Random graphs with a fixed maximum degree

Alan Frieze*and Tomasz Tkocz
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh PA15217
U.S.A.

March 13, 2019

Abstract

We study the component structure of the random graph $G = G_{n,m,d}$. Here $d = O(1)$ and G is sampled uniformly from $\mathcal{G}_{n,m,d}$, the set of graphs with vertex set $[n]$, m edges and maximum degree at most d. If $m = \mu n/2$ then we establish a threshold value $\mu_\star$ such that if $\mu < \mu_\star$ then w.h.p. the maximum component size is $O(\log n)$. If $\mu > \mu_\star$ then w.h.p. there is a unique giant component of order n and the remaining components have size $O(\log n)$.

2010 Mathematics Subject Classification. 05C80.

Key words. Random Graphs, Maximum Degree.

1 Introduction

We study the evolution of the component structure of the random graph $G_{n,m,d}$. Here $d = O(1)$ and G is sampled uniformly from $\mathcal{G}_{n,m,d}$, the set of graphs with vertex set $[n]$, m edges and maximum degree at most d. In the past the first author has studied properties of sparse random graphs with a lower bound on minimum degree, see for example [6]. In this paper we study sparse random graphs with a bound on the maximum degree. The model we study is close to, but distinct from that studied by Alon, Benjamini and Stacey [1] and Nachmias and Peres [12]. They studied the following model: begin with a random d-regular graph and then delete edges with probability $1 - p$. They show in [1] that for $d \geq 3$ there is a critical probability $p_c = \frac{1}{d-1}$ such that w.h.p. there is a "double jump" from components of maximum size $O(\log n)$ for $p < p_c$, a unique giant for $p > p_c$ and a mximum component size of order $n^{2/3}$ for $p = p_c$. The paper [12] does a detailed analysis of the scaling window around $p = p_c$.

*Research supported in part by NSF grant DMS1661063

Naively, one might think that this analysis covers $G_{n,m,d}$. We shall see however that $G_{n,m,d}$ and random subgraphs of random regular graphs have distinct degree sequence distributions. In the latter the number of vertices of degree $i = 0, 1, 2, \ldots, d$ will be n times a binomial random variable, whereas in $G_{n,m,d}$ this number will be asymptotic to n times a Poisson random variable, truncated from above.

We will write that $A_n \approx B_n$ if $A_n = (1 + o(1))B_n$ and $A_n \lesssim B_n$ if $A_n \leq (1 + o(1))B_n$ as $n \to \infty$.

For $d \geq 1$ and $\lambda > 0$ define

$$s_d(\lambda) = \sum_{j=0}^{d} \frac{\lambda^j}{j!} \qquad \text{and} \qquad f_d(\lambda) = \lambda \frac{s_{d-1}(\lambda)}{s_d(\lambda)}. \tag{1}$$

Theorem 1. *Let $d \geq 2$ and $\mu \in (0, d)$. Let $m = \lceil \frac{\mu n}{2} \rceil$. Let $G = G_{n,m,d}$ be a random graph chosen uniformly at random from the graphs with n vertices, m edges and maximum degree at most d. Let*

$$\mu_\star(d) = f_d(f_{d-1}^{-1}(1)), \qquad \textit{functional inverse being used here,}$$

where the functions f_k are defined in (1) and let λ satisfy

$$f_d(\lambda) = \mu. \tag{2}$$

The following hold w.h.p.

(a) The number $\nu_i, i = 0, 1, \ldots, d$ of vertices of degree i in G satisfies

$$\nu_i \approx \lambda_i n \text{ where } \lambda_i = \frac{1}{s_d(\lambda)} \frac{\lambda^i}{i!}. \tag{3}$$

(b) If $\mu < \mu_\star(d)$, then G has all components of size $O(\log n)$.

(c) If $\mu > \mu_\star(d)$, then G has a unique giant component of linear size Θn, where Θ is defined as follows: let $D = \sum_{i=1}^{L} i\lambda_i$ and

$$g(x) = D - 2x - \sum_{i=1}^{L} i\lambda_i \left(1 - \frac{2x}{D}\right)^{i/2}. \tag{4}$$

Let ψ be the smallest positive solution to $g(x) = 0$. Then

$$\Theta = 1 - \sum_{i=1}^{L} \lambda_i \left(1 - \frac{2\psi}{D}\right)^{i/2}.$$

All the other components are of size $O(\log n)$.

Remark 2. Numerical values of the threshold point $\mu_\star(d)$ for the average degree for small values of d are gathered in Table 1. Note that we have an exact expression for the case $d = 3$. We use $f_2(\lambda) = \frac{\lambda(1+\lambda)}{1+\lambda+\lambda^2/2}$ to see that $f_2^{-1}(1) = \sqrt{2}$. And then $\mu_\star(3) = \frac{\lambda(1+\lambda+\lambda^2/2)}{1+\lambda+\lambda^2/2+\lambda^3/6} = 3(\sqrt{2} - 1)$.

Moreover, if we consider large d, then we have, as a function of d,

$$\mu_\star(d) = 1 + \frac{1}{e(d-1)!} - \frac{1}{ed!} + O\left(\frac{1}{(d-1)!^2}\right). \tag{5}$$

Comparing to the percolation model considered in [1] and [12], where $\mu_\star(d) = 1 + \frac{1}{d-1}$, we see that in our model a giant occurs *significantly earlier* for large d. Approximation (5) can be justified as follows. We have

$$f_d(1) = \frac{s_{d-1}(1)}{s_d(1)} = 1 - \frac{1}{d!s_d(1)} = 1 - \frac{1}{ed!} + O\left(\frac{1}{d!^2}\right)$$

and

$$f_d'(1) = \frac{(s_{d-1}(1) + s_{d-2}(1))s_d(1) - s_{d-1}(1)^2}{s_d(1)^2} = 1 - \frac{1}{ed!} + O\left(\frac{1}{d!^2}\right),$$

(Express here s_{d-1} and s_{d-2} in terms of s_d and use $s_d(1) = e - O(1/d!)$).

If $f_{d-1}^{-1}(1) = 1 + \varepsilon$, then

$$1 = f_{d-1}(1+\varepsilon) = f_{d-1}(1) + f_{d-1}'(1)\varepsilon + O(\varepsilon^2),$$

which gives

$$\varepsilon + O(\varepsilon^2) = \frac{1 - f_{d-1}(1)}{f_{d-1}'(1)} = \frac{1}{e(d-1)!} + O\left(\frac{1}{(d-1)!d!}\right).$$

Consequently,

$$\mu_\star(d) = f_d(1+\varepsilon) = f_d(1) + f_d'(1)\frac{1 - f_{d-1}(1)}{f_{d-1}'(1)} + O(\varepsilon^2).$$

and (5) follows.

d	$\mu_\star(d)$
2	∞
3	$3(\sqrt{2}-1) = 1.23264\ldots$
4	1.05783
5	1.01309
6	1.00259
7	1.00044
8	1.00006

Table 1: Numerical values of $\mu_\star(d)$ for small d.

2 Proof of Theorem 1

The main idea is to estimate the degree distribution of $G_{n,m,d}$ and then apply the results of Molloy and Reed [10], [11].

2.1 Technical Lemmas

The following lemmas will be needed for the proof of part (a).

Lemma 3. *Let $\lambda > 0$, $d \geq 1$. Let $Z_1, Z_2, \ldots$ be i.i.d. random variables with*

$$\mathbb{P}\left(Z_i = k\right) = c_\lambda \frac{\lambda^k}{k!}, \qquad k = 0, 1, \ldots, d, \tag{6}$$

where

$$c_\lambda = \frac{1}{s_d(\lambda)}. \tag{7}$$

(a truncated Poisson distribution). Let $(x_1, \ldots, x_n)$ be a random vector of occupancies of boxes when m distinguishable balls are placed uniformly at random into n labelled boxes, each with capacity d. Then the vector $(Z_1, \ldots, Z_n)$ conditioned on $\sum_{j=1}^n Z_j = m$ has the same distribution as $(x_1, \ldots, x_n)$.

Proof. Let A be the set of vectors $z = (z_1, \ldots, z_n)$ of non-negative integers z_j such that $\sum_{j=1}^n z_j = m$ and $z_j \leq d$ for every j. Fix $z \in A$. We have

$$\begin{aligned}
\mathbb{P}\left((Z_1, \ldots, Z_n) = z \;\middle|\; \sum_{j=1}^n Z_j = m \right) &= \frac{\mathbb{P}\left((Z_1, \ldots, Z_n) = z\right)}{\mathbb{P}\left(\sum_{j=1}^n Z_j = m\right)} \\
&= \frac{\prod_{j=1}^n c_\lambda \frac{\lambda^{z_j}}{z_j!}}{\sum_{z \in A} \prod_{j=1}^n c_\lambda \frac{\lambda^{z_j}}{z_j!}} = \frac{\frac{1}{z_1! \cdot \ldots \cdot z_n!}}{\sum_{z \in A} \frac{1}{z_1! \cdot \ldots \cdot z_n!}}.
\end{aligned}$$

On the other hand, there are $\frac{m!}{z_1! \cdot \ldots \cdot z_n!}$ ways to place m balls into n labelled boxes in such a way that the jth box gets z_j balls. Therefore,

$$\mathbb{P}\left((x_1, \ldots, x_n) = z\right) = \frac{\frac{m!}{z_1! \cdot \ldots \cdot z_n!}}{\sum_{z \in A} \frac{m!}{z_1! \cdot \ldots \cdot z_n!}} = \mathbb{P}\left((Z_1, \ldots, Z_n) = z \;\middle|\; \sum_{j=1}^n Z_j = m \right).$$

□

Remark 4. The same argument can be adapted to different constraints for the occupancies of the boxes. In general, we can replace $k \in \{0, 1, \ldots, d\}$ by $k \in I$ for some set of non-negative integers I. For example, instead of restricting the maximal occupancy, we can require a minimal occupancy (which has appeared in Lemma 4 in [2]), or that the occupancy is even, etc.

A straightforward consequence of a standard i.i.d. case of the local central limit theorem (see, e.g. Theorem 3.5.2 in [5]) is the following lemma which will help us get rid of the conditioning from Lemma 3.

Lemma 5. *Let $\lambda > 0$, $d \geq 1$. Let $Z_1, Z_2, \ldots$ be i.i.d. truncated Poisson random variables defined by* (6) *and* (7)*. Then*

$$\sup_{m=0,1,2,\ldots} \sqrt{n} \left| \mathbb{P}\left(Z_1 + \ldots + Z_n = m\right) - \frac{1}{\sqrt{2\pi n \sigma^2}} \exp\left\{ -\frac{(m - \mu n)^2}{2n\sigma^2} \right\} \right| \xrightarrow[n \to \infty]{} 0, \tag{8}$$

where $\mu = \mathbb{E} Z_1$ and $\sigma^2 = \mathrm{Var}(Z_1)$.

We shall also need two lemmas concerning the function s_d from (1). A function f is log-concave if $\log f$ is concave.

Lemma 6. *For every $\lambda > 0$, the sequence $(s_d(\lambda))_{d=0}^\infty$ defined by* (1) *is log-concave, that is $s_{d-1}(\lambda)s_{d+1}(\lambda) \leq s_d(\lambda)^2$, $d \geq 1$.*

Proof. First note that the product of log-concave functions is log-concave. Integration by parts yields

$$e^{-\lambda}s_d(\lambda) = \int_\lambda^\infty \frac{t^d}{d!}e^{-t}\mathrm{d}t. \tag{9}$$

Given this integral representation, the log-concavity of $(s_d(\lambda))_{d=0}^\infty$ follows from a more general result saying that if $f : (0,\infty) \to [0,\infty)$ is log-concave, then the function $(0,+\infty) \ni p \mapsto \int_0^\infty \frac{t^p}{\Gamma(p+1)}f(t)\mathrm{d}t$ is also log-concave (apply to $f(t) = e^{-t}\mathbf{1}_{(\lambda,\infty)}(t)$). This result goes back to Borell's work [4] (for this exact formulation see, e.g. Corollary 5.13 in [8] or Theorem 5 in [13] containing a direct proof). □

Remark 7. The above theorem and proof uses two related notions of log-concavity. They are reconciled by the fact that if $f : (0,\infty) \to [0,\infty)$ is log-concave then the sequence $f(i), i = 0, 1, \ldots$ is also log-concave.

Lemma 8. *For every $k \geq 1$, the function f_k is strictly increasing on $(0,\infty)$ and onto $(0,k)$. In particular, the functional inverse, $f_k^{-1} : (0,k) \to (0,\infty)$ is well-defined, also strictly increasing.*

Proof. Fix $k \geq 1$ and consider f_k: rewriting (9) in terms of the upper incomplete gamma function $\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\mathrm{d}t$, we have

$$f_k(x) = k\frac{x\Gamma(k,x)}{\Gamma(k+1,x)}.$$

Differentiating,

$$\frac{\Gamma(k+1,x)^2}{k}\frac{\mathrm{d}}{\mathrm{d}x}f_{k+1}(x) = (\Gamma(k,x) - x^k e^{-x})\Gamma(k+1,x) + x^{k+1}e^{-x}\Gamma(k,x).$$

Using $\Gamma(k+1,x) = k\Gamma(k,x) + x^k e^{-x}$ we can express the condition $\frac{\mathrm{d}}{\mathrm{d}x}f_{k+1}(x) > 0$ as a quadratic inequality for $\Gamma(k,x)$:

$$k\Gamma(k,x)^2 + x^k e^{-x}(x-k+1)\Gamma(k,x) - x^{2k}e^{-2x} > 0,$$

or

$$\left(\Gamma(k,x) + \frac{x^k e^{-x}(x-k+1)}{2k}\right)^2 > \frac{x^{2k}e^{-2x}}{k} + \left(\frac{x^k e^{-x}(x-k+1)}{2k}\right)^2$$

or

$$\Gamma(k,x) > \frac{x^k e^{-x}}{2k}\left(\sqrt{(x-k+1)^2 + 4k} - (x-k+1)\right). \tag{10}$$

Let $h(x)$ be the left hand side minus the right hand side of (10). Clearly, $h(0) = (k-1)! > 0$. Moreover, using a standard asymptotic expansion

$$\Gamma(k,x) \approx x^{k-1}e^{-x}\left(1 + \frac{k-1}{x} + \frac{(k-1)(k-2)}{x^2} + \ldots\right), \text{ as } x \to \infty,$$

we can check that $h(x) \approx x^{k-1}e^{-x}(\frac{1}{x^2} + \ldots)$, so $h(x) \to 0$ as $x \to \infty$. Thus to see that $h(x) > 0$ for $x > 0$, it suffices to check that $h'(x) < 0$ for $x > 0$. We have,

$$\begin{aligned}
h'(x) &= -x^{k-1}e^{-x} - \frac{x^{k-1}e^{-x}}{2k}(k-x)\left(\frac{x-k+1}{\sqrt{(x-k+1)^2+4k}} - 1\right) \\
&= -\frac{x^{k-1}e^{-x}}{2k\sqrt{(x-k+1)^2+4k}}\Big(2k\sqrt{(x-k+1)^2+4k} + (k-x)\big((x-k+1) \\
&\qquad\qquad - \sqrt{(x-k+1)^2+4k}\big)\Big) \\
&= -\frac{x^{k-1}e^{-x}}{2k\sqrt{(x-k+1)^2+4k}}\left((k+x)\sqrt{(x-k+1)^2+4k} + (k-x)(x-k+1)\right),
\end{aligned}$$

so $h'(x) < 0$ is equivalent to

$$(k+x)\sqrt{(x-k+1)^2+4k} > (x-k)(x-k+1).$$

When $k-1 < x < k$, the right hand side is negative, so the inequality is clearly true. Otherwise, squaring it, we equivalently get

$$(k+x)^2((x-k+1)^2+4k) > (x-k)^2(x-k+1)^2$$

which is clearly true because $(k+x)^2 > (x-k)^2$ for $x > 0$.

It is clear from (7) and (1) that f_k is a ratio of two polynomials, each of degree k and $f_k(x) = \frac{\frac{x^k}{(k-1)!}+\ldots}{\frac{x^k}{k!}+\ldots}$, so $f_k(x) \to k$ as $x \to \infty$. This combined with the monotonicity and $f_k(0) = 0$ justifies that f_k is a bijection onto $(0, k)$. □

2.2 Main elements of the proof

Let $\mathcal{D}$ be the set of all sequences of nonnegative integers $x_1, \ldots, x_n \le d$ such that $\sum x_i = 2m$ (possible degrees). For $x \in \mathcal{D}$, let $\mathcal{G}_{n,x}$ be the set of all simple graphs on vertex set $[n]$ such that vertex i has degree x_i, $i = 1, 2, \ldots, n$. We study graphs in $\mathcal{G}_{n,x}$ via the Configuration Model of Bollobás [3]. We do this as follows: let Z_x be the multi-set consisting of x_i copies of i, for $i = 1, 2, \ldots, n$ and let $z = z_1, z_2, \ldots, z_{2m}$ be a random permutation of Z_x. We then define Γ_z to be the (configuration) multigraph with vertex set $[n]$ and edges $\{z_{2i-1}, z_{2i}\}$ for $i = 1, 2, \ldots, m$. It is a classical fact that conditional on being simple, Γ_z is distributed as a uniform random member of $\mathcal{G}_{n,x}$, see for example Section 11.1 of [7].

Let $\alpha_x = \frac{\sum_i x_i(x_i-1)}{2m}$. Note that $0 \le \alpha_x \le d$. It is known that

$$|\mathcal{G}_{n,x}| \approx e^{-\alpha_x(\alpha_x+1)}\frac{(2m)!}{\prod_i x_i!}$$

as $n \to \infty$ with the $o(1)$ term being uniform in x (in fact, depending only on $\Delta = \max_i x_i$). Here the term $e^{-\alpha_x(\alpha_x+1)}$ is the asymptotic probability that Γ_z is simple. Therefore, for any $x \in \mathcal{D}$, we have

$$\mathbb{P}\left(G_{n,m,d} \in \mathcal{G}_{n,x}\right) = \frac{|\mathcal{G}_{n,x}|}{\sum_{y\in\mathcal{D}}|\mathcal{G}_{n,y}|} \lesssim e^{d(d+1)}\frac{\frac{(2m)!}{\prod_i x_i!}}{\sum_{y\in\mathcal{D}}\frac{(2m)!}{\prod_i y_i!}},$$

which by Lemma 3 gives

$$\mathbb{P}\left(G_{n,m,d} \in \mathcal{G}_{n,x}\right) \lesssim e^{d(d+1)} \mathbb{P}\left(Z = x \;\Big|\; \sum_i Z_i = 2m\right),$$

where $Z_1, \ldots, Z_n$ are i.i.d. truncated Poisson random variables defined in (6).

For any graph property $\mathcal{P}$, we thus have

$$\begin{aligned}
\mathbb{P}\left(G_{n,m,d} \in \mathcal{P}\right) &= \sum_{x \in \mathcal{D}} \mathbb{P}\left(G_{n,m,d} \in \mathcal{P} \mid G_{n,m,d} \in \mathcal{G}_{n,x}\right) \mathbb{P}\left(G_{n,m,d} \in \mathcal{G}_{n,x}\right) \\
&= \sum_{x \in \mathcal{D}} \mathbb{P}\left(G_{n,x} \in \mathcal{P}\right) \mathbb{P}\left(G_{n,m,d} \in \mathcal{G}_{n,x}\right) \\
&\lesssim e^{d(d+1)} \sum_{x \in \mathcal{D}} \mathbb{P}\left(G_{n,x} \in \mathcal{P}\right) \mathbb{P}\left(Z = x \;\Big|\; \sum_i Z_i = 2m\right), \qquad (11)
\end{aligned}$$

where $G_{n,x}$ denotes a random graph selected uniformly at random from $\mathcal{G}_{n,x}$.

To handle the conditioning, we have chosen λ so that $\mu = \mathbb{E}Z_1$, that is the value of λ given by (2).

From Lemma 5 we get that for arbitrary $\delta > 0$, for sufficiently large n,

$$\mathbb{P}\left(Z_1 + \ldots + Z_n = 2m\right) \geq -\frac{\delta}{\sqrt{n}} + \frac{1}{\sqrt{2\pi n \sigma^2}} \exp\left\{-\frac{(2m - \mu n)^2}{2n\sigma^2}\right\}.$$

Since $2m - \mu n = 2\lceil \frac{\mu n}{2} \rceil - \mu n \leq 2$ and $\sigma^2 = \mathrm{Var}(Z_1)$ depends only on λ and d, hence only on μ and d, for sufficiently large n, the exponential factor is greater than, say 1/2. Adjusting δ appropriately and using that $\sigma^2 \leq \mu$, in fact,

$$\mathrm{Var}(Z_1) = \mathbb{E}Z_1(Z_1 - 1) - (\mathbb{E}Z_1)^2 + \mathbb{E}Z_1 = \lambda^2 \frac{s_{d-2}(\lambda) s_d(\lambda) - s_{d-1}(\lambda)^2}{s_d(\lambda)} + \mathbb{E}Z_1,$$

which by Lemma 6 is bounded by $\mathbb{E}Z_1 = \mu$, we get for sufficiently large n,

$$\mathbb{P}\left(Z_1 + \ldots + Z_n = 2m\right) \geq \frac{1}{10\sqrt{\mu n}}. \qquad (12)$$

Thus, for every $x \in \mathcal{D}$,

$$\mathbb{P}\left(Z = x \;\Big|\; \sum_i Z_i = 2m\right) \leq \frac{\mathbb{P}\left(Z = x\right)}{\mathbb{P}\left(\sum_i Z_i = 2m\right)} \leq 10\sqrt{\mu n}\, \mathbb{P}\left(Z = x\right). \qquad (13)$$

The next step is to break the sum in (11) into likely and unlikely degree sequences. Note that $\mathbb{E} \sum_{j=1}^d \mathbf{1}_{\{Z_j = i\}} = n\mathbb{P}\left(Z_1 = i\right) = n\lambda_i$. By Hoeffding's inequality,

$$\mathbb{P}\left(\left|\sum_{j=1}^n \mathbf{1}_{\{Z_j = i\}} - n\lambda_i\right| > \varepsilon n \lambda_i\right) \leq 2e^{-\varepsilon^2 n \lambda_i / 3}, \qquad \varepsilon > 0.$$

Put $\varepsilon = n^{-1/3} \frac{1}{\max_i \lambda_i}$. The union bound yields

$$\mathbb{P}\left(\exists i \leq d \left|\sum_{j=1}^n \mathbf{1}_{\{Z_j = i\}} - n\lambda_i\right| > n^{2/3}\right) \leq 2d \exp\left\{-n^{1/3} \frac{\min_i \lambda_i}{3(\max_i \lambda_i)^2}\right\}. \qquad (14)$$

This proves (a). It also shows that w.h.p. $n\lambda_i, i = 0, 1, \ldots, d$ asymptotically defines the degree distribution of $G_{n,m,d}$. Also, given that x is chosen uniformly at random from $\mathcal{D}$, we see that the distribution of $G_{n,x}$ in this case is the same as the distribution of the configuration model for the given degree sequence.

To prove (b) and (c), we will use the Molloy-Reed criterion (see [10],[11] and Theorem 11.11 in [7] for the exact formulation we shall use). First define

$$\mathcal{A} = \left\{ x = (x_1, \ldots, x_n) \in \mathcal{D},\ \exists i \le d \ \left| \sum_{j=1}^{n} \mathbf{1}_{\{x_j = i\}} - n\lambda_i \right| > n^{2/3} \right\}.$$

Then, using (13) and (14),

$$\begin{aligned}\sum_{x \in \mathcal{A}} \mathbb{P}\left(G_{n,x} \in \mathcal{P}\right) \mathbb{P}\left(Z = x \;\middle|\; \sum_i Z_i = 2m\right) &\le 10\sqrt{\mu n} \sum_{x \in \mathcal{A}} \mathbb{P}\left(Z = x\right) \\ &= 10\sqrt{\mu n}\mathbb{P}\left(Z \in \mathcal{A}\right) \\ &\le 20d\sqrt{\mu n} \exp\left\{-n^{1/3} \frac{\min_i \lambda_i}{3(\max_i \lambda_i)^2}\right\}.\end{aligned}$$

It remains to handle the typical terms $x \in \mathcal{D} \setminus \mathcal{A}$ in (11). For such x, we now estimate $p_x = \mathbb{P}\left(G_{n,x} \in \mathcal{P}\right)$ in two cases: for $\mathcal{P}$ being the complement of (i) "there are only small components", and (ii) "there is a giant" depending on the behaviour of the degree sequences.

Let $Q = \sum_{i=0}^{d} i(i-2)\lambda_i$. Note that by the definition of $\mathcal{A}$, for every $x \in \mathcal{D} \setminus \mathcal{A}$, the number of vertices in $G_{n,x}$ is $n\lambda_i + O(n^{2/3})$, so it is justified to use the Molloy-Reed criterion and we obtain that: if $Q < 0$, then $\max_x p_x \to 0$ in the case (i), and the same if $Q > 0$ in the case (ii). Finally note that

$$Q = \lambda^2 \frac{s_{d-2}(\lambda)}{s_d(\lambda)} - \lambda \frac{s_{d-1}(\lambda)}{s_d(\lambda)} = f_d(\lambda)(f_{d-1}(\lambda) - 1)$$

and Lemma 8 together with the definition of λ, that is (2), finishes the proof. The expression for Θ is in [11]. (One can also find a simplified proof of the Molloy-Reed results in [7], Theorem 11.11.)

3 Conclusions

We have found tight expressions for the degree sequence of $G_{n,m,d}$ and we have used the Molloy-Reed results to exploit them. In future work, we plan to study the scaling window around Q close to zero. Hatami and Molloy [9] consider this case and their results show that we can expect a maximum component size close to $n^{2/3}$ in this case. They deal with a general degree sequence and perhaps we can prove tighter results for our specific case.

References

[1] N. Alon, I. Benjamini and A. Stacey, Percolation on finite graphs and isoperimetric inequalities, *The Annals of Probability* 32 (2004) 1727-1745.

[2] J. Aronson, A.M. Frieze and B. Pittel, Maximum matchings in sparse random graphs: Karp-Sipser revisited, *Random Structures and Algorithms* **12** (1998), 111-178.

[3] B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled graphs, *European Journal on Combinatorics* 1 (1980) 311-316.

[4] C. Borell, Complements of Lyapunov's inequality., *Math. Ann.* 205 (1973) 323-331.

[5] R. Durrett, Probability: theory and examples, Fourth edition, Cambridge Series in Statistical and Probabilistic Mathematics, 31. *Cambridge University Press, Cambridge*, 2010.

[6] A.M. Frieze, On a Greedy 2-Matching Algorithm and Hamilton Cycles in Random Graphs with Minimum Degree at Least Three, *Random structures and Algorithms* 45 (2014) 443-497.

[7] A.M. Frieze and M. Karoński, Introduction to random graphs. *Cambridge University Press, Cambridge*, 2016.

[8] O. Guédon, P. Nayar and T. Tkocz, Concentration inequalities and geometry of convex bodies, Analytical and probabilistic methods in the geometry of convex bodies, 9–86, IMPAN Lect. Notes, 2, *Polish Acad. Sci. Inst. Math., Warsaw*, 2014.

[9] H. Hatami and M. Molloy. The scaling window for a random graph with a given degree sequence, *Random Structures and Algorithms* 41 (2012) 99-123.

[10] M. Molloy and B. Reed, A critical point for random graphs with a given degree sequence. *Random Structures Algorithms* 6 (1995) 161-179.

[11] M. Molloy and B. Reed, The size of the giant component of a random graph with a given degree sequence, *Combinatorics. Probability and Computing* 7 (1998) 295-305.

[12] A. Nachmias and Y. Peres, Critical percolation on random regular graphs, *Random Structures and Algorithms* 36 (2010) 111-148.

[13] P. Nayar and K. Oleszkiewicz, Khinchine type inequalities with optimal constants via ultra log-concavity, *Positivity* 16 (2012) 359-371.