{(4n-1)!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{x^{4n-3}}{(4n-3)!} - \frac{x^{4n-1}}{(4n-1)!} + \frac{x^{4n+1}}{(4n+1)!} \).

(b) \( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{x^{4n-2}}{(4n-2)!} < \cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{x^{4n-2}}{(4n-2)!} + \frac{x^{4n}}{(4n)!} \).

Do these inequalities hold for \( x \geq \pi/2 \) as well?

**Question 3** (♥). Prove that \( e^x \geq 1 + x \) for \( x \in \mathbb{R} \) and then derive the inequality between means

\[ \frac{x_1 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 \cdot \ldots \cdot x_n}. \]

**Question 4** (♥). Prove the inequality

(a) \( 1 - 1/x \leq \ln x \leq x/e \) for \( x > 0 \).

(b) \( 2 \tan x > \sinh x \) for \( x \in (0, \pi/2) \).

*Remark.* Combining the inequalities \( \cosh x \leq e^{x^2/2} \) and \( \cos x \leq e^{-x^2/2} \) (see Support class 7, Question 6), one can actually show \( c \tan x > \sinh x \) with \( c = 1 \) which is sharp.