A note on suprema of canonical processes based on random variables with regular moments

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Abstract

We derive two-sided bounds for expected values of suprema of canonical processes based on random variables with moments growing regularly. We also discuss a Sudakov-type minoration principle for canonical processes.

1 Introduction

In many problems arising in probability theory and its applications one needs to estimate the supremum of a stochastic process. In particular it is very useful to be able to find two-sided bounds for the mean of the supremum. The modern approach to this challenge is based on the chaining methods, see monograph [16].

In this note we study the class of *canonical processes* (X_t) of the form

$$X_t = \sum_{i=1}^{\infty} t_i X_i,$$

where X_i are independent random variables. If X_i are *standardized*, i.e. have mean zero and variance one, then this series converges a.s. for $t \in \ell^2$ and we may try to estimate $\mathbb{E}\sup_{t\in T} X_t$ for $T \subset \ell^2$. To avoid measurability questions we either assume that the index set T is countable or define in a general situation

$$\mathbb{E}\sup_{t\in T} X_t = \sup\left\{\mathbb{E}\sup_{t\in F} X_t \colon\ F\subset T \text{ finite}\right\}.$$

It is also more convenient to work with the quantity $\mathbb{E}\sup_{s,t\in T}(X_t - X_s)$ rather than $\mathbb{E}\sup_{t\in T} X_t$. Observe however that if the set T or the variables X_i are symmetric then

$$\mathbb{E}\sup_{s,t\in T}(X_s - X_t) = \mathbb{E}\sup_{s\in T} X_s + \mathbb{E}\sup_{t\in T}(-X_t) = 2\mathbb{E}\sup_{t\in T} X_t.$$

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For instance, in the case when X_i are i.i.d. $\mathcal{N}(0,1)$ r.v.s, X_t is the canonical Gaussian process. Moreover, any centred separable Gaussian process has the Karhunen-Loève representation of such form (see e.g. Corollary 5.3.4 in [10]). In the Gaussian case the behaviour of the supremum of the process is related to the geometry of the metric space (T, d_2), where d_2 is the ℓ^2-metric $d(s,t) = (\mathbb{E}|X_s - X_t|^2)^{1/2}$. The celebrated Fernique-Talagrand majorizing measure bound (cf. [2, 14]) can be expressed in the form

$$\frac{1}{C}\gamma_2(T) \le \mathbb{E}\sup_{t\in T} X_t \le C\gamma_2(T),$$

where here and in the sequel C denotes a universal constant,

$$\gamma_2(T) := \inf \sup_{t\in T} \sum_{n=0}^{\infty} 2^{n/2} \Delta_2(A_n(t)),$$

the infimum runs over all admissible sequences of partitions $(\mathcal{A}_n)_{n\ge 0}$ of the set T, $A_n(t)$ is the unique set in $\mathcal{A}_n$ which contains t, and Δ_2 denotes the ℓ^2-diameter. An increasing sequence of partitions $(\mathcal{A}_n)_{n\ge 0}$ of T is called *admissible* if $\mathcal{A}_0 = \{T\}$ and $|\mathcal{A}_n| \le N_n := 2^{2^n}$ for $n \ge 1$.

Let us emphasise that Talagrand's γ_2 functional is tailored to govern the behaviour of suprema of specifically Gaussian processes. Since we want to study canonical processes for a wide class of random variables, we shall discuss now some general ideas developed to obtain bounds on suprema of stochastic processes.

To motivate our first definition, let us look at the following easy estimate based on the union bound; for $p \ge 1$ and a finite set T we have

$$\begin{aligned} \mathbb{E}\sup_{s,t\in T}(X_s - X_t) &\le \Big(\mathbb{E}\sup_{s,t\in T}|X_s - X_t|^p\Big)^{1/p} \le \Big(\mathbb{E}\sum_{s,t\in T}|X_s - X_t|^p\Big)^{1/p} \\ &\le |T|^{2/p}\sup_{s,t\in T}\|X_s - X_t\|_p. \end{aligned} \tag{1}$$

If $|T| \le e^p$, we get that the expectation of the supremum is controlled above up to a constant by the diameter $\Delta_p(T)$ of the metric space (T, d_p), where $d_p(s,t) = \|X_s - X_t\|_p$. Can this be reversed? Following [8] (see also [11]) we say that:

a process $(X_t)_{t\in S}$ satisfies *the Sudakov minoration principle with constant* $\kappa > 0$ if for any $p \ge 1$, $T \subset S$ with $|T| \ge e^p$ such that $\|X_s - X_t\|_p \ge u$ for all $s, t \in T$, $s \ne t$, we have $\mathbb{E}\sup_{s,t\in T}(X_s - X_t) \ge \kappa u$. (2)

Establishing the Sudakov minoration principle is usually a crucial step in deriving lower bounds for suprema of stochastic processes.

Let us try to soup up the previous bound employing this time a chaining argument. We will follow closely Talagrand's construction of the γ_2 functional mentioned earlier (see Section 2.2 in [16]). Let $(X_t)_{t\in T}$ be a general process with T finite (for simplicity). The main idea of the chaining technique is to build finer and finer levels of approximations $\mathcal{A}_n$ in order to gather together those t's for which X_t are close. Then we apply union bounds along *chains*, built across the levels $\mathcal{A}_n$ which comprise at each step variables that are rather close and crucially, there are not too many of them. We fix an increasing sequence of admissible partitions $(\mathcal{A}_n)_{n\geq 0}$. For each n we construct a set T_n by picking exactly one point from every set A of the partition $\mathcal{A}_n$. Hence, $|T_n| \leq 2^{2^n}$. At level n we use the metric d_{2^n} to measure the order of magnitude of variables as this will let us capture properly probabilities via moment estimates. This is the key subtle distinction we have to make between a general case and the Gaussian case where we precisely know all the moments, so a good scaling of the d_2 metric suffices. We pick $\pi_n(t) \in T_n$ in such a way that t and $\pi_n(t)$ belong to the same set in the partition $\mathcal{A}_n$. The chain we build is this:

$$X_t - X_{\pi_1(t)} = \sum_{n\geq 1} \big(X_{\pi_{n+1}(t)} - X_{\pi_n(t)}\big).$$

Let $A_{n,t,u}$ be the event $\{|X_{\pi_{n+1}(t)} - X_{\pi_n(t)}| \leq u \cdot d_{2^n}(\pi_{n+1}(t), \pi_n(t))\}$. By Chebyshev's inequality, $\mathbb{P}(A^c_{n,t,u}) \leq u^{-2^n}$, so if we set $\Omega_u = \bigcap_{n\geq 1}\bigcap_t A_{n,t,u}$, by the union bound we easily find that

$$\mathbb{P}(\Omega_u^c) \leq \sum_{n\geq 1} |T_{n+1}||T_n| u^{-2^n} \leq \sum_{n\geq 1} \left(\frac{8}{u}\right)^{2^n} \leq \frac{128}{u^2}, \quad u \geq 16.$$

Since on Ω_u we have

$$\sup_{t\in T} |X_t - X_{\pi_1(t)}| \leq u \cdot S,$$

where

$$S = \sup_{t\in T} \sum_{n\geq 1} d_{2^n}(\pi_{n+1}(t), \pi_n(t)),$$

we obtain

$$\mathbb{P}\left(\frac{1}{S}\sup_{t\in T}|X_t - X_{\pi_1(t)}| > u\right) \leq \frac{128}{u^2}, \quad u \geq 16.$$

This readily yields that the expectation of

$$\sup_{t,s\in T}(X_t - X_s) \leq \sup_{t\in T}|X_t - X_{\pi_1(t)}| + \sup_{s,t\in T}|X_{\pi_1(t)} - X_{\pi_1(s)}| + \sup_{s\in T}|X_s - X_{\pi_1(s)}|$$

can be bounded by

$$C\cdot S + \mathbb{E}\sup_{s,t\in T}|X_{\pi_1(t)} - X_{\pi_1(s)}| \leq C\cdot S + |T_1|^2\cdot \Delta_1(T).$$

By the triangle inequality $d_{2^n}(\pi_{n+1}(t),\pi_n(t)) \le d_{2^{n+1}}(t,\pi_{n+1}(t)) + d_{2^n}(t,\pi_n(t))$, so we can control S as follows

$$S \le 2\sup_{t\in T}\sum_{n\ge 1} d_{2^n}(t,\pi_n(t)) \le 2\sup_{t\in T}\sum_{n\ge 1}\Delta_{2^n}(A_n(t)),$$

where $\Delta_{2^n}(A_n(t))$ is the diameter of the unique set $A_n(t)$ from $\mathcal{A}_n$ containing t.

This argument motivates the following definition

$$\gamma_X(T) = \inf\sup_{t\in T}\sum_{n=0}^{\infty}\Delta_{2^n}(A_n(t)), \tag{3}$$

where the infimum runs over all admissible sequences of partitions $(\mathcal{A}_n)$ of the set T. The reasoning above shows that for *any* process $(X_t)_{t\in T}$,

$$\mathbb{E}\sup_{s,t\in T}(X_s - X_t) \le C\gamma_X(T). \tag{4}$$

This was noted independently by Mendelson and the first named author (see, e.g. [16, Exercise 2.2.25]). Similar chaining ideas have also been used in [12].

Plainly, the bound (4) is less crude that (1). Therefore, we expect that a bound reverse to (4) should imply the Sudakov minoration principle. We make two remarks.

Remark 1. Suppose that for any finite $T \subset \ell^2$ we have $\mathbb{E}\sup_{s,t\in T}(X_s - X_t) \ge \kappa\gamma_X(T)$. Assume moreover that for any $p \ge 1$ and $t \in \ell^2$, $\|X_t\|_{2p} \le \gamma\|X_t\|_p$. Then X satisfies the Sudakov minoration principle with constant κ/γ.

Proof. Let $p \ge 1$ and $T \subset \ell^2$ of cardinality at least e^p be such that $\|X_s - X_t\|_p \ge u$ for any $s,t \in T$, $s \ne t$. Let $2^k \le p < 2^{k+1}$ and $(\mathcal{A}_n)$ be an admissible sequence of partitions of the set T. Then there is $A \in \mathcal{A}_k$ which contains at least two points of T. Hence

$$\mathbb{E}\sup_{s,t\in T}(X_s - X_t) \ge \kappa\gamma_X(T) \ge \kappa\Delta_{2^k}(A) \ge \kappa\Delta_{\max\{p/2,1\}}(A) \ge \kappa u/\gamma.$$

□

In fact, in the i.i.d. case we do not need the regularity assumption $\|X_t\|_{2p} \le \gamma\|X_t\|_p$.

Remark 2. Let $X_t = \sum_{i=1}^{\infty} t_i X_i$, $t \in \ell^2$, where X_i are i.i.d. standardized r.v.s. Suppose that $\mathbb{E}\sup_{t,s\in T} X_t \ge \kappa\gamma_X(T)$ for all finite $T \subset \ell^2$. Then $(X_t)_{t\in\ell^2}$ satisfies the Sudakov minoration principle with constant $\kappa/2$.

Proof. Fix $p \ge 1$ and $T \subset \ell^2$ such that $|T| \ge e^p$ and $\|X_s - X_t\|_p \ge u$ for distinct points $s,t \in T$. For $t^1, t^2 \in T$ define a new point in ℓ^2 by $t(t^1,t^2) := (t^1_1, t^2_1, t^1_2, t^2_2, \ldots)$. Put also $\widetilde{T} := \{t(t^1,t^2)\colon t^1,t^2 \in T\}$. It is not hard to see that $\|X_s - X_t\|_p \ge u$ for $t,s \in \widetilde{T}$, $t \ne s$.

Choose an integer k such that $2^k \leq p < 2^{k+1}$ and let $(\mathcal{A}_n)$ be an admissible sequence of partitions of the set $\widetilde{T}$. Since $|\widetilde{T}| = |T|^2 \geq e^{2p} > 2^{2^{k+1}}$, there is $A \in \mathcal{A}_k$ which contains at least two points of $\tilde{T}$. Hence

$$u \leq \Delta_{2^k}(A) \leq \gamma_X(\widetilde{T}) \leq \frac{1}{\kappa}\mathbb{E}\sup_{s,t\in\widetilde{T}}(X_s - X_t) \leq \frac{2}{\kappa}\mathbb{E}\sup_{s,t\in T}(X_s - X_t).$$

□

There are two goals of this note. First, we would like to find fairly general assumptions that will allow us to reverse inequality (1), that is we want to obtain the Sudakov minoration principle for a large class of canonical processes based on i.i.d. variables. Second, possibly assuming more, we want to derive lower bounds for suprema of canonical processes in terms of the γ_X functional, that is we want to reverse inequality (4). Let us collect known results in these directions.

In [15] Talagrand derived two-sided bounds for suprema of the canonical processes based on i.i.d. symmetric r.v.s X_i such that $\mathbb{P}(|X_i| > t) = \exp(-|t|^p)$, $1 \leq p < \infty$. This result was later extended in [7] to the case of variables with (not too rapidly decreasing) log-concave tails, i.e. to the case when X_i are symmetric, independent, $\mathbb{P}(|X_i| \geq t) = \exp(-N_i(t))$, $N_i\colon [0,\infty) \to [0,\infty)$ are convex and $N_i(2t) \leq \gamma N_i(t)$ for $t > 0$ and some constant γ. The relevant results can be restated as follows (see Theorems 1 and 3 in [7]).

Theorem 1 ([7])**.** *Let $X_t = \sum_{i=1}^{\infty} t_i X_i$, $t \in \ell^2$ be the canonical process based on independent symmetric r.v.s X_i with log-concave tails. Then $(X_t)_{t\in\ell^2}$ satisfies the Sudakov minoration principle with a universal constant $\kappa_{\mathrm{lct}} > 0$.*

Remark 3. Since we may normalize X_i we need not assume that they have variance one. It suffices to have $\sup_i \mathrm{Var}(X_i) < \infty$ in order for X_t to be well defined for $t \in \ell^2$.

Theorem 3 in [7] (see also Theorem 10.2.7 and Exercise 10.2.14 in [16]) implies the following result.

Theorem 2 ([7])**.** *Let $X_t = \sum_{i=1}^{\infty} t_i X_i$, $t \in \ell^2$ be the canonical process based on independent symmetric r.v.s X_i with log-concave tails. Assume moreover that there exists γ such that $N_i(2t) \leq \gamma N_i(t)$ for all i and $t > 0$, where $N_i(t) = -\ln \mathbb{P}(|X_i| > t)$. Then there exists a constant $C_{\mathrm{lct}}(\gamma)$, which depends only on γ such that for any $T \subset \ell^2$,*

$$\mathbb{E}\sup_{s,t\in T}(X_s - X_t) = 2\mathbb{E}\sup_{t\in T} X_t \geq \frac{1}{C_{\mathrm{lct}}(\gamma)}\gamma_X(T).$$

Remark 4. Theorem 3 in [7] and Theorem 10.2.7 in [16] were formulated in a slightly different language. It is rather technical to see how they imply the formulation presented here. The dedicated reader who is not afraid of technical subtleties is encouraged to check the details. One way to do it is to see that the latter theorem states that there exist

$r > 2$, an admissible sequence of partitions $(\mathcal{A}_n)$ and numbers $j_n(A)$ for $A \in \mathcal{A}_n$ such that $\varphi_{j_n(A)}(s, s') \le 2^{n+1}$ for all $s, s' \in A$ and

$$\sup_{t\in T} \sum_{n=0}^{\infty} 2^n r^{-j_n(A_n(t))} \le C(\gamma)\mathbb{E}\sup_{t\in T} X_t.$$

(For the definition of φ see [16] — it precedes the statement of Theorem 10.2.7.) However, the condition $\varphi_{j_n(A)}(s, s') \le 2^{n+1}$ yields that $\|X_s - X_{s'}\|_{2^n} \le C2^n r^{-j_n(A)}$ (see [3] for the i.i.d. case and Example 3 in [6] for the general situation), so $\Delta_{2^n}(A_n(t)) \le C2^n r^{-j_n(A_n(t))}$ and

$$\gamma_X(T) \le C \sup_{t\in T} \sum_{n=0}^{\infty} 2^n r^{-j_n(A_n(t))} \le C_{\mathrm{lct}}(\gamma)\mathbb{E}\sup_{t\in T} X_t.$$

This paper is organized as follows. In the next section we present our results. Then we gather some general facts. The last section is devoted to the proofs. We will frequently use various constants. By a letter C we denote universal constants. Value of a constant C may differ at each occurrence. Whenever we want to fix the value of an absolute constant we use letters $C_1, C_2, \ldots$. We write $C(\alpha)$ (resp. $C(\alpha, \beta)$, etc.) for constants depending only on parameters α (resp. α, β etc.). We will also frequently work with a Bernoulli sequence ε_i of i.i.d. symmetric r.v.s taking values ± 1. We assume that variables ε_i are independent of other r.v.s.

2 Results

2.1 The Sudakov minoration principle

Our first main result concerns the Sudakov minoration principle (2). Recall that it has been established for canonical processes based on indpendent random variables with log-concave tails (Theorem 1). It is easy to check that for a symmetric variable Y with a log-concave tail $\exp(-N(t))$, we have $\|Y\|_p \le C\frac{p}{q}\|Y\|_q$ for $p \ge q \ge 2$. This motives the following definition. For $\alpha \ge 1$ we say that moments of a random variable X *grow α-regularly* if

$$\|X\|_p \le \alpha\frac{p}{q}\|X\|_q \quad \text{for } p \ge q \ge 2.$$

The class of all standardized random variables with the α-regular growth of moments will be denoted by $\mathcal{R}_\alpha$. It turns out that this condition suffices to obtain the Sudakov minoration principle for canonical processes.

Theorem 3. *Suppose that $X_1, X_2, \ldots$ are independent standardized r.v.s and moments of X_i grow α-regularly for some $\alpha \ge 1$. Then the canonical process $X_t = \sum_{i=1}^{\infty} t_i X_i$, $t \in \ell^2$ satisfies the Sudakov minoration principle with constant $\kappa(\alpha)$, which depends only on α.*

In fact the assumption on regular growth of moments is necessary for the Sudakov minoration principle in the i.i.d. case.

Proposition 4. *Suppose that a canonical process $X_t = \sum_{i=1}^{\infty} t_i X_i$, $t \in \ell^2$ based on i.i.d. standardized random variables X_i satisfies the Sudakov minoration with constant $\kappa > 0$. Then moments of X_i grow C/κ-regularly.*

Methods developed to prove Theorem 3 also enable us to establish the following comparison of weak and strong moments of the canonical processes based on variables with regular growth of moments.

Theorem 5. *Let X_t be as in Theorem 3. Then for any nonempty $T \subset \ell_2$ and $p \geq 1$,*

$$\left(\mathbb{E}\sup_{t\in T}|X_t|^p\right)^{1/p} \leq C(\alpha)\left(\mathbb{E}\sup_{t\in T}|X_t| + \sup_{t\in T}(\mathbb{E}|X_t|^p)^{1/p}\right).$$

2.2 Lower bounds

Our next main result concerns reversing the bound (4). As we indicated in the introduction (Remarks 1 and 2), such an inequality will be a refinement to the Sudakov minoration principle. We shall need more regularity. Recall that in the case of independent random variables X_i with log-concave tails $\exp(-N_i(t))$ (Theorem 2), the additional condition $N_i(2t) \leq \gamma N_i(t)$ was relevant. It is readily checked that this condition yields $\|Y\|_{\beta p} \geq 2\|Y\|_p$ for $p \geq 2$ and a constant β which depends only on γ. This motivates our next definition. For $\beta < \infty$ we say that moments of a random variable X *grow with speed* β if

$$\|X\|_{\beta p} \geq 2\|X\|_p \quad \text{for } p \geq 2.$$

The class of all standardized random variables with the moments growing with speed β will be denoted by $\mathcal{S}_\beta$.

Theorem 6. *Let $X_t = \sum_{i=1}^{\infty} t_i X_i$, $t \in \ell^2$ be the canonical process based on independent standardized r.v.s X_i with moments growing α-regularly with speed β for some $\alpha \geq 1$ and $\beta > 1$. Then for any $T \subset \ell^2$,*

$$\frac{1}{C(\alpha,\beta)}\gamma_X(T) \leq \mathbb{E}\sup_{s,t\in T}(X_s - X_t) \leq C\gamma_X(T).$$

The above result easily yields the following comparison result for suprema of processes.

Corollary 7. *Let X_t be as in Theorem 6. Then for any nonempty $T \subset \ell^2$ and any process $(Y_t)_{t\in T}$ such that $\|Y_s - Y_t\|_p \leq \|X_s - X_t\|_p$ for $p \geq 1$ and $s, t \in T$ we have*

$$\mathbb{E}\sup_{s,t\in T}(Y_s - Y_t) \leq C(\alpha,\beta)\mathbb{E}\sup_{s,t\in T}(X_s - X_t).$$

Proof. The assumption implies $\gamma_Y(T) \le \gamma_X(T)$ and the result immediately follows by the lower bound in Theorem 6 and estimate (4) used for the process Y. $\square$

In fact one may show a stronger result.

Corollary 8. *Let X_t and Y_t be as in Corollary 7. Then for $u \ge 0$,*

$$\mathbb{P}\left(\sup_{s,t\in T}(Y_s - Y_t) \ge u\right) \le C(\alpha,\beta)\mathbb{P}\left(\sup_{s,t\in T}(X_s - X_t) \ge \frac{1}{C(\alpha,\beta)}u\right).$$

Another consequence of Theorem 6 is the following striking bound for suprema of some canonical processes.

Corollary 9. *Let X_t be as in Theorem 6 and $T \subset \ell^2$ be such that $\mathbb{E}\sup_{s,t\in T}(X_s - X_t) < \infty$. Then there exist $t^1, t^2, \ldots \in \ell^2$ such that $T - T \subset \overline{\mathrm{conv}}\{\pm t^n\colon\ n \ge 1\}$ and $\|X_{t^n}\|_{\log(n+2)} \le C(\alpha,\beta)\mathbb{E}\sup_{s,t\in T}(X_s - X_t)$.*

Remark 5. The reverse statement easily follows by the union bound and Chebyshev's inequality. Namely, for any canonical process $(X_t)_{t\in\ell^2}$ and any nonempty set $T \subset \ell^2$ such that $T - T \subset \overline{\mathrm{conv}}\{\pm t^n\colon\ n \ge 1\}$ and $\|X_{t^n}\|_{\log(n+2)} \le M$ one has $\mathbb{E}\sup_{s,t\in T}(X_s - X_t) \le CM$. For details see the argument after Corollary 1.2 in [1].

Remark 6. Let $(\varepsilon_i)_{i\ge1}$ be i.i.d. symmetric ±1-valued r.v.s, $X_t = \sum_{i=1}^{\infty} t_i\varepsilon_i$, $t \in \ell^2$ and $T = \{e_n\colon\ n \ge 1\}$, where (e_n) is the canonical basis of ℓ^2. Then obviously $\mathbb{E}\sup_{s,t\in T}(X_s - X_t) = 2$, moreover for any $A \subset T$ with cardinality at least 2, we have $\Delta_{2^k}(T) \ge \Delta_2(T) = \sqrt{2}$, hence $\gamma_X(T) = \infty$. Therefore one cannot reverse bound (4) for Bernoulli processes, so some assumptions on the nontrivial speed of growth of moments are necessary in Theorem 6. However, Corollary 9 holds for Bernoulli processes (cf. [1]) and we believe that in that statement the assumption of the β-speed of the moments growth is not needed.

3 Preliminaries

In this section we gather basic facts used in the sequel. We start with the contraction principle for Bernoulli processes (see e.g. [9, Theorem 4.4]).

Theorem 10 (Contraction principle)**.** *Let $(a_i)_{i=1}^n$, $(b_i)_{i=1}^n$ be two sequences of real numbers such that $|a_i| \le |b_i|$, $i = 1, \ldots, n$. Then*

$$\mathbb{E}F\left(\left|\sum_{i=1}^n a_i\varepsilon_i\right|\right) \le \mathbb{E}F\left(\left|\sum_{i=1}^n b_i\varepsilon_i\right|\right), \tag{5}$$

where $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function. In particular,

$$\left\| \sum_{i=1}^n a_i \varepsilon_i \right\|_p \leq \left\| \sum_{i=1}^n b_i \varepsilon_i \right\|_p . \tag{6}$$

Moreover, for a nonempty subset T of $\mathbb{R}^n$,

$$\mathbb{E} \sup_{t\in T} \sum_{i=1}^n t_i a_i \varepsilon_i \leq \mathbb{E} \sup_{t\in T} \sum_{i=1}^n t_i b_i \varepsilon_i . \tag{7}$$

The next Lemma is a standard symmetrization argument (see e.g. [9, Lemma 6.3]).

Lemma 11 (Symmetrization)**.** *Let X_i be independent standardized r.v.s and (ε_i) be a Bernoulli sequence independent of (X_i). Define two canonical processes $X_t = \sum_{i=1}^{\infty} t_i X_i$ and its symmetrized version $\tilde{X}_t = \sum_{i=1}^{\infty} t_i \varepsilon_i X_i$. Then*

$$\frac{1}{2}\|X_s - X_t\|_p \leq \|\tilde{X}_s - \tilde{X}_t\|_p \leq 2\|X_s - X_t\|_p \quad \text{for } s,t \in \ell^2$$

and for any $T \subset \ell^2$,

$$\frac{1}{2}\mathbb{E} \sup_{s,t\in T} (X_s - X_t) \leq \mathbb{E} \sup_{s,t\in T} (\tilde{X}_s - \tilde{X}_t) = 2\mathbb{E} \sup_{t\in T} \tilde{X}_t \leq 2\mathbb{E} \sup_{s,t\in T} (X_s - X_t).$$

Let us also recall the Paley-Zygmund inequality (cf. [4, Lemma 0.2.1]) which goes back to work [13] on trigonometric series.

Lemma 12 (Paley-Zygmund inequality)**.** *For any nonnegative random variable S and $\lambda \in (0,1)$,*

$$\mathbb{P}(S \geq \lambda \mathbb{E}S) \geq (1-\lambda)^2 \frac{(\mathbb{E}S)^2}{\mathbb{E}S^2}. \tag{8}$$

The next lemma shows that convolution preserves (up to a universal constant) the property of the α-regular growth of moments.

Lemma 13. *Let $S = \sum_{i=1}^n X_i$, where X_i are independent mean zero r.v.s with moments growing α-regularly. Then moments of S grow $C\alpha$-regularly. In particular, if (X_t) is a canonical process based on r.v.s from $\mathcal{R}_\alpha$, then $\|X_t\|_{4p} \leq C\alpha \|X_t\|_p$ for $p \geq 2$.*

Proof. We are to show that $\|S\|_p \leq C\alpha \frac{p}{q} \|S\|_q$ for $p \geq q \geq 2$. By Lemma 11 we may assume that the r.v.s X_i are symmetric. Moreover, by monotonicity of moments, it is enough to consider only the case when p and q are even integers and $p \geq 2q$. In [6] it was shown that for $r \geq 2$,

$$\frac{e-1}{2e^2} |||(X_i)|||_r \leq \|S\|_r \leq e|||(X_i)|||_r,$$

where

$$|||(X_i)|||_r := \inf\left\{u>0\colon \quad \prod_i \mathbb{E}\left|1+\frac{X_i}{u}\right|^r \le e^r\right\}.$$

Therefore it is enough to prove that $|||(X_i)|||_p \le \frac{4e\alpha p}{q}|||(X_i)|||_q$, which follows by the following claim.

Claim. Suppose that Y is a symmetric r.v. with moments growing α-regularly. Let p, q be positive even integers such that $p \ge 2q$ and $\mathbb{E}|1+Y|^q \le e^A$ for some $A \le q$. Then $\mathbb{E}|1+\frac{q}{4e\alpha p}Y|^p \le e^{pA/q}$.

To show the claim first notice that

$$\mathbb{E}|1+Y|^q = 1+\sum_{k=1}^{q/2}\binom{q}{2k}\mathbb{E}|Y|^{2k} \ge 1+\sum_{k=1}^{q/2}\left(\frac{q}{2k}\right)^{2k}\mathbb{E}|Y|^{2k} \ge 1+\mathbb{E}|Y|^q.$$

In particular, $\|Y\|_q \le (e^A-1)^{1/q} \le e$. On the other hand,

$$\mathbb{E}\left|1+\frac{q}{4e\alpha p}Y\right|^p = 1+\sum_{k=1}^{p/2}\binom{p}{2k}\mathbb{E}\left|\frac{q}{4e\alpha p}Y\right|^{2k} \le 1+\sum_{k=1}^{p/2}\left(\frac{q}{8\alpha k}\right)^{2k}\mathbb{E}|Y|^{2k}.$$

Since $\alpha \ge 1$ we obviously have

$$1+\sum_{k=1}^{q/2}\left(\frac{q}{8\alpha k}\right)^{2k}\mathbb{E}|Y|^{2k} \le \mathbb{E}|1+Y|^q \le e^A.$$

The α-regularity of moments of Y yields

$$\sum_{k=q/2+1}^{p/2}\left(\frac{q}{8\alpha k}\right)^{2k}\mathbb{E}|Y|^{2k} \le \sum_{k=q/2+1}^{p/2}\left(\frac{1}{4}\|Y\|_q\right)^{2k} \le \left(\frac{1}{4}\|Y\|_q\right)^q\sum_{l=1}^{\infty}\left(\frac{e}{4}\right)^{2l} \le \|Y\|_q^q.$$

Thus

$$\mathbb{E}\left|1+\frac{q}{4e\alpha p}Y\right|^p \le e^A+\|Y\|_q^q \le 2e^A-1 \le e^{2A} \le e^{pA/q},$$

which completes the proof of the claim and of the lemma. □

We finish this section with the observation that will allow us to compare regular r.v.s with variables with log-concave tails.

Lemma 14. *Let a nondecreasing function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfy*

$$f(c\lambda t) \ge \lambda f(t), \qquad \text{for } \lambda \ge 1,\ t \ge t_0,$$

where $t_0 \ge 0$, $c \ge 2$ are some constants. Then there is a convex function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g(t) \le f(t) \le g(c^2 t), \qquad \text{for } t \ge ct_0,$$

and $g(t) = 0$ for $t \in [0, ct_0]$.

Proof. We set $g(t) = 0$ for $t \in [0, ct_0]$ and

$$g(t) := \int_{ct_0}^{t} \sup_{ct_0 \le y \le x} \frac{f(y/c)}{y} \mathrm{d}x \quad \text{for } t \ge ct_0.$$

Then g is convex as an integral of a nondecreasing function. For $t \ge x \ge ct_0$ we have $\sup_{ct_0 \le y \le x} f(y/c)/y \le f(t)/t$, as $f(\lambda y)/(\lambda y) \ge f(y/c)/y$ for $y \ge ct_0$ and $\lambda \ge 1$. Thus

$$g(t) \le (t - ct_0)\frac{f(t)}{t} \le f(t), \qquad \text{for } t \ge ct_0.$$

Moreover, for $t \ge ct_0$

$$\begin{aligned} g(ct) &= \int_{ct_0}^{ct} \sup_{ct_0 \le y \le x} \frac{f(y/c)}{y} \mathrm{d}x \ge \int_{t}^{ct} \sup_{ct_0 \le y \le x} \frac{f(y/c)}{y} \mathrm{d}x \\ &\ge (ct - t)\frac{f(t/c)}{t} = (c-1)f(t/c) \ge f(t/c), \end{aligned}$$

hence $g(c^2 t) \ge f(t)$ for $t \ge ct_0$. □

4 Proofs

4.1 Sudakov minoration principle

The main goal of this section is to prove Theorem 3. The strategy of the proof is to reduce the problem involving random variables with moments growing regularly to the case of random variables with log-concave tails, for which the minoration is known (Theorem 1). This reduction hinges on the idea that the tail functions of random variables with regular growth of moments ought to be close to log-concave functions as, conversely, log-concave random variables are regular.

Proposition 15. *Let $\alpha \ge 1$. There exist constants T_α, L_α such that for any $X \in \mathcal{R}_\alpha$ there is a nondecreasing function $M\colon [0,\infty) \to [0,\infty]$ which is convex, $M(T_\alpha) = 0$, and satisfies*

$$M(t) \le N(t) \le M(L_\alpha t), \qquad \text{for } t \ge T_\alpha, \tag{9}$$

where $N(t) = -\ln \mathbb{P}(|X| > t)$.

Proof. Fix $\alpha \ge 1$. We begin with showing that there is a constant κ_α such that for any $X \in \mathcal{R}_\alpha$,

$$N(\kappa_\alpha \lambda t) \ge \lambda N(t), \qquad \lambda \ge 1,\ t \ge 1 - 1/e. \tag{10}$$

When $\|X\|_\infty < \infty$ it is enough to prove this assertion for $t < (1-1/e)\|X\|_\infty$ as, providing that $\kappa_\alpha \ge (1-1/e)^{-1}$, for $t \ge (1-1/e)\|X\|_\infty$ we have $N(\kappa_\alpha \lambda t) \ge N(\|X\|_\infty) = \infty$.

So, fix $\lambda \geq 1$ and $1 - 1/e \leq t < (1-1/e)\|X\|_\infty$. There exists $q \geq 2$ such that $t = (1-1/e)\|X\|_q$. Pick also $p \geq q$ so that $\lambda = p/q$. By the Paley-Zygmund inequality (8) and by the assumption that $X \in \mathcal{R}_\alpha$ we obtain

$$
\begin{aligned}
N(t) &= N\left((1-1/e)\|X\|_q\right) \leq N\left((1-1/e)^{1/q}\|X\|_q\right) \\
&= -\ln \mathbb{P}(|X|^q > (1-1/e)\mathbb{E}|X|^q) \leq -\ln\left(\frac{1}{e^2}\left(\frac{\|X\|_q}{\|X\|_{2q}}\right)^{2q}\right) \\
&\leq 2 + q\ln\left[(2\alpha)^2\right] \leq q\ln\left(e(2\alpha)^2\right) =: qb_\alpha. \qquad (11)
\end{aligned}
$$

On the other hand, setting $\kappa_\alpha = e^{b_\alpha}(1-1/e)^{-1}\alpha$, with the aid of the assumption that $X \in \mathcal{R}_\alpha$ and Chebyshev's inequality, we get

$$
\begin{aligned}
N(\kappa_\alpha \lambda t) &= N\left(e^{b_\alpha}\alpha\frac{p}{q}\|X\|_q\right) \geq N\left(e^{b_\alpha}\|X\|_p\right) \\
&= -\ln \mathbb{P}(|X|^p > e^{pb_\alpha}\mathbb{E}|X|^p) \geq pb_\alpha = \lambda q b_\alpha. \qquad (12)
\end{aligned}
$$

Joining inequalities (11) and (12) we get (10) with $\kappa_\alpha = \frac{4e^2}{e-1}\alpha^3$.

By virtue of this *sublinear* property (10), Lemma 14 applied to $f = N$, $c = \kappa_\alpha$, and $t_0 = 1 - 1/e$ finishes the proof, providing the constants

$$
L_\alpha = \kappa_\alpha^2 = \left(\frac{4e^2}{e-1}\alpha^3\right)^2, \qquad T_\alpha = \kappa_\alpha t_0 = 4e\alpha^3.
$$

□

Proof of Theorem 3. We fix $p \geq 2$, $T \subset \ell^2$ such that $|T| \geq e^p$ and $\|X_s - X_t\|_p \geq u$ for all distinct $s, t \in T$. We are to show that $\mathbb{E}\sup_{s,t\in T}(X_s - X_t) \geq \kappa_\alpha u$ for a constant κ_α which depends only on α. By Lemma 11 we may assume that r.v.s X_i are symmetric.

Proposition 15 yields that the tail functions $N_i(t) := -\ln \mathbb{P}(|X_i| > t)$ of the variables X_i are controlled by the convex functions $M_i(t)$, apart from $t \leq T_\alpha$, i.e. we have $M_i(t) \leq N_i(t) \leq M_i(L_\alpha t)$ only for $t \geq T_\alpha$. To gain control also for $t \leq T_\alpha$, define the symmetric random variables

$$
\widetilde{X}_i = (\operatorname{sgn} X_i)\max\{|X_i|, T_\alpha\},
$$

so that their tail functions $\widetilde{N}_i(t) = -\ln \mathbb{P}(|\widetilde{X}_i| > t)$,

$$
\widetilde{N}_i(t) = \begin{cases} 0, & t < T_\alpha \\ N_i(t), & t \geq T_\alpha \end{cases},
$$

satisfy

$$
M_i(t) \leq \widetilde{N}_i(t) \leq M_i(L_\alpha t) \qquad \text{for all } t \geq 0. \qquad (13)
$$

This allows us to construct a sequence $Y_1, Y_2, \dots$ of independent symmetric r.v.s with log-concave tails given by $\mathbb{P}(|Y_i| > t) = e^{-M_i(t)}$ such that

$$|Y_i| \geq |\widetilde{X}_i| \geq \frac{1}{L_\alpha}|Y_i|. \tag{14}$$

Define the canonical processes $\widetilde{X}_t := \sum_{i=1}^{\infty} t_i \widetilde{X}_i$ and $Y_t := \sum_{i=1}^{\infty} t_i Y_i$, $t \in \ell^2$.

Since $|Y_i| \geq |X_i|$ and variables Y_i and X_i are symmetric we get for $s, t \in T$, $s \neq t$,

$$\|Y_s - Y_t\|_p = \left\|\sum_{i=1}^{\infty}(s_i - t_i)|Y_i|\varepsilon_i\right\|_p \geq \left\|\sum_{i=1}^{\infty}(s_i - t_i)|X_i|\varepsilon_i\right\|_p = \|X_s - X_t\|_p \geq u,$$

where the first inequality follows by contraction principle (6) as $|Y_i| \geq |\widetilde{X}_i| \geq |X_i|$. Hence we can apply Theorem 1 to the canonical process (Y_t) and obtain

$$2\mathbb{E}\sup_{t\in T} Y_t = \mathbb{E}\sup_{s,t\in T}(Y_s - Y_t) \geq \kappa_{\mathrm{lct}} u. \tag{15}$$

To finish the proof it suffices to show that $\mathbb{E}\sup_{t\in T} X_t$ majorizes $\mathbb{E}\sup_{t\in T} Y_t$. Clearly,

$$\mathbb{E}\sup_{t\in T} X_t \geq \mathbb{E}\sup_{t\in T}\widetilde{X}_t - \mathbb{E}\sup_{t\in T}(\widetilde{X}_t - X_t). \tag{16}$$

Recall that by the definition of $\widetilde{X}_i$, $|\widetilde{X}_i - X_i| = |T_\alpha - X_i|\mathbf{1}_{\{|X_i|\leq T_\alpha\}} \leq T_\alpha$. As a consequence, the supremum of the canonical process $\mathbb{E}\sup_{t\in T}(\widetilde{X}_t - X_t)$ is bounded by the supremum of the Bernoulli process $\mathbb{E}\sup_{t\in T}\sum t_i T_\alpha \varepsilon_i$. Indeed, using the symmetry of the distribution of the variables $\widetilde{X}_i - X_i$ and contraction principle (7),

$$\mathbb{E}\sup_{t\in T}(\widetilde{X}_t - X_t) = \mathbb{E}_X\mathbb{E}_\varepsilon\sup_{t\in T}\sum_{i=1}^{\infty} t_i|\widetilde{X}_i - X_i|\varepsilon_i \leq \mathbb{E}_\varepsilon\sup_{t\in T}\sum_{i=1}^{\infty} t_i T_\alpha\varepsilon_i.$$

Since $X_i \in \mathcal{R}_\alpha$ we get by Hölder's inequality,

$$1 = \mathbb{E}X_i^2 = \mathbb{E}X_i^{4/3}X_i^{2/3} \leq \|X_i\|_4^{4/3}\|X_i\|_1^{2/3} \leq (2\alpha\|X_i\|_2)^{4/3}\|X_i\|_1^{2/3} = (2\alpha)^{4/3}(\mathbb{E}|X_i|)^{2/3}$$

and thus $\mathbb{E}|X_i| \geq (2\alpha)^{-2}$. Hence by Jensen's inequality

$$\mathbb{E}\sup_{t\in T} X_t = \mathbb{E}_\varepsilon\mathbb{E}_X\sup_{t\in T}\sum_{i=1}^{\infty} t_i|X_i|\varepsilon_i \geq \mathbb{E}_\varepsilon\sup_{t\in T}\sum_{i=1}^{\infty} t_i\mathbb{E}_X|X_i|\varepsilon_i \geq \frac{1}{(2\alpha)^2}\mathbb{E}\sup_{t\in T}\sum_{i=1}^{\infty} t_i\varepsilon_i.$$

As a result,

$$\mathbb{E}\sup_{t\in T}(\widetilde{X}_t - X_t) \leq (2\alpha)^2 T_\alpha\mathbb{E}\sup_{t\in T} X_t,$$

and by (16),

$$\mathbb{E}\sup_{t\in T} X_t \geq \frac{1}{1+(2\alpha)^2 T_\alpha}\mathbb{E}\sup_{t\in T}\widetilde{X}_t. \tag{17}$$

Finally, notice that, by virtue of contraction principle (7), the second inequality of (14) implies that

$$\mathbb{E}\sup_{t\in T}\widetilde{X}_t \geq \frac{1}{L_\alpha}\mathbb{E}\sup_{t\in T} Y_t. \tag{18}$$

Estimates (15), (17) and (18) yield

$$\mathbb{E}\sup_{s,t\in T}(X_s - X_t) = 2\mathbb{E}\sup_{t\in T} X_t \geq \frac{2}{L_\alpha(1+(2\alpha)^2 T_\alpha)}\mathbb{E}\sup_{t\in T} Y_t \geq \frac{\kappa_{\mathrm{lct}}}{L_\alpha(1+(2\alpha)^2 T_\alpha)} u.$$

□

Proof of Theorem 5. Using a symmetrization argument we may assume that the variables X_i are symmetric. Let variables $\widetilde{X}_i, Y_i$ and the related canonical processes be as in the proof of Theorem 3. Since the variables Y_i have log-concave tails we get by [5]

$$\left(\mathbb{E}\sup_{t\in T}|Y_t|^p\right)^{1/p} \leq C\left(\mathbb{E}\sup_{t\in T}|Y_t| + \sup_{t\in T}(\mathbb{E}|Y_t|^p)^{1/p}\right).$$

Estimate $|Y_i| \geq |X_i|$ and the contraction principle yield

$$\mathbb{E}\sup_{t\in T}|X_t|^p \leq \mathbb{E}\sup_{t\in T}|Y_t|^p.$$

We showed above that

$$\mathbb{E}\sup_{t\in T}|Y_t| \leq L_\alpha(1+(2\alpha)^2 T_\alpha)\mathbb{E}\sup_{t\in T}|X_t|.$$

Finally, the contraction principle together with the bounds $|Y_i| \leq L_\alpha|\widetilde{X}_i|$, $|X_i - \widetilde{X}_i| \leq T_\alpha$ and $\mathbb{E}|X_i| \geq (2\alpha)^{-2}$ imply

$$\|Y_t\|_p \leq L_\alpha\|\widetilde{X}_t\|_p \leq L_\alpha\|X_t\|_p + L_\alpha T_\alpha\left\|\sum_{i=1}^{\infty} t_i\varepsilon_i\right\|_p \leq L_\alpha(1+T_\alpha(2\alpha)^2)\|X_t\|_p.$$

□

We conclude this section with the proof of Proposition 4 showing that in the i.i.d. case the Sudakov minoration principle and the α-regular growth of moments are equivalent.

Proof of Proposition 4. Let us fix $p \geq q \geq 2$ and for $1 \leq m \leq n$ consider the following subset of ℓ^2

$$T = T(m,n) = \left\{ t \in \{0,1\}^{\mathbb{N}} \colon \ \sum_{i=1}^{n} t_i = m,\ t_i = 0, i > n \right\}.$$

Then $|T| = \binom{n}{m} \geq (n/m)^m \geq e^p$ if $n \geq m e^{p/m}$. Moreover, for any $s, t \in T$, $s \neq t$, say with $s_j \neq t_j$ we have $\|X_s - X_t\|_p \geq \|X_j\|_p$. Thus the Sudakov minoration principle yields for any $n \geq m e^{p/m}$,

$$\kappa \|X_i\|_p \leq \mathbb{E} \sup_{s,t \in T} (X_s - X_t) \leq 2\mathbb{E} \sup_{\substack{I \subset [n] \\ |I| = m}} \sum_{i \in I} |X_i| = 2\mathbb{E} \sum_{k=1}^{m} X_k^*, \tag{19}$$

where $(X_1^*, X_2^*, \ldots, X_n^*)$ is the nonincreasing rearrangement of $(|X_1|, |X_2|, \ldots, |X_n|)$.

We have

$$\mathbb{P}(X_k^* \geq t) = \mathbb{P}\left(\sum_{i=1}^{n} \mathbf{1}_{\{|X_i| \geq t\}} \geq k \right) \leq \frac{1}{k} \sum_{i=1}^{n} \mathbb{E}\mathbf{1}_{\{|X_i| \geq t\}} = \frac{n}{k} \mathbb{P}(|X_i| \geq t) \leq \frac{n}{k} \frac{\|X_i\|_q^q}{t^q}.$$

Integration by parts shows that

$$\mathbb{E} X_k^* = \int_0^\infty \mathbb{P}(X_k^* \geq t) \leq \int_0^\infty \min\left\{1, \frac{n}{k} \frac{\|X_i\|_q^q}{t^q}\right\} \leq C \left(\frac{n}{k}\right)^{1/q} \|X_i\|_q.$$

Combining this with (19) we get (recall that $q \geq 2$ and constant C may differ at each occurrence)

$$\kappa \|X_i\|_p \leq C \sum_{k=1}^{m} \left(\frac{n}{k}\right)^{1/q} \|X_i\|_q \leq C n^{1/q} m^{1-1/q} \|X_i\|_q.$$

Taking $m = \lceil p/q \rceil$ and $n = \lceil m e^{p/m} \rceil$ we find that $n^{1/q} m^{1-1/q} \leq 4ep/q$. Hence

$$\|X_i\|_p \leq \frac{C}{\kappa} \frac{p}{q} \|X_i\|_q$$

which finishes the proof. □

4.2 Lower bounds

As in the case of the Sudakov minoration principle the proof of the lower bound in Theorem 6 is based on the corresponding result for the canonical processes built on variables with log-concave tails, that is Theorem 2.

Proposition 16. *Let $\alpha \geq 1, \beta > 1$. For any $r > 1$ there exists a constant $C(\alpha, \beta, r)$ such that for $X \in \mathcal{R}_\alpha \cap \mathcal{S}_\beta$ we have*

$$N(rt) \leq C(\alpha, \beta, r) N(t), \qquad t \geq 2, \tag{20}$$

where $N(t) := -\ln \mathbb{P}(|X| > t)$.

Proof. Fix $t \geq 2$ and define

$$q := \inf\{p \geq 2 : \ \|X\|_{\beta p} \geq t\}.$$

Since $X \in \mathcal{R}_\alpha \cap \mathcal{S}_\beta$, the function $p \longmapsto \|X\|_p$ is finite and continuous on $[2, \infty)$, moreover $\|X\|_2 = 1$ and $\|X\|_\infty = \infty$. Hence, if $t \geq \|X\|_{2\beta}$, we have $t = \|X\|_{\beta q}$ and by Chebyshev's inequality,

$$N(t) = N(\|X\|_{\beta q}) \geq N(2\|X\|_q) = -\ln \mathbb{P}(|X|^q > 2^q \mathbb{E}|X|^q) \geq q \ln 2.$$

If $2 \leq t < \|X\|_{2\beta}$, then $q = 2$ and

$$N(t) \geq N(2) = -\ln \mathbb{P}(|X|^2 > 4\mathbb{E}|X|^2) \geq \ln 4 = q \ln 2.$$

Set an integer k such that $r \leq 2^{k-2}$. Then, using consecutively the definition of q, the assumption that $X \in \mathcal{S}_\beta$, the Paley-Zygmund inequality, and the assumption that $X \in \mathcal{R}_\alpha$, we get the estimates

$$\begin{aligned} N(rt) &\leq N\left(2^{k-2}\|X\|_{\beta q}\right) \leq N\left(\frac{1}{2}\|X\|_{\beta^k q}\right) = -\ln \mathbb{P}\left(|X|^{\beta^k q} > 2^{-\beta^k q}\mathbb{E}|X|^{\beta^k q}\right) \\ &\leq -\ln\left(\frac{1}{4}\left(\frac{\|X\|_{\beta^k q}}{\|X\|_{2\beta^k q}}\right)^{2\beta^k q}\right) \leq \ln 4 + 2\beta^k q \ln(2\alpha) \leq q(\ln 2 + 2\beta^k \ln(2\alpha)). \end{aligned} \tag{21}$$

Combining the above estimates we obtain the assertion with $C(\alpha, \beta, r) = (\ln 2 + 2\beta^k \ln(2\alpha))/\ln 2$ and $k = k(r)$ being an integer such that $2^{k-2} \geq r$. □

Remark 7. Taking in (21) $t = 2$ which corresponds to $q = 2$ we find that

$$N(s) \leq 2(\ln 2 + 2\beta^k \ln(2\alpha)), \qquad \text{for } s < 2^{k-1},$$

which means that the tail distribution function of a variable $X \in \mathcal{R}_\alpha \cap \mathcal{S}_\beta$ at a certain value s is bounded with a constant not depending on the distribution of X but only on the parameters α, β and of course the value of s.

Proof of Theorem 6. In view of (4) we are to address only the lower bound on $\mathbb{E}\sup_{t \in T} X_t$. A symmetrization argument shows that we may assume that variables X_i are symmetric.

Let L_α, T_α be the constants as in Proposition 15. Given symmetric X_i let Y_i be random variables defined as in the proof of Theorem 3, i.e. Y_i's are independent symmetric r.v.s having log-concave tails $\mathbb{P}(|Y_i| > t) = e^{-M_i(t)}$. Due to Proposition 16 for $r = 2L_\alpha$ we know that the functions $N_i(t) := -\mathbb{P}(|X_i| > t)$ satisfy

$$N_i(2L_\alpha t) \le \gamma N(t), \qquad t \ge 2,$$

where $\gamma = \gamma(\alpha, \beta) := C(\alpha, \beta, 2L_\alpha)$.

What then can be said about M_i? Using (9) we find that for $t \ge \tilde{T}_\alpha := \max\{2, T_\alpha\}$

$$M_i(2L_\alpha t) \le N_i(2L_\alpha t) \le \gamma N_i(t) \le \gamma M_i(L_\alpha t),$$

which means that M_i are almost of moderate growth, namely for $t_\alpha := L_\alpha \tilde{T}_\alpha$ we have

$$M_i(2t) \le \gamma M_i(t), \qquad t \ge t_\alpha.$$

Therefore, we improve the function M_i putting on the interval $[0, t_\alpha]$ an artificial linear piece $t \mapsto \lambda(i, \alpha)t$, where $\lambda(i, \alpha) := M_i(t_\alpha)/t_\alpha$. In other words, take the numbers $p(i, \alpha) := \mathbb{P}(|Y_i| > t_\alpha) = e^{-M_i(t_\alpha)}$ and let U_i be a sequence of independent random variables with the following symmetric *truncated* exponential distribution,

$$\mathbb{P}(|U_i| > t) = \begin{cases} \frac{e^{-\lambda(i,\alpha)t} - p(i,\alpha)}{1 - p(i,\alpha)}, & t \le t_\alpha \\ 0, & t > t_\alpha \end{cases},$$

which are in addition independent of the sequences (X_i) and (Y_i). Define

$$Z_i := Y_i \mathbf{1}_{\{|Y_i| > t_\alpha\}} + U_i \mathbf{1}_{\{|Y_i| \le t_\alpha\}}.$$

Let

$$\widetilde{M}_i(t) := -\ln \mathbb{P}(|Z_i| > t) = \begin{cases} \lambda(i, \alpha)t, & t \le t_\alpha, \\ M_i(t), & t > t_\alpha. \end{cases}$$

Then $\widetilde{M}_i$ are convex functions of moderate growth

$$\widetilde{M}_i(2t) \le 2\gamma \widetilde{M}_i(t), \qquad t \ge 0.$$

Thus Theorem 2 can be applied to the canonical process $Z_t := \sum_i t_i Z_i$ and we get

$$\mathbb{E} \sup_{t \in T} Z_t \ge \frac{1}{C_1(\alpha, \beta)} \gamma_Z(T),$$

where $C_1(\alpha, \beta) = C_{\mathrm{lct}}(2\gamma)$.

What is left is to compare both the suprema and the functionals γ's of the processes (X_t) and (Z_t). The former is easy, because we have $M_i(t) \le \widetilde{M}_i(t)$, $t \ge 0$, which allows to take samples such that $|Y_i| \ge |Z_i|$, and consequently, thanks to contraction principle (7), $\mathbb{E} \sup_{t \in T} Z_t \le \mathbb{E} \sup_{t \in T} Y_t$. Joining this with estimates (18) and (17) we derive

$$\mathbb{E} \sup_{t \in T} Z_t \le L_\alpha (1 + (2\alpha)^2 T_\alpha) \mathbb{E} \sup_{t \in T} X_t.$$

For the latter, we would like to show $C(\alpha,\beta)\gamma_Z \geq \gamma_X$. It is enough to compare the metrics, i.e. to prove that $C(\alpha,\beta)\|Z_s - Z_t\|_p \geq \|X_s - X_t\|_p$ for $p \geq 1$. We proceed as in the proof of Theorem 3. We have

$$\|Z_s - Z_t\|_p \geq \|Y_s - Y_t\|_p - \|(Y_s - Z_s) - (Y_t - Z_t)\|_p. \tag{22}$$

In the proof of Theorem 3 it was established that $\|Y_s - Y_t\|_p \geq \|X_s - X_t\|_p$. For the second term we use the symmetry of the variables $Y_i - Z_i$, contraction principle (6), and the fact that $|Y_i - Z_i| \leq 2t_\alpha$, obtaining

$$\|(Y_s - Z_s) - (Y_t - Z_t)\|_p = \left\|\sum_i (s_i - t_i)|Y_i - Z_i|\varepsilon_i\right\|_p \leq 2t_\alpha \left\|\sum_i (s_i - t_i)\varepsilon_i\right\|_p. \tag{23}$$

Now we compare $\|Z_s - Z_t\|_p$ with moments of increments of the Bernoulli process. By Jensen's inequality we get

$$\|Z_s - Z_t\|_p = \left\|\sum_i (s_i - t_i)|Z_i|\varepsilon_i\right\|_p \geq \min_i \mathbb{E}|Z_i| \left\|\sum_i (s_i - t_i)\varepsilon_i\right\|_p. \tag{24}$$

Combining (22), (23), and (24) yields

$$\|Z_s - Z_t\|_p \geq \left(1 + \frac{2t_\alpha}{\min_i \mathbb{E}|Z_i|}\right)^{-1} \|X_s - X_t\|_p.$$

To finish it suffices to prove that $\mathbb{E}|Z_i| \geq c_{\alpha,\beta}$ for some positive constant $c_{\alpha,\beta}$, which depends only on α and β. This is a cumbersome yet simple calculation. Recall the distributions of the variables Y_i and U_i, the fact that they are independent, and observe that

$$\begin{aligned}
\mathbb{E}|Z_i| &= \mathbb{E}|Y_i|\mathbf{1}_{\{|Y_i|>t_\alpha\}} + \mathbb{E}|U_i|\mathbf{1}_{\{|Y_i|\leq t_\alpha\}} \\
&\geq t_\alpha \mathbb{P}(|Y_i| > t_\alpha) + (\mathbb{E}|U_i|)\,\mathbb{P}(|Y_i| \leq t_\alpha) \\
&= t_\alpha p(i,\alpha) + (1 - p(i,\alpha)) \int_0^{t_\alpha} \frac{e^{-\lambda(i,\alpha)t} - p(i,\alpha)}{1 - p(i,\alpha)} \mathrm{d}t \\
&= \frac{1}{\lambda(i,\alpha)}\left(1 - e^{-\lambda(i,\alpha)t_\alpha}\right) = \frac{t_\alpha}{M_i(t_\alpha)}\left(1 - e^{-M_i(t_\alpha)}\right).
\end{aligned}$$

The last expression is nonincreasing with respect to $M_i(t_\alpha)$. Since $M_i(t_\alpha) \leq N_i(t_\alpha)$ (see (9)), we are done provided that we can bound $N_i(t_\alpha)$ above. Thus, Remark 7 completes the proof. □

Proof of Corollary 8. Proposition 6.1 in [8] yields for $p \geq 1$,

$$\begin{aligned}\left(\mathbb{E}\sup_{s,t\in T}|Y_t-Y_s|^p\right)^{1/p} &\leq C(\gamma_Y(T)+\sup_{s,t\in T}\|Y_s-Y_t\|_p)\leq C(\gamma_X(T)+\sup_{s,t\in T}\|X_s-X_t\|_p)\\ &\leq C(\alpha,\beta)\left(\mathbb{E}\sup_{s,t\in T}|X_s-X_t|+\sup_{s,t\in T}\|X_s-X_t\|_p\right)\\ &\leq (C(\alpha,\beta)+1)\left\|\sup_{s,t\in T}|X_s-X_t|\right\|_p,\end{aligned}$$

where the third inequality follows by Theorem 6. Hence by Chebyshev's inequality we obtain

$$\mathbb{P}\left(\sup_{s,t\in T}|Y_t-Y_s|\geq C_1(\alpha,\beta)\left\|\sup_{s,t\in T}|X_s-X_t|\right\|_p\right)\leq e^{-p}\quad \text{for } p\geq 1. \tag{25}$$

Theorem 5 (used for the set $T-T$) and Lemma 13 yield for $p \geq q \geq 1$,

$$\left\|\sup_{s,t\in T}|X_s-X_t|\right\|_p\leq C_2(\alpha)\frac{p}{q}\left\|\sup_{s,t\in T}|X_s-X_t|\right\|_q.$$

Hence, by the Paley-Zygmund inequality we get for $q \geq 1$,

$$\mathbb{P}\left(\sup_{s,t\in T}|X_t-X_s|\geq\frac{1}{2}\left\|\sup_{s,t\in T}|X_s-X_t|\right\|_q\right)\geq\frac{1}{4}\left(\frac{1}{2C_2(\alpha)}\right)^{2q}.$$

Applying the above estimate with $q = p/(2\ln(2C_2(\alpha)))$ we get

$$\mathbb{P}\left(\sup_{t,s\in T}|X_t-X_s|\geq\frac{1}{2C_2(\alpha)\ln(2C_2\alpha)}\left\|\sup_{s,t\in T}|X_s-X_t|\right\|_p\right)\geq\frac{1}{4}e^{-p}\ \ \text{for } p\geq 2\ln(2C_2(\alpha)). \tag{26}$$

The assertion easily follows by (25) and (26).

□

Proof of Corollary 9. By Theorem 6 we may find an admissible sequence of partitions $(\mathcal{A}_n)$ such that

$$\sup_{t\in T}\sum_{n=0}^{\infty}\Delta_{2^n}(A_n(t))\leq C(\alpha,\beta)\mathbb{E}\sup_{s,t\in T}(X_s-X_t). \tag{27}$$

For any $A \in \mathcal{A}_n$ let us choose a point $\pi_n(A) \in A$ and set $\pi_n(t) := \pi_n(A_n(t))$.

Let $M_n := \sum_{j=0}^{n} N_j$ for $n = 0, 1, \ldots$ (recall that we denote $N_j = 2^{2^j}$ for $j \geq 1$ and $N_0 = 1$). Then $\log(M_n + 2) \leq 2^{n+1}$. Notice that there are $|\mathcal{A}_n| \leq N_n$ points of the form $\pi_n(t) - \pi_{n-1}(t)$, $t \in T$. So we may set $s^1 := 0$ and for $n = 1, 2, \ldots$ define s^k,

$M_{n-1} < k \leq M_n$ as some rearrangement (with repetition if $|\mathcal{A}_n| < N_n$) of points of the form $(\pi_n(t) - \pi_{n-1}(t))/d_{2^{n+1}}(\pi_n(t), \pi_{n-1}(t))$, $t \in T$. Then $\|X_{s^k}\|_{\log(k+2)} \leq 1$ for all k.

Observe that

$$\|t - \pi_n(t)\|_2 = \|X_t - X_{\pi_n(t)}\|_2 \leq \Delta_2(A_n(t)) \leq \Delta_{2^n}(A_n(t)) \to 0 \text{ for } n \to \infty.$$

For any $s, t \in T$ we have $\pi_0(s) = \pi_0(t)$ and thus

$$s - t = \lim_{n\to\infty} (\pi_n(s) - \pi_n(t)) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} (\pi_k(s) - \pi_{k-1}(s)) - \sum_{k=1}^{n} (\pi_k(t) - \pi_{k-1}(t)) \right).$$

This shows that

$$T - T \subset R\, \overline{\mathrm{conv}}\{\pm s^k\colon\ k \geq 1\},$$

where

$$\begin{aligned} R &:= 2 \sup_{t\in T} \sum_{n=1}^{\infty} d_{2^{n+1}}(\pi_n(t), \pi_{n-1}(t)) \leq 2 \sup_{t\in T} \sum_{n=1}^{\infty} \Delta_{2^{n+1}}(A_{n-1}(t)) \\ &\leq C(\alpha) \sup_{t\in T} \sum_{n=1}^{\infty} \Delta_{2^{n-1}}(A_{n-1}(t)) \leq C(\alpha, \beta) \mathbb{E} \sup_{s,t\in T} (X_s - X_t), \end{aligned}$$

where the second inequality follows by Lemma 13 and the last one by (27). Thus it is enough to define $t^k := R s^k$, $k \geq 1$. □

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References

[1] W. Bednorz and R. Latała, *On the boundedness of Bernoulli processes*, Ann. of Math. (2) 180 (2014), 1167–1203.

[2] X. Fernique, *Regularité des trajectoires des fonctions aléatoires gaussiennes*, École d'Été de Probabilités de Saint-Flour, IV-1974, Lecture Notes in Mathematics 480, 1–96, Springer, Berlin, 1975.

[3] E. D. Gluskin and S. Kwapień *Tail and moment estimates for sums of independent random variables with logarithmically concave tails*, Studia Math. **114** (1995), 303–309.

[4] S. Kwapień and W. A Woyczyński, *Random Series and Stochastic Integrals. Single and Multiple.*, Birkhauser, Boston 1992.

[5] R. Latała, *Tail and moment estimates for sums of independent random vectors with logarithmically concave tails*, Studia Math. **118** (1996), 301–304.

[6] R. Latała, *Estimation of moments of sums of independent real random variables*, Ann. Probab. **25** (1997), 1502–1513.

[7] R. Latała, *Sudakov minoration principle and supremum of some processes*, Geom. Funct. Anal. **7** (1997), 936–953.

[8] R. Latała, *Sudakov-type minoration for log-concave vectors*, Studia Math. 223 (2014), 251–274.

[9] M. Ledoux and M. Talagrand, *Probability in Banach spaces. Isoperimetry and processes.*, Springer-Verlag, Berlin 1991.

[10] M. Marcus and J. Rosen, *Markov processes, Gaussian processes, and local times*, Cambridge Studies in Advanced Mathematics, 100, Cambridge University Press, Cambridge, 2006.

[11] S. Mendelson, E. Milman and G. Paouris, *Towards a generalization of Sudakov's inequality via separation dimension reduction*, in preparation.

[12] S. Mendelson, G. Paouris, *On generic chaining and the smallest singular values of random matrices with heavy tails*, Journal of Functional Analysis, 262(9), 3775-3811, 2012.

[13] R.E.A.C. Paley and A. Zygmund, *A note on analytic functions in the unit circle*, Proc. Camb. Phil. Soc. **28** (1932), 266–272.

[14] M. Talagrand, *Regularity of Gaussian processes*, Acta Math. **159** (1987), 99–149.

[15] M. Talagrand, *The supremum of some canonical processes*, Amer. J. Math. **116** (1994), 283–325.

[16] M. Talagrand, *Upper and lower bounds for stochastic processes. Modern methods and classical problems*, Springer-Verlag, 2014.

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