How fast does the Gaussian measure of a convex and symmetric set grow?

Tomasz Tkocz

College of Inter-Faculty Individual Studies in Mathematics and Natural Sciences
and
Institute of Mathematics
University of Warsaw

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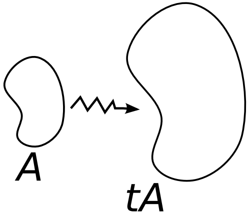
Outline

Main question

Dilation of a Borel set $A \subset \mathbb{R}^n$ is

$$A \rightsquigarrow tA, \qquad t > 0.$$

What happens with its measure?



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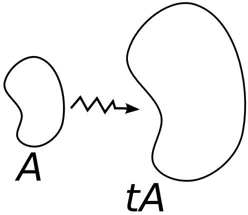

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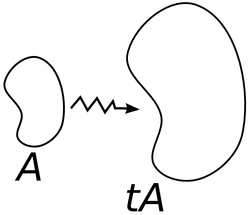

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$$|tA| = t^n |A|.$$

- Gaussian measure?

$$\gamma_n(tA) = ?$$

Reminder

The standard Gaussian measure γ_n it is a measure with the density $\frac{1}{\sqrt{2\pi}^n} e^{-|x|^2/2}$.

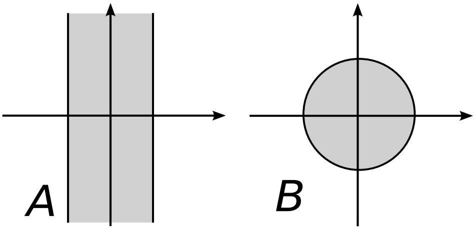
Examples on the plane ($n = 2$)

1) $A = (-1, 1) \times \mathbb{R}$. Then

$$f_A(t) = \gamma_2(tA) = \gamma_2((-t, t) \times \mathbb{R}) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-x^2/2} \mathrm{d}x.$$

2) $B = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$. Then

$$f_B(t) = \gamma_2(tB) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{t} e^{-r^2/2} r \mathrm{d}r \mathrm{d}\varphi = 1 - e^{-t^2/2}.$$



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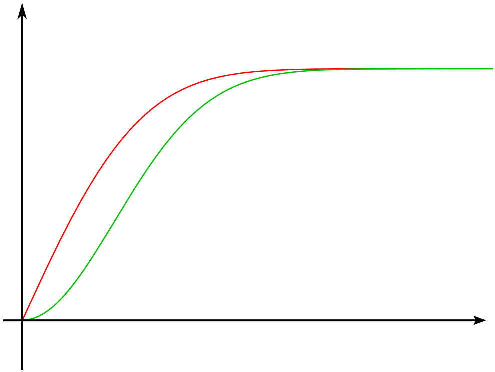

Figure: Graphs of f_A and f_B

Main target

Problem

Give the optimal bounds from above and from below on the function

$$t \overset{f_A}{\longmapsto} \gamma_n(tA), \qquad t \geq 1.$$

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Remark

We have to assume something on the set A as there are inconvenient examples

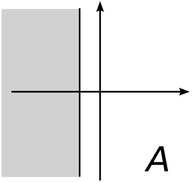

Figure: The function f_A is decreasing for $t \geq 1$

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Give the optimal bounds from above and from below on the function

$$t \xmapsto{f_A} \gamma_n(tA), \qquad t \geq 1.$$

Assumption

We restrict ourself to the sets which are

- convex,
- symmetric ($A = -A$).

That is, A is a ball with respect to some norm on $\mathbb{R}^n$.

Bounds

Bound from above = Homework

Let $A \subset \mathbb{R}^n$ be convex and symmetric and $B = \{|x| \leq R\}$ be an euclidean ball such that $\gamma_n(A) = \gamma_n(B)$. Then

$$\gamma_n(tA) \leq \gamma_n(tB), \qquad t \geq 1.$$

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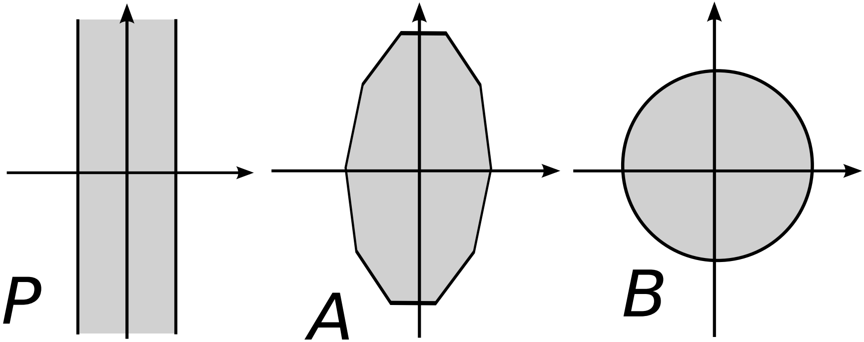
Bound from below = the deep result of Latała and Oleszkiewicz

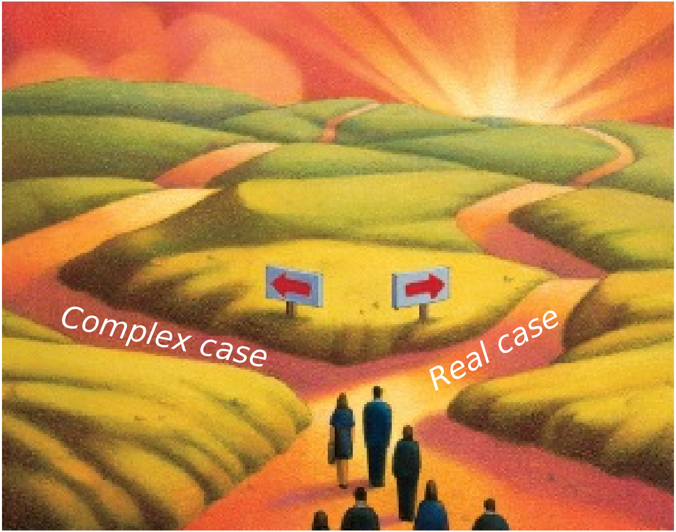
Let $A \subset \mathbb{R}^n$ be convex and symmetric and $P = \{|x_1| \leq p\}$ be a strip such that $\gamma_n(A) = \gamma_n(P)$. Then

$$\gamma_n(tA) \geq \gamma_n(tP), \qquad t \geq 1.$$

Bounds in the picture



Further Questions



Complex case

$$\mathbb{C}^n \approx \mathbb{R}^{2n} \quad \Longrightarrow \quad \text{there is the Gaussian measure } \nu_n \text{ on } \mathbb{C}^n$$

Question

What can one say about the function

$$t \overset{f_A}{\longmapsto} \nu_n(tA), \qquad t \geq 1$$

for the set $A \subset \mathbb{C}^n$ which is

- convex,
- **rotationally** symmetric, i.e.

$$A = \lambda A, \qquad \text{for any } \lambda \in \mathbb{C} \text{ such that } |\lambda| = 1$$

Partial Results in Complex Case

Upper bound

Again balls are optimal — their measure grows the fastest.

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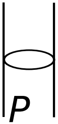
Lower bound — conjecture



Cylinders (complex counterpart of strips) are optimal. A cylinder with the radius p is the set of the form

$$P = \{z \in \mathbb{C}^n \mid |z_1| \leq p\}.$$

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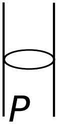

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Theorem (TT)

Let $A \subset \mathbb{C}^n$ be a convex and rotationally symmetric set and P be a cylinder such that $\nu_n(A) = \nu_n(P)$. Then

$$\nu_n(tA) \geq \nu_n(tP), \qquad t \in [1, t_0],$$

where $t_0 = t_0(A)$ is such that $\nu_n(t_0 A) = c$ for some absolute constant $c \approx 0.64$.

Wandering towards Proof

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Theorem (Latała & Oleszkiewicz)

Let $A \subset \mathbb{R}^n$ be convex and symmetric and $P = \{|x_1| \leq p\}$ be a strip such that $\gamma_n(A) = \gamma_n(P)$. Then

$$\gamma_n(tA) \geq \gamma_n(tP), \qquad t \geq 1.$$

Sketch of the Proof

Step I

Let us remind the notation $f_A(t) = \gamma_n(tA)$. An easy observation is that

$$f_A(t) \geq f_P(t) \quad \Longleftrightarrow \quad f'_A(1) \geq f'_P(1).$$

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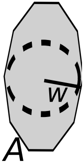
Step II



Define a radius of the set A as

$$w = \sup\{r > 0 \mid rB \subset A\}.$$

Then

$$\text{convexity} \implies f'_A(1) \geq w\gamma_n^+(A),$$
$$P \text{ is a strip} \implies f'_P(1) = p\gamma_n^+(P).$$

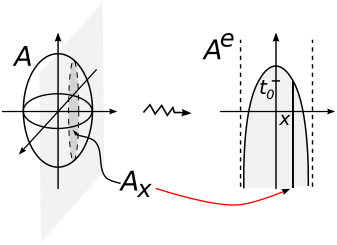
Sketch of the Proof

Step III

We apply the Ehrhard symmetrization so as to reduce the dimension

$$A \rightsquigarrow A^e = \{(x,t) \in \mathbb{R}^2 \mid t \le t_0(x) \text{ chosen so that } \gamma_{n-1}(A_x) = \gamma_1((-\infty, t_0))\}.$$

A_x denotes a cut of the set A at the level x, i.e. the set $\{(x_2, \ldots, x_n) \mid (x, x_2, \ldots, x_n) \in A\}$.



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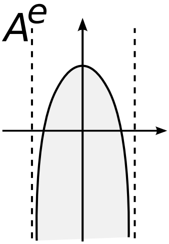

The key properties of the Ehrhard symmetrization

- it does not increase the measure of the boundary

$$\gamma_n^+(A) \geq \gamma_n^+(A^e),$$
$$\gamma_n^+(P) = \gamma_2^+(P^e),$$

- A^e lies under the graph of concave function (Ehrhard inequality)

Hence we are to prove the isoperimetric-like inequality in $\mathbb{R}^2$

$$w\gamma_2^+(A^e) \geq p\gamma_2^+(P^e),$$

which reduces to (only?) hard calculations.

References

R. Latała and K. Oleszkiewicz, *Gaussian measures of dilatations of convex symmetric sets*, The Annals of Probability, **27** (1999), 1922–1938

T. Tkocz, *Gaussian measures of dilations of convex rotationally symmetric sets in C^n*, arXiv:1007.2907v1 [math.PR] (2010)

Thanks!

Questions?