Khinchin inequalities for uniforms on spheres with a deficit

Jacek Jakimiuk[†], Colin Tang[*], Tomasz Tkocz[*]

[†]*Institute of Mathematics, University of Warsaw, 02-097 Warsaw, Poland.*

[*]*Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA*

Abstract

We sharpen the moment comparison inequalities with sharp constants for sums of random vectors uniform on Euclidean spheres, providing a deficit term (optimal in high dimensions).

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1 Introduction

Sharp moment comparison inequalities (a.k.a. of the Khinchin-type, [22]) have been extensively studied (see [1, 2, 3, 5, 6, 10, 11, 12, 17, 18, 19, 23, 24, 25, 26, 27, 31, 33, 34, 35, 37]), nonetheless the investigation of their stability presently appears to be in its nascent stages and has been focused so far only on the Rademacher sums (see [2, 9, 21]), as constituting, arguably, the most fundamental case. This note makes a first step towards widening the scope of this

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investigation and is devoted to sharpening sharp moment comparison inequalities for sums of random vectors uniform on Euclidean spheres, which provide a natural compelling generalisation of the Rademacher distribution to Euclidean space.

1.1 New results

Cutting to the chase, our main results read as follows. (We work in $\mathbb{R}^d$, equipped with the standard inner product $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$, $x, y \in \mathbb{R}^d$, and the endowed Euclidean norm $|x| = \sqrt{\langle x, x\rangle}$.)

Theorem 1. *Let $d \geq 2$ be a fixed dimension, let $\xi_1, \xi_2, \ldots$ be independent random vectors uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$ and let Z be a Gaussian random vector in $\mathbb{R}^d$ with mean 0 and covariance $\frac{1}{d}I_d$. Let $p \geq 2$. For every $n \geq 1$, every real scalars $a_1, \ldots, a_n$ with $\sum_{j=1}^n a_j^2 = 1$, we have*

$$\mathbb{E}\left|\sum_{j=1}^n a_j \xi_j\right|^p \leq \mathbb{E}|Z|^p - c_{p,d}\sum_{j=1}^n a_j^4, \tag{1}$$

where

$$c_{p,d} = \frac{(p+d-2)(p+d-4)}{24d^2(d+2)} \cdot \begin{cases} 3p(p-2), & 2 \leq p \leq 4, \\ 1, & p > 4. \end{cases}$$

Remark. For a fixed p, as $d \to \infty$, we have $c_{p,d} = \Theta_p(1/d)$. This is best possible, since the special case $n = 1$ gives the bound

$$c_{p,d} \leq \mathbb{E}|Z|^p - 1 = \frac{\Gamma(\frac{p+d}{2})}{(\frac{d}{2})^{p/2}\Gamma(\frac{d}{2})} - 1 = O_p(1/d),$$

as can be checked for instance by a direct calculation using that $d \cdot |Z|^2$ follows the chi-squared distribution $\chi^2(d)$ with d-degrees of freedom which has density $\frac{1}{2^{d/2}\Gamma(d/2)}x^{d/2-1}e^{-x/2}$ on $(0, +\infty)$.

We emphasise that the Gaussian distribution is normalised so that $\mathbb{E}|Z|^2 = 1 = \mathbb{E}|\xi|^2$, that is there is equality when $p = 2$, so necessarily $c_{2,d} = 0$. When $d = 1$, the ξ_j are Rademacher random variables, that is random signs uniform on $\{-1, 1\}$ and the Z_j are standard Gaussian random variables $N(0, 1)$, and when

$d = 2$, the ξ_j are Steinhaus random variables (upon the usual identification $\mathbb{R}^2 \simeq \mathbb{C}$), both distributions playing a pivotal role in Banach space theory, see [20, 28, 39].

For coefficient vectors $a = (a_1, \ldots, a_n)$ of a *fixed* length n, we offer the following stability result.

Theorem 2. *Let $d \geq 2$ be a fixed dimension, let $\xi_1, \xi_2, \ldots$ be independent random vectors uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$. Let $p \geq 2$. For every $n \geq 2$, every real scalars $a_1, \ldots, a_n$ with $\sum_{j=1}^n a_j^2 = 1$, we have*

$$\mathbb{E}\left|\sum_{j=1}^n a_j \xi_j\right|^p \leq \mathbb{E}\left|\sum_{j=1}^n \frac{1}{\sqrt{n}}\xi_j\right|^p - \tilde{c}_{p,d}\sum_{j=1}^n \left(\frac{1}{n} - a_j^2\right)^2, \tag{2}$$

where

$$\tilde{c}_{p,d} = \frac{p(p-2)}{4d}\begin{cases} \frac{(p+d-2)(p+d-4)}{d(d+2)}, & 2 \leq p \leq 4, \\ 0.385, & p > 4. \end{cases}$$

1.2 Previous work

Inequalities (1) and (2) have been previously proved without the deficit terms $O_{p,d}(\sum_{j=1}^n a_j^4)$ and $O_{p,d}(\sum_{j=1}^n (1/n - a_j^2)^2)$ respectively, by König and Kwapień in [24], and, independently, by Baernstein II and Culverhouse in [1], who de facto established a more general convexity-type result which in particular asserts that the function $(x_1, \ldots, x_n) \mapsto \mathbb{E}\left|\sqrt{x_j}\xi_j\right|^p$ is Schur-concave on $\mathbb{R}_+^n$ when $p \geq 2$ (see [1] for background and details). The inequality is often restated equivalently in a homogeneous form as the sharp $L_p - L_2$ moment comparison inequality,

$$\mathbb{E}\left|\sum_{j=1}^n a_j \xi_j\right|^p \leq (\mathbb{E}|Z|^p)\left(\mathbb{E}\left|\sum_{j=1}^n a_j \xi_j\right|^2\right)^{p/2}, \qquad p \geq 2, \tag{3}$$

for all $a_1, \ldots, a_n \in \mathbb{R}$. The multiplicative constant $\mathbb{E}|Z|^p$ is sharp, as follows from the case $a_1 = \cdots = a_n = \frac{1}{\sqrt{n}}$, $n \to \infty$ in view of the central limit theorem.

As hinted earlier, there are few stability results, and only for Rademacher sums, that is in the case $d = 1$, when the ξ_j are i.i.d. uniform random signs. In this classical setting, inequalities (1) and (2) are known to hold for all $p \geq 3$, as has been recently established by Jakimiuk in [21], albeit with a worse dependence on p of the constant $c_{p,1}$ for large p; see also Corollary 1 in [2], where this is derived as a by product of the sharp $L_p - L_4$ inequality for $p \geq 4$. Prior to Jakimiuk's work, there had been one more stability result, viz. De, Diakonikolas and Servedio in [9] found a deficit term in the celebrated Szarek's $L_1 - L_2$ inequality from [37] (see also [13] for a different approach and explicit constants). This was paralleled in [7, 16, 32] in the geometric context of stability results for maximal volume sections of ℓ_p-balls, polydisc and simplex, respectively (which themselves can be viewed as the moment comparison inequalities but in "L_{-1}"), and has found interesting applications, see [13, 30]. In a different spirit, "distributional stability" has been recently investigated in [14].

We recommend, e.g. [2, 21, 33] for further references on the pursuit of the sharp constants in the classical Khinchin inequality for random signs, as well as [19] for an account on what is known for other distributions, and [8] specifically for spherically symmetric random vectors.

2 Proofs

As in the statement of Theorems 1 and 2, throughout the rest of this paper, $\xi, \xi_1, \xi_2, \ldots$ are independent identically distributed (i.i.d.) random vectors uniform on the unit sphere $S^{d-1} = \{x \in \mathbb{R}^d, \ |x| = 1\}$ in $\mathbb{R}^d$, and $Z, Z_1, Z_2, \ldots$ are independent of them i.i.d. Gaussian random vectors in $\mathbb{R}^d$ with mean 0 and covariance $\frac{1}{d} I_d$.

First, we focus solely on Theorem 1. Second, having established and building on the important ideas and auxiliary lemmas, we shall prove Theorem 2.

2.1 Overview

At a high level, the main idea to tackle Theorem 1 is reminiscent of Lindeberg's swapping argument from his work [29] on the central limit theorem, which has been widely used in a variety of contexts (see e.g. [36] for historical accounts), in particular for moment comparison inequalities (see, e.g. [3, 4, 12, 18]). This has also been Jakimiuk's approach in [21].

Specifically, for $p > 0$, we define the deficit,

$$D_p(a, v) = \mathbb{E}|aZ + v|^p - \mathbb{E}|a\xi + v|^p, \qquad a \in \mathbb{R},\ v \in \mathbb{R}^d. \tag{4}$$

Suppose that it is nonnegative, $D_p(a, v) \geq 0$, for all $p \geq 2$, $a \in \mathbb{R}$ and $v \in \mathbb{R}^d$. Then, the proof of (3) goes by repeatedly swapping each ξ_j with Z_j (relying on the independence of the summands which allows in turn to condition on all but one summand that is being swapped),

$$\begin{aligned} \mathbb{E}\,|a_1\xi_1 + a_2\xi_2 + \cdots + a_n\xi_n|^p &\leq \mathbb{E}\,|a_1 Z_1 + a_2\xi_2 + \cdots + a_n\xi_n|^p \leq \ldots \\ &\leq \mathbb{E}\,|a_1 Z_1 + a_2 Z_2 + \cdots + a_n Z_n|^p \\ &= \mathbb{E}|(a_1^2 + \cdots + a_n^2)^{1/2} Z|^p. \end{aligned}$$

Now, to make some savings and obtain a deficit term in this bound, compellingly, we would like to sharpen the bound $D_p(a, v) \geq 0$.

This task brings us asking: why is $D_p(a, v)$ nonnegative (when $p \geq 2$)? Fix $a \in \mathbb{R}, v \in \mathbb{R}^d$ and consider the function

$$h_{a,v}(t) = \mathbb{E}|v + at^{1/2}\xi|^p, \qquad t > 0.$$

The heart of the matter in both [1] and [24] is the convexity of $h_{a,v}$ on $(0, +\infty)$, for then decomposing the distribution of Z as $|Z|\xi$ (the magnitude and independent uniform direction), Jensen's inequality yields

$$\begin{aligned} \mathbb{E}|v + aZ|^p = \mathbb{E}|v + a\sqrt{|Z|^2}\xi|^p &= \mathbb{E}_{|Z|} h_{a,v}(|Z|^2) \\ &\geq h_{a,v}(\mathbb{E}|Z|^2) = h_{a,v}(1) = \mathbb{E}|v + a\xi|^p, \end{aligned}$$

which is $D_p(a, v) \geq 0$.

As a side note, this argument is robust enough to treat arbitrary rotationally invariant random vectors whose magnitudes are comparable in the stochastic convex ordering, as done in [24], as well as other functionals than just the moments, as in [1].

2.2 Main lemmas

Our argument therefore begins with a derivation of an exact expression for the second derivative of functions $h_{a,v}$, amenable to quantitative improvements on their convexity. For greater transparency of the ensuing calculations, we follow [1] and treat arbitrary (smooth) functionals.

Lemma 3. *Let Ψ be a smooth function on $\mathbb{R}^d$. For a fixed vector $v \in \mathbb{R}^d$, define*

$$f(t) = \int_{S^{d-1}} \Psi\big(v + t^{1/2}x\big)\mathrm{d}x, \qquad t > 0.$$

Then

$$f'(t) = \frac{1}{2}\int_{B_2^d} \Delta\Psi\big(v + t^{1/2}x\big)\mathrm{d}x, \tag{5}$$

$$f''(t) = \frac{1}{4}\int_0^1 r^{d+1}\left(\int_{B_2^d} \Delta\Delta\Psi\big(v + t^{1/2}rx\big)\mathrm{d}x\right)\mathrm{d}r. \tag{6}$$

Proof. The approach we use was indicated in Remark 15 in [8] and is in the spirit of Lemma 4 from [1]. Plainly,

$$f'(t) = \frac{1}{2}t^{-1/2}\int_{S^{d-1}} \Big\langle \nabla\Psi\big(v + t^{1/2}x\big), x\Big\rangle\,\mathrm{d}x.$$

Since x is the outer-normal unit vector, the divergence theorem gives

$$f'(t) = \frac{1}{2}t^{-1/2}\int_{B_2^d} \mathrm{div}_x\Big(\nabla\Psi\big(v + t^{1/2}x\big)\Big)\mathrm{d}x = \frac{1}{2}\int_{B_2^d} \Delta\Psi\big(v + t^{1/2}x\big)\mathrm{d}x.$$

To find the next derivative, note that using polar coordinates,

$$f'(t) = \frac{1}{2}\int_0^1 r^{d-1}\left(\int_{S^{d-1}} \Delta\Psi\big(v + t^{1/2}rx\big)\mathrm{d}x\right)\mathrm{d}r$$

and the point is that the integral over the sphere, as the function in t is of the same form as f, with the variable rescaled by r^2. Therefore, applying the previous calculation, we get

$$f''(t) = \frac{1}{2}\int_0^1 r^{d-1}\left(\frac{1}{2}\int_{B_2^d} \Delta\Delta\Psi\big(v + t^{1/2}rx\big)\cdot r^2\mathrm{d}x\right)\mathrm{d}r. \qquad \square$$

The workhorse of our proof will be the following quantitative bound on the deficit introduced in (4).

Lemma 4. *Let $p \geq 2$. For $a \in \mathbb{R}, v \in \mathbb{R}^d$, $d \geq 2$, we have*

$$D_p(a, v) \geq \kappa_{p,d} a^4 \cdot \begin{cases} (|v|^2 + 2a^2)^{\frac{p-4}{2}}, & 2 \leq p \leq 4, \\ |v|^{p-4}, & p > 4, \end{cases}$$

where

$$\kappa_{p,d} = \frac{p(p-2)(p+d-2)(p+d-4)}{4d^2(d+2)}.$$

Proof. If $a = 0$, there is equality. Otherwise, by homogeneity, we can assume without loss of generality that $a = 1$. Fix a vector v in $\mathbb{R}^d$ and consider the function

$$g_v(t) = \mathbb{E}|v + \sqrt{t}\xi|^p, \qquad t > 0.$$

By the rotational invariance of the Gaussian distribution, Z has the same distribution as $|Z|\xi$. Thus,

$$D_p(1, v) = \mathbb{E}\Big[g_v(|Z|^2) - g_v(1)\Big].$$

Using Taylor's expansion with Lagrange's remainder, for $t > 0$, there is η_t between 1 and t such that

$$g_v(t) - g_v(1) = (t-1)g_v'(1) + \frac{1}{2}(t-1)^2 g_v''(\eta_t).$$

As a result (recall $\mathbb{E}|Z|^2 = 1$),

$$D_p(1, v) = \frac{1}{2}\mathbb{E}\Big[(|Z|^2 - 1)^2 g_v''(\eta_{|Z|^2})\Big]. \tag{7}$$

An application of (6) of Lemma 3 with $\Psi(x) = |x|^p$ yields for every $\eta > 0$,

$$g_v''(\eta) = \frac{p(p-2)(p+d-2)(p+d-4)}{4|S^{d-1}|} \int_0^1 r^{d+1} \left(\int_{B_2^d} |v + xr\sqrt{\eta}|^{p-4} \mathrm{d}x \right) \mathrm{d}r$$

(sweeping under the rug inessential regularity issues caused by the singularity at the origin, overcome e.g. by taking the smooth approximations $\Psi_\delta(x) = (|x|^2 + \delta)^{p/2}$, $\delta \downarrow 0$, see [1], eq. (1.5) for details). Writing the integral over B_2^d

in polar coordinates leads to

$$g_v''(\eta) = \beta_{p,d} \int_0^1 \int_0^1 \mathbb{E}_\xi |v + r_1 r_2 \sqrt{\eta}\xi|^{p-4} \mathrm{d}r_1^d \mathrm{d}r_2^{d+2}, \qquad \eta > 0, \tag{8}$$

with

$$\beta_{p,d} = \frac{p(p-2)(p+d-2)(p+d-4)}{4d(d+2)}. \tag{9}$$

Note that for convenience, we have renormalised the double integral $\int_0^1 \int_0^1$ so that it is against the probability measure on $[0,1]^2$ with density $d(d{+}2)r_1^{d-1}r_2^{d+1}$.

Our argument lower bounding g'' now differs depending on whether $p > 4$.

Case 1: $2 \le p \le 4$. Let $q = \frac{p-4}{2} \le 0$. Here, we lean on the convexity of the function $x \mapsto x^q$ on $(0, +\infty)$. Rewriting (8) and using Jensen's inequality,

$$\begin{aligned} g_v''(\eta) &= \beta_{p,d} \int_0^1 \int_0^1 \mathbb{E}_\xi \big(|v|^2 + 2r_1 r_2 \sqrt{\eta}\, \langle v, \xi \rangle + r_1^2 r_2^2 \eta\big)^q \mathrm{d}r_1^d \mathrm{d}r_2^{d+2} \\ &\ge \beta_{p,d} \left(|v|^2 + \frac{d}{d+4}\eta\right)^q. \end{aligned} \tag{10}$$

Plugging this back into (7) and using a crude bound $\eta_{|Z|^2} \le \max\{1, |Z|^2\}$, we arrive at

$$D_p(1,v) \ge \frac{1}{2}\beta_{p,d}\mathbb{E}\left[(|Z|^2-1)^2\left(|v|^2 + \frac{d}{d+4}\max\{1,|Z|^2\}\right)^q\right].$$

To finish off with a clean bound, we use Jensen's inequality yet again with respect to the probability measure $\frac{(|Z|^2-1)^2}{2/d}\mathrm{d}\mathbb{P}$ and obtain

$$D_p(1,v) \ge \frac{1}{d}\beta_{p,d}\left(|v|^2 + \frac{d}{d+4}\mathbb{E}\left[\max\{1,|Z|^2\}\frac{(|Z|^2-1)^2}{2/d}\right]\right)^q.$$

Finally, crudely $\max\{1,|Z|^2\} \le 1 + |Z|^2$, thus

$$\mathbb{E}\left[\max\{1,|Z|^2\}\frac{(|Z|^2-1)^2}{2/d}\right] \le \frac{d}{2}\left(\mathbb{E}(|Z|^2-1)^2 + \mathbb{E}\Big[|Z|^2(|Z|^2-1)^2\Big]\right) = 1 + \frac{d+4}{d},$$

where the last two expectations are calculated directly using that $d \cdot |Z|^2$ follows the $\chi^2(d)$ distribution. Consequently,

$$D_p(1,v) \geq \frac{1}{d}\beta_{p,d}\left(|v|^2 + \frac{d}{d+4} + 1\right)^q \geq \frac{1}{d}\beta_{p,d}\left(|v|^2+2\right)^q.$$

Case 2: $p > 4$. We simply use monotonicity asserted by the following immediate consequence of (5).

Claim. Let $q > 0$, $v \in \mathbb{R}^d$, $d \geq 2$. Then

$$t \mapsto \mathbb{E}|v + \sqrt{t}\xi|^q \qquad \text{increases on } (0,+\infty).$$

In particular,

$$\mathbb{E}|v + \sqrt{t}\xi|^q \geq |v|^q, \tag{11}$$

and, thanks to rotational invariance,

$$(v,t) \mapsto \mathbb{E}|v+\sqrt{t}\xi|^q = \mathbb{E}\big||v|\xi' + \sqrt{t}\xi\big|^q \quad \text{increases both in } t \text{ and } |v|. \tag{12}$$

Indeed, by (5),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}|v+\sqrt{t}\xi|^q = \frac{q(q+d-2)}{2|S^{d-1}|}\int_{B_2^d}|v+\sqrt{t}x|^{q-2}\mathrm{d}x \geq 0. \qquad \square$$

Therefore, using (8),

$$g_v''(\eta) \geq \beta_{p,d}|v|^{p-4}$$

(deterministically, for every $\eta > 0$). Plugging this pointwise bound into (7) gives

$$D_p(1,v) \geq \frac{1}{2}\mathbb{E}\Big[(|Z|^2-1)^2\Big]\beta_{p,d}|v|^{p-4}.$$

Since $\mathbb{E}\Big[(|Z|^2-1)^2\Big] = \frac{2}{d}$, we obtain $D_p(1,v) \geq \frac{1}{d}\beta_{p,d}|v|^{p-4}$, that is the lemma holds with $\kappa_{p,d} = \frac{1}{d}\beta_{p,d}$, as desired.

2.3 An auxiliary lemma

In order to handle the averages of the terms $\mathbb{E}|v|^{p-4}$ coming from the bound on D_p in the case $p > 4$, with v being sums of uniforms of spheres, we shall need some sort of concentration. For simplicity, we choose to use Khinchin-type inequalities (which in fact yields explicit and decent values of the constants involved).

Lemma 5. *Let $d \geq 2$. There is a universal constant $c_{\mathrm{Kh}} > 0$ such that for all real numbers $a_1, a_2, \ldots$ and $q > 0$, we have*

$$\mathbb{E}\left|\sum_{j=1}^{n} a_j \xi_j\right|^q \geq c_{\mathrm{Kh}} \left(\sum_{j=1}^{n} a_j^2\right)^{q/2}.$$

One can take $c_{\mathrm{Kh}} = 0.77$. In particular, the same inequality holds if any of the variables ξ_j is replaced by Z_j.

Proof. When $q \geq 2$, the inequality plainly holds with constant 1 (by Jensen's inequality). When $0 < q < 2$, let $c_{d,q}$ be the best constant such that the Khinchin-type inequality

$$\mathbb{E}\left|\sum_{j=1}^{n} a_j \xi_j\right|^q \geq c_{d,q} \left(\sum_{j=1}^{n} a_j^2\right)^{q/2}$$

holds for all $n \geq 1$ and all scalars a_j. It is the main result of [5, 23, 24] that

$$c_{d,q} = \min\{2^{-q/2}\mathbb{E}|\xi_1 + \xi_2|^q, \mathbb{E}|Z|^q\}.$$

(see also [8]), and it is known that when $d \geq 3$, this minimum is attained at the second term, $\mathbb{E}|Z|^q$, which we now lower bound. We have,

$$\mathbb{E}|Z|^q = \frac{\Gamma(\frac{q+d}{2})}{\left(\frac{d}{2}\right)^{q/2}\Gamma(\frac{d}{2})}.$$

By the log-convexity of the Gamma function, for $x > 0$ and $0 < s < 1$,

$$\begin{aligned}\Gamma(x+1) &= \Gamma\big(s(x+s) + (1-s)(x+s+1)\big) \\ &\leq \Gamma(x+s)^s \Gamma(x+s+1)^{1-s} = (x+s)^{1-s}\Gamma(x+s)\end{aligned} \tag{13}$$

which is Wendel's inequality, [38], resulting in

$$\frac{\Gamma(x+s)}{x^s\Gamma(x)} \geq \left(\frac{x}{x+s}\right)^{1-s}.$$

Applied to $x = \frac{d}{2} \geq 1$, $s = \frac{q}{2}$, we obtain

$$\mathbb{E}|Z|^q \geq \frac{1}{(1+s)^{1-s}} \geq e^{-s(1-s)} \geq e^{-1/4} > 0.778.$$

It remains to lower bound the first term $2^{-q/2}\mathbb{E}|\xi_1 + \xi_2|^q$ when $d = 2$. We have,

$$2^{-q/2}\mathbb{E}|\xi_1 + \xi_2|^q = 2^{q/2}\frac{\Gamma(\frac{q}{2}+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{q}{2}+1)}.$$

Again, by virtue of (13), applied this time with $x = \frac{q}{2}$, $s = \frac{1}{2}$, we get

$$2^{-q/2}\mathbb{E}|\xi_1 + \xi_2|^q \geq \frac{2^{\frac{q+1}{2}}}{\sqrt{\pi(q+1)}}.$$

The right hand side is minimised at $q = \frac{1}{\log 2} - 1$ attaining value 0.774.., which finishes the proof. $\square$

2.4 Proof of Theorem 1

We shall follow the traditional notation $\|a\|_p = (\sum |a_j|^p)^{1/p}$, $\|a\|_\infty = \max_j |a_j|$ for the ℓ_p norms of a vector $a = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$.

The proof uses two different arguments: when $\|a\|_\infty$ is bounded away from 1, we shall (iteratively) use the pointwise bounds from Lemma 4 resulting with the deficit of the order $\|a\|_4$, whereas in the oppose case we easily get a constant deficit (i.e. independent of a), leveraging the Schur concavity of the moment functional. With hindsight, we choose the following cut-off for the ℓ_∞ norm,

$$m_p = \begin{cases} 1, & 2 \leq p \leq 4, \\ \sqrt{1 - 2^{-\frac{1}{p-4}}}, & p > 4. \end{cases}$$

Case 1: $\|a\|_\infty \leq m_p$. (Clarification: when $2 \leq p \leq 4$ this case is exhaustive, since $m_p = 1$.) We use the classical Lindeberg's swapping argument. To this end, we define

$$
\begin{aligned}
S_0 &= \sum_{j=1}^{n} a_j \xi_j, \\
S_k &= \sum_{j=1}^{k} a_j Z_j + \sum_{j=k+1}^{n} a_j \xi_j, \qquad k = 1, \dots, n
\end{aligned}
$$

and break the deficit up with a telescoping sum,

$$
\mathbb{E}|Z|^p - \mathbb{E}|S_0|^p = \sum_{k=1}^{n} \big(\mathbb{E}|S_k|^p - \mathbb{E}|S_{k-1}|^p\big).
$$

Note that the sums S_k and S_{k-1} only differ by the k-th term which is $a_k Z_k$ and $a_k \xi_k$, respectively. Letting

$$
v_k = \sum_{j=1}^{k-1} a_j Z_j + \sum_{j=k+1}^{n} a_j \xi_j
$$

and using the notation from (4), we have

$$
\mathbb{E}|S_{k+1}|^p - \mathbb{E}|S_k|^p = \mathbb{E} D_p(a_k, v_k).
$$

By Lemma 4,

$$
\mathbb{E} D_p(a_k, v_k) \geq \kappa_{p,d} a_k^4 \begin{cases} \mathbb{E}(|v_k|^2 + 2a_k^2)^{\frac{p-4}{2}}, & 2 \leq p \leq 4, \\ \mathbb{E}|v_k|^{p-4}, & p > 4. \end{cases}
$$

Observe that $\mathbb{E}|v_k|^2 = \sum_{j \neq k} a_j^2 = 1 - a_k^2$.

When $2 \leq p \leq 4$, Jensen's inequality yields

$$
\mathbb{E} D_p(a_k, v_k) \geq \kappa_{p,d} a_k^4 (1 - a_k^2 + 2a_k^2)^{\frac{p-4}{2}} = \kappa_{p,d} a_k^4 (1 + a_k^2)^{\frac{p-4}{2}} \geq \frac{1}{2} \kappa_{p,d} a_k^4.
$$

Summing these bounds over $1 \leq k \leq n$ gives the result and finishes the proof.

When $p > 4$, Lemma 5 and a crude bound $1 - a_k^2 \geq 1 - \|a\|_\infty^2 \geq 1 - m_p^2 = 2^{-\frac{1}{p-4}}$ yield

$$
\mathbb{E} D_p(a_k, v_k) \geq c_{\mathrm{Kh}} \kappa_{p,d} a_k^4 (1 - a_k^2)^{\frac{p-4}{2}} \geq 2^{-1/2} c_{\mathrm{Kh}} \kappa_{p,d} a_k^4 > \frac{1}{2} \kappa_{p,d} a_k^4.
$$

Summing these bounds over $1 \le k \le n$ gives the result.

Case 2: $\|a\|_\infty > m_p$. Clarification: when $2 \le p \le 4$, $m_p = 1$, this case is empty, and the proof has already been completed, so we implicitly assume that $p > 4$. Here we simply use the Schur concavity of

$$\mathbb{R}_+^n \ni x \mapsto \mathbb{E}\left|\sum_{j=1}^n \sqrt{x_j}\xi_j\right|^p$$

known from [1] to hold in every dimension $d \ge 2$ as long as $p \ge 2$. Say $a_1 = \|a\|_\infty$. Then the vector $(a_1^2, \dots, a_n^2)$ majorises the vector $(m_p^2, \frac{1-m_p^2}{n-1}, \dots, \frac{1-m_p^2}{n-1})$, provided that $m_p^2 \ge \frac{1-m_p^2}{n-1}$, equivalently $nm_p^2 \ge 1$, and we obtain

$$\mathbb{E}\left|\sum_{j=1}^n a_j\xi_j\right|^p \le \mathbb{E}\left|m_p\xi_1 + \sqrt{\frac{1-m_p^2}{n-1}}(\xi_2 + \dots + \xi_{n-1})\right|^p.$$

We can apply Case 1 to the right hand side, which results in

$$\mathbb{E}\left|\sum_{j=1}^n a_j\xi_j\right|^p \le \mathbb{E}|Z|^p - \frac{1}{2}\kappa_{p,d}\left(m_p^4 + \frac{(1-m_p^2)^2}{n-1}\right) \le \mathbb{E}|Z|^p - \frac{1}{2}\kappa_{p,d}m_p^4$$

Note that $\|a\|_4 \le \|a\|_2 = 1$. For cosmetics, say $m_p^4 = (1-2^{-1/(p-4)})^2 > \frac{1}{3p(p-2)}$ which gives the constant $c_{p,d}$ from the statement of the theorem. Finally, if $nm_p^2 < 1$, we take an integer $N \ge 2$ such that $N \ge \frac{1}{m_p^2}$ and observe that $(a_1^2, \dots, a_n^2, \underbrace{0, \dots, 0}_{N-n})$ majorises the vector $(m_p^2, \frac{1-m_p^2}{N-1}, \dots, \frac{1-m_p^2}{N-1})$, so that we can repeat the last part of the argument verbatim to finish the proof. $\square$

2.5 Proof of Theorem 2

Exactly as for Theorem 1, the argument here will be driven by tracking the deficit along *local* changes to the coefficient vector $a = (a_1, \dots, a_n)$ performed to make it progressively closer to the extremising diagonal one $(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$. The next lemma will facilitate that. We denote the deficit term from Theorem 2 by $\delta(a)$,

$$\delta(a) = \sum_{j=1}^{n} \left(\frac{1}{n} - a_j^2 \right)^2$$

and note that with the ℓ_2 normalisation $\sum_{j=1}^n a_j^2 = 1$,

$$\delta(a) = \sum_{j=1}^{n} a_j^4 - \frac{1}{n} = \sum_{j=1}^{n} a_j^4 - \sum_{j=1}^{n} \frac{1}{\sqrt{n}^4}, \tag{14}$$

that is the deficit term can be equivalently derived by comparing the changes of the ℓ_4 norms of the coefficient vectors.

Lemma 6. *Let $p \geq 2$. Suppose that $a_1 \geq b_1 \geq b_2 \geq a_2 > 0$ with $a_1^2 + a_2^2 = b_1^2 + b_2^2 = \sigma^2$ for some $\sigma > 0$. For every vector v in $\mathbb{R}^d$, $d \geq 2$, we have*

$$\begin{aligned} &\mathbb{E}\,|b_1\xi_1 + b_2\xi_2 + v|^p - \mathbb{E}\,|a_1\xi_1 + a_2\xi_2 + v|^p \\ &\qquad \geq \frac{1}{d}(a_1^4 + a_2^4 - b_1^4 - b_2^4) \cdot \begin{cases} \beta_{p,d}(|v|^2 + \sigma^2)^{\frac{p-4}{2}}, & 2 \leq p \leq 4, \\ \frac{p(p-2)}{8}\mathbb{E}|v + \sigma\xi|^{p-4}, & p > 4, \end{cases} \end{aligned} \tag{15}$$

with $\beta_{p,d}$ defined in (9).

Proof. Recall that Lemma 3 provides us with good expressions for the derivatives of the function

$$g_v(t) = \mathbb{E}|v + \sqrt{t}\xi|^p, \qquad t \geq 0$$

and this very function emerges naturally: thanks to independence and rotational invariance of the ensuing random vectors,

$$\begin{aligned} \mathbb{E}|a_1\xi_1 + a_2\xi_2 + v|^p = \mathbb{E}\big||a_1\xi_1 + a_2\xi_2|\xi + v\big|^p &= \mathbb{E}g_v\big(|a_1\xi_1 + a_2\xi_2|^2\big) \\ &= \mathbb{E}g_v\big(\sigma^2 + 2a_1a_2\theta\big), \end{aligned}$$

where the last expectation is with respect to a random variable θ which has the same distribution as a one-dimensional marginal of ξ, say $\theta = \langle \xi, e_1 \rangle$. With $\sigma > 0$ fixed, we set

$$h(u) = \mathbb{E}\big[g_v(\sigma^2 + 2u\theta)\big], \qquad u > 0.$$

Then the deficit of interest becomes

$$\Delta = \mathbb{E}\left|b_1\xi_1 + b_2\xi_2 + v\right|^p - \mathbb{E}\left|a_1\xi_1 + a_2\xi_2 + v\right|^p = h(b_1b_2) - h(a_1a_2)$$
$$= \int_{a_1a_2}^{b_1b_2} h'(u)\mathrm{d}u.$$

Plainly,

$$h'(u) = 2\mathbb{E}\big[g_v'(\sigma^2 + 2u\theta)\theta\big].$$

To manoeuvrer this into a more convenient expression, we shall use an integration by parts formula for θ.

Claim. Let $d \geq 2$ and $\xi, \tilde{\xi}$ be uniform on S^{d-1}, S^{d+1} respectively. For random variables $\theta = \langle \xi, e_1 \rangle$, $\tilde{\theta} = \left\langle \tilde{\xi}, e_1 \right\rangle$ and a function f differentiable on $(-1,1)$ such that $f(x)(1-x^2)^{\frac{d-1}{2}} \to 0$ as $x \to \pm 1$, we have

$$\mathbb{E}\big[f(\theta)\theta\big] = \frac{1}{d}\mathbb{E}\big[f'(\tilde{\theta})\big]. \tag{16}$$

Proof. One can check that θ has density $\frac{1}{A_d}(1-x^2)^{\frac{d-3}{2}}\mathbf{1}_{(-1,1)}(x)$ with the normalising constant $A_d = \sqrt{\pi}\frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})}$, $d \geq 2$. Integration by parts yields

$$\begin{aligned}
\mathbb{E}\big[f(\theta)\theta\big] &= \frac{1}{A_d}\int_{-1}^{1} f(x)x(1-x^2)^{\frac{d-3}{2}}\mathrm{d}x = \frac{1}{A_d}\int_{-1}^{1} f(x)\Big(-\frac{1}{d-1}(1-x^2)^{\frac{d-1}{2}}\Big)'\mathrm{d}x \\
&= -\frac{1}{(d-1)A_d}f(x)(1-x^2)^{\frac{d-1}{2}}\Bigg|_{-1}^{1} + \frac{1}{(d-1)A_d}\int_{-1}^{1} f'(x)(1-x^2)^{\frac{d-1}{2}}\mathrm{d}x \\
&= \frac{A_{d+2}}{(d-1)A_d}\mathbb{E}\big[f'(\tilde{\theta})\big]
\end{aligned}$$

and $\frac{A_{d+2}}{(d-1)A_d} = \frac{1}{d}$, as claimed. □

Applying the claim to $f(x) = g_v'(\sigma^2 + 2ux)$ gives

$$h'(u) = \frac{2}{d}\mathbb{E}\Big[g_v''(\sigma^2 + 2u\tilde{\theta}) \cdot (2u)\Big]$$

where the expectation is over the distribution of $\tilde{\theta}$ from the statement of the claim. Moreover, evoking (8),

$$g_v''(t) = \beta_{p,d}\mathbb{E}|v + R_1 R_2 \sqrt{t}\xi|^{p-4},$$

where, to compactify the notation, we let (R_1, R_2) be random variables with joint density $\mathrm{d}r_1^d \mathrm{d}r_2^{d+2}$ on $(0,1)^2$.

Putting these together,

$$h'(u) = \tilde{\beta}_{p,d}\mathbb{E}\big[|v + R_1 R_2(\sigma^2 + 2u\tilde{\theta})^{1/2}\xi|^{p-4}\big] \cdot (2u)$$

with

$$\tilde{\beta}_{p,d} = \frac{2}{d}\beta_{p,d}$$

and the expectation taken over the product distribution of $R_1, R_2, \tilde{\theta}, \xi$. Thus, we arrive at

$$\Delta = \tilde{\beta}_{p,d}\int_{a_1 a_2}^{b_1 b_2} \mathbb{E}\big[|v + R_1 R_2(\sigma^2 + 2u\tilde{\theta})^{1/2}\xi|^{p-4}\big] \cdot (2u)\mathrm{d}u.$$

As before, we now break the analysis into two cases depending on whether $p > 4$.

Case 1: $2 \le p \le 4$. As in Lemma 4, letting $q = \frac{p-4}{2}$ and using the point-wise bound (10), we get

$$\Delta \ge \tilde{\beta}_{p,d}\int_{a_1 a_2}^{b_1 b_2} (2u) \cdot \mathbb{E}\left[\left(|v|^2 + \frac{d}{d+4}(\sigma^2 + 2u\tilde{\theta})\right)^q\right]\mathrm{d}u.$$

Since $\mathbb{E}\tilde{\theta} = 0$, Jensen's inequality and a further simple cosmetic bound $\frac{d}{d+4} < 1$ allow to lower-bound the expectation by $(|v|^2 + \sigma^2)^q$ which results in

$$\Delta \ge \tilde{\beta}_{p,d}(|v|^2 + \sigma^2)^q(b_1^2 b_2^2 - a_1^2 a_2^2).$$

Finally, as a result of the constraint $a_1^2 + a_2^2 = b_1^2 + b_2^2$,

$$0 = (b_1^2 + b_2^2)^2 - (a_1^2 + a_2^2)^2 = b_1^4 + b_2^4 - (a_1^4 + a_2^4) + 2(b_1^2 b_2^2 - a_1^2 a_2^2),$$

so

$$b_1^2 b_2^2 - a_1^2 a_2^2 = \frac{1}{2}(a_1^4 + a_2^4 - b_1^4 - b_2^4) \tag{17}$$

and we are done with the proof in this case with constant $\frac{1}{2}\tilde{\beta}_{p,d}$, as desired.

Case 2: $p > 4$. We can crudely bound the expectation by restricting it to the positive values of $\tilde{\theta}$,

$$\Delta \geq \tilde{\beta}_{p,d} \int_{a_1 a_2}^{b_1 b_2} \mathbb{E}\big[|v + R_1 R_2(\sigma^2 + 2u\tilde{\theta})^{1/2}\xi|^{p-4}\mathbf{1}_{\{\tilde{\theta}>0\}}\big] \cdot (2u)\mathrm{d}u$$

Thanks to (12), used conditioning on the positive values of $R_1, R_2, \tilde{\theta}$,

$$\begin{aligned}\Delta &\geq \tilde{\beta}_{p,d} \int_{a_1 a_2}^{b_1 b_2} \mathbb{E}\big[|v + R_1 R_2 \sigma\xi|^{p-4}\big] \cdot \mathbb{P}\left(\tilde{\theta} > 0\right) \cdot (2u)\mathrm{d}u \\ &= \frac{1}{2}\tilde{\beta}_{p,d}\mathbb{E}\big[|v + R_1 R_2 \sigma\xi|^{p-4}\big](b_1^2 b_2^2 - a_1^2 a_2^2).\end{aligned}$$

Since $R_1 R_2 \leq 1$ a.s., using (12) once more, we get a bound

$$\mathbb{E}\big[|v + R_1 R_2 \sigma\xi|^{p-4}\big] \geq \mathbb{E}\big[|R_1 R_2 v + R_1 R_2 \sigma\xi|^{p-4}\big] = \mathbb{E}(R_1 R_2)^{p-4}\mathbb{E}\big[|v + \sigma\xi|^{p-4}\big].$$

Consequently, employing (17),

$$\Delta \geq \frac{1}{4}\tilde{\beta}_{p,d}\mathbb{E}(R_1 R_2)^{p-4}\mathbb{E}\big[|v + \sigma\xi|^{p-4}\big](a_1^4 + a_2^4 - b_1^4 - b_2^4).$$

By a direct calculation, $\mathbb{E}(R_1 R_2)^{p-4} = \frac{d(d+2)}{(p-4+d)(p-2+d)}$, so after tracing the constants and simplifying, we have

$$\frac{1}{4}\tilde{\beta}_{p,d}\mathbb{E}(R_1 R_2)^{p-4} = \frac{p(p-2)}{8d}. \qquad \square$$

Proof of Theorem 2. When $n = 2$, we apply Lemma 6 with $v = 0$ directly to the coefficient vectors $b = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $a = (a_1, a_2)$ for which $\sigma = 1$. We get

$$\mathbb{E}\left|\frac{1}{\sqrt{2}}\xi_1 + \frac{1}{\sqrt{2}}\xi_2\right|^p - \mathbb{E}|a_1\xi_1 + a_2\xi_2|^p \geq \tilde{c}_{p,d}\delta(a),$$

as desired in view of (14), where

$$\tilde{c}_{p,d} = \frac{1}{d} \cdot \begin{cases} \beta_{p,d}, & 2 \leq p \leq 4, \\ \frac{p(p-2)}{8}, & p > 4. \end{cases}$$

Now suppose that $n \geq 3$, $a_1 \geq \cdots \geq a_n > 0$ and that $a \neq (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$ (in particular, $a_n < \frac{1}{\sqrt{n}}$). We follow a strategy from [21], namely we repetitively perform the following local operation on the coefficient vector a until it becomes the diagonal vector $(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$: we take its largest coefficient a_1, the smallest one a_n, replace them with $a_1' = \sqrt{a_1^2 + a_n^2 - \frac{1}{n}}$ and $a_n' = \frac{1}{\sqrt{n}}$, and finally rearrange the coefficients of $(a_1', a_2, \ldots, a_{n-1}, a_n')$ to be nonincreasing, calling the resulting vector b. Since this operation strictly increases the number of coefficients equal to $\frac{1}{\sqrt{n}}$, after finitely many operations, we arrive at the diagonal vector $(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$. Plainly, this operation preserves the ℓ_2 norm. This allows to apply Lemma 6 (conditioning on the value of $v = \sum_{j=2}^{n-1} a_j \xi_j$) which yields the following bound on the deficit coming from one operation transforming a to b,

$$
\begin{aligned}
&\mathbb{E}\,|a_1'\xi_1 + a_n'\xi_2 + v|^p - \mathbb{E}\,|a_1\xi_1 + a_n\xi_2 + v|^p \\
&\qquad \geq \frac{1}{d}(a_1^4 + a_n^4 - a_1'^4 - a_n'^4) \cdot \begin{cases} \beta_{p,d}(|v|^2 + \sigma^2)^{\frac{p-4}{2}}, & 2 \leq p \leq 4, \\ \frac{p(p-2)}{8}\mathbb{E}|v + \sigma\xi|^{p-4}, & p > 4, \end{cases}
\end{aligned}
$$

where, as in the lemma, $\sigma^2 = a_1^2 + a_n^2 = a_1'^2 + a_n'^2$. Moreover, averaging over the distribution of v gives

- when $2 \leq p \leq 4$, by Jensen's inequality,

$$
\mathbb{E}(|v|^2 + \sigma^2)^{\frac{p-4}{2}} \geq \left(\mathbb{E}|v|^2 + \sigma^2\right)^{\frac{p-4}{2}} = \left(\sum_{j=1}^{n} a_j^2\right)^{\frac{p-4}{2}} = 1,
$$

- when $p > 4$, by Lemma 5 (Khinchin's inequality),

$$
\mathbb{E}|v + \sigma\xi|^{p-4} \geq c_{\mathrm{Kh}} \left(\sum_{j=1}^{n} a_j^2\right)^{p-4} = c_{\mathrm{Kh}}.
$$

Notably, $a_1^4 + a_n^4 - a_1'^4 - a_n'^4 = \|a\|_4^4 - \|b\|_4^4$, thus adding the contributions over all operations, these ℓ_4 deficit differences add over a telescoping-type sum, leading to the bound

$$
\mathbb{E}\left|\sum_{j=1}^{n} \frac{1}{\sqrt{n}}\xi_j\right|^p - \mathbb{E}\left|\sum_{j=1}^{n} a_j\xi_j\right|^p \geq \tilde{c}_{p,d}\left(\|a\|_4^4 - \left\|\left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)\right\|_4^4\right) = \tilde{c}_{p,d}\delta(a),
$$

with

$$\tilde{c}_{p,d} = \frac{1}{d} \begin{cases} \beta_{p,d}, & 2 \le p \le 4, \\ c_{\mathrm{Kh}} \frac{p(p-2)}{8}, & p > 4, \end{cases}$$

which after plugging in the value of $\beta_{p,d}$ from (9) finishes the proof. □

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