How fast does the Gaussian measure of a convex and symmetric set grow?

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Definition
Dilation of a Borel set $A \subset \mathbb{R}^n$ is $A \mapsto tA$, $t > 0$.

What happens with its measure?
- Lebesgue measure — it is trivial $|tA| = t^n |A|$.
- Gaussian measure — there is a question $\gamma_n(tA) =$?

Problem
Give the optimal bounds from above and from below on the function $t \mapsto \gamma_n(tA)$, $t \geq 1$.

Reminder
The Gaussian measure $\gamma_n$ has the density $\frac{1}{\sqrt{2\pi}} e^{-|x|^2/2}$.

Real case

Assumptions
We restrict ourself to the sets which are
- convex,
- symmetric ($A = -A$).
That is, $A$ is a ball with respect to some norm on $\mathbb{R}^n$.

Complex case

Assumptions
Let $A \subset \mathbb{C}^n$ be
- convex (in the usual real sense),
- rotationally symmetric ($A = \lambda A$, for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$).

Question
How fast does the function $t \mapsto \nu_n(tA)$ grow?

Upper bound
Let $A \subset \mathbb{R}^n$ ($A \subset \mathbb{C}^n$) be convex and (rotationally) symmetric and let $B = \{|x| \leq R\}$ be an euclidean ball such that $\gamma_n(A) = \gamma_n(B)$ ($\nu_n(A) = \nu_n(B)$).

Then $f_A(t) \leq f_B(t)$, $t \geq 1$.
It means that the measure of balls grows the fastest.

Lower bound \([1]\)
Let $A \subset \mathbb{R}^n$ be convex and symmetric and $P = \{|x| \leq p\}$ be a strip such that $\gamma_n(A) = \gamma_n(P)$. Then $f_A(t) \geq f_P(t)$, $t \geq 1$.

Conjecture
Cylinders are optimal, i.e. the above inequality holds for all $t \geq 1$.

Lower bound \([2]\)
Let $A \subset \mathbb{C}^n$ be convex and rotationally symmetric and $P = \{|x| \leq p\}$ be a cylinder such that $\nu_n(A) = \nu_n(P)$. Then $f_A(t) \geq f_P(t)$, $t \in [1, t_0]$, where $t_0 = t_0(A)$ is such that $\nu_n(t_0A) = c$ for some absolute constant $c \approx 0.64$.

References