HAAGERUP'S PHASE TRANSITION AT POLYDISC SLICING

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Abstract. We establish a sharp comparison inequality between the negative moments and the second moment of the magnitude of sums of independent random vectors uniform on three-dimensional Euclidean spheres. This provides a probabilistic extension of the Oleszkiewicz-Pełczyński polydisc slicing result. The Haagerup-type phase transition occurs exactly when the p-norm recovers volume, in contrast to the real case. We also obtain partial results in higher dimensions.

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1. Introduction

Khinchin-type inequalities concern estimates on L_p norms of (weighted) sums of independent random variables, typically involving a norm which is easily understood (or explicit in given parameters) such as the L_2 norm. They can be traced back to Khinchin's work [25] on the law of the iterated logarithm, where he established such bounds for Rademacher random variables (random signs). Beyond their original use, most notably, such inequalities have played an important role in Banach space theory (in connection with topics such as unconditional convergence or type and cotype), see [13, 22, 34, 50]. Considerable work has been devoted to the pursuit of sharp constants in Khinichin-type inequalities, see for instance [3, 6, 15, 16, 19, 21, 31, 32, 33, 37, 38, 39, 40, 41, 42, 44, 46, 49, 51], in particular for sums of random vectors uniform on Euclidean spheres [4, 9, 10, 26, 29] (as a natural generalisation of Rademacher and Steinhaus random variables, intimately related to uniform convergence in real and complex Banach spaces, respectively). This paper continues that line of research.

Throughout, $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^d$, inherited from the standard inner product $\langle\cdot,\cdot\rangle$. For a random vector X in $\mathbb{R}^d$ and a real parameter p, we write $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$ for the L_p-norm (p-th moment) of the magnitude

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of X (whenever the expectation exists, with $p = 0$ understood as usual as $\|X\|_0 = e^{\mathbb{E}\log|X|}$, arising from taking the limit as $p \to 0$).

Let $\xi_1, \xi_2, \ldots$ be independent random vectors, each uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$. In particular, when $d = 1$, these are Rademacher random variables, that is symmetric random signs in $\mathbb{R}$, whereas when $d = 2$, they are often referred to as Steinhaus random variables (especially when $\mathbb{R}^2$ is treated as $\mathbb{C}$). For $q > -(d-1)$, let $c_d(q)$ be the best positive constant such that the following Khinchin-type inequality holds: for every $n \geqslant 1$ and real scalars $a_1, \ldots, a_n$, we have

$$(1) \qquad \left\|\sum_{k=1}^n a_k\xi_k\right\|_q \geqslant c_d(q)\left\|\sum_{k=1}^n a_k\xi_k\right\|_2.$$

In other words, thanks to homogeneity, $c(q)$ is the infimal value of $\left\|\sum_{k=1}^n a_k\xi_k\right\|_q$ over all $n \geqslant 1$ and $a_1, \ldots, a_n \in \mathbb{R}$ with $\sum a_k^2 = 1$. We stress that this L_q norm exists only when $q > -(d-1)$.

Plainly, $c_d(q) = 1$ for $q \geqslant 2$ (by the monotonicity of $p \mapsto \|\cdot\|_p$). When $q \geqslant 2$, the reverse inequality to (1) is nontrivial and interesting, but we do not discuss it here at all, referring instead to, for instance [4, 20, 38] for a comprehensive account of known as well as recent results.

From now on we consider $-(d-1) < q < 2$. We define two constants arising from two particular choices of weights in (1): $a_1 = a_2 = \frac{1}{\sqrt{2}}$ with $n = 2$ and $a_1 = \cdots = a_n = \frac{1}{\sqrt{n}}$ with $n \to \infty$,

$$(2) \qquad c_{d,2}(q) = \left\|\frac{\xi_1+\xi_2}{\sqrt{2}}\right\|_q = \frac{1}{\sqrt{2}}\left(\frac{\Gamma\left(\frac{d}{2}\right)\Gamma(d+q-1)}{\Gamma\left(\frac{d+q}{2}\right)\Gamma\left(d+\frac{q}{2}-1\right)}\right)^{1/q},$$

$$(3) \qquad c_{d,\infty}(q) = \lim_{n\to\infty}\left\|\frac{\xi_1+\cdots+\xi_n}{\sqrt{n}}\right\|_q = \left\|\frac{Z}{\sqrt{d}}\right\|_q = \sqrt{\frac{2}{d}}\left(\frac{\Gamma\left(\frac{d+q}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right)^{1/q},$$

where Z is a standard Gaussian random vector in $\mathbb{R}^d$ (emerging by the central limit theorem). The expression for $c_{d,2}(q)$ will be justified later (see Corollary 14), whereas the expression for $c_{d,\infty}(q)$ follows by a simple integration in polar coordinates. Note that

$$(4) \qquad c_d(q) \geqslant \min\{c_{d,2}(q), c_{d,\infty}(q)\}.$$

It can be checked that in fact

$$(5) \qquad \min\{c_{d,2}(q), c_{d,\infty}(q)\} = \begin{cases} c_{d,2}(q), & -(d-1) < q \leqslant q_d^*, \\ c_{d,\infty}(q), & q_d^* \leqslant q \leqslant 2, \end{cases}$$

where q_d^* is the unique solution of the equation $c_{d,2}(q) = c_{d,\infty}(q)$ in $(-(d-1), 2)$. We have included a sketch of the proof of this fact in the appendix. In Table 1 below we list some numerical values of q_d^*. We are grateful to Hermann König for sharing his notes on these topics, [27].

1.1. **Known results.** The pursuit of the value of $c_d(q)$ has a rich history which can be summarised in one simple statement that in all known cases, the trivial lower bound (4) is tight, in that there is equality. Of course, the history begins with the one dimensional case of Rademacher random variables. In his study [35] on bilinear forms, Littlewood conjectured that $c_1(1) = c_{1,2}(1) = \frac{1}{\sqrt{2}}$, which was confirmed by Szarek in [46] (see also [32] and [47]). Haagerup's pivotal work [19] addressed the entire range $0 < q < 2$, showing the following phase transition in the behaviour of $c_1(q)$:

$$c_1(q) = \begin{cases} c_{1,2}(q), & 0 < q \leqslant q_1^*, \\ c_{1,\infty}(q), & q_1^* \leqslant q < 2, \end{cases}$$

where $q_1^* = 1.84..$ is the unique solution of the equation $c_{1,2}(q) = c_{1,\infty}(q)$ in $(0, 2)$; in particular, when $d = 1$, we have equality in (4). We also refer to Nazarov and Podkorytov's paper [39] which offered great simplifications. Haagerup devised a very efficient argument, crucially relying on Fourier-analytic formulae for L_p-norms, which together with [39] paved the path for many further results.

That a similar behaviour occurs in the case $d = 2$ (Steinhaus variables) was conjectured by Haagerup, later confirmed by König in [26]: when $d = 2$, $0 \leqslant q < 2$, we have equality in (4) and the phase transition occurs now at $q_2^* = 0.47...$ The range $1 \leqslant q < 2$ was in fact earlier dealt with by König and Kwapień in [29] (with $q = 1$ handled even earlier by Sawa in [45]), whereas $-1 < q < 0$ (to the best of our knowledge) appears to be left open, with a natural conjecture that $c(q) = c_{2,2}(q)$.

For the case $d = 3$: Latała and Oleszkiewicz showed in [33] that $c_3(q) = c_{3,\infty}(q)$ for $1 \leqslant q < 2$ which was extended to $0 < q < 1$ in our joint work [9] with Gurushankar (see Proposition 3 below for a connection to uniform distribution on intervals). The phase transition occurs in the range $-1 < q < 0$ at $q_3^* = -0.79..$, as established in our joint work [10] with König, so when $d = 3$ and $-1 < q < 2$, (4) holds with equality. Again, $-2 < q < -1$ appears to be open with a natural conjecture that $c(q) = c_{3,2}(q)$.

In higher dimensions $d \geqslant 4$, there are precise Schur-convexity results available for positive moments due to Baerstein II and Culverhouse from [4] and, independently König and Kwapień from [29]: when $0 \leqslant q < 2$, it follows in particular that $c_d(q) = c_{d,\infty}(q)$. However, nothing seems to be known about the value of $c_d(q)$ for negative q, except it being (nontrivially) finite, as shown by Gorin and Favorov in [18] (in a much more general setting). This paper partially fills out this gap.

1.2. **Our contribution.** Our first result concerns the best constant $c_d(q)$ in the inequality (1) when $q > -(d-4)$. It turns out that this is a consequence of a Schur-concavity type statement that follows directly from the main result of [4] (see Theorem 6 below).

Theorem 1. *For every $d \geqslant 5$ and $-(d-4) \leqslant q < 0$, we have $c_d(q) = c_{d,\infty}(q)$.*

This is only meaningful for dimensions $d \geqslant 5$. Our second result covers the entire range $-3 < q < 0$ for dimension $d = 4$, which exhibits Haagerup's phase transition at exactly $q_4^* = -2$ (see also Table 1 for other values of q_d^* and a summary of known results and open questions).

Theorem 2. *For $-3 < q < 0$, we have*

$$c_4(q) = \begin{cases} c_{4,2}(q), & -3 < q \leqslant -2, \\ c_{4,\infty}(q), & -2 \leqslant q < 0. \end{cases}$$

Table 1. Numerical values of q_d^* (see (35) for its asymptotics), known results and open questions about the best constant in Khinchin inequality (1).

d	q_d^*	Range where $c(q)$ known	Phase transition	Left open
1	1.82..	$0 < q < 2$ ([19])	[19]	–
2	0.47..	$0 < q < 2$ ([4, 26, 29])	[26]	$-1 < q < 0$
3	$-0.79..$	$-1 < q < 2$ ([9, 10, 33])	[10]	$-2 < q < -1$
4	-2	$-3 < q < 2$ (Thm. 2)	Thm. 2	–
5	$-3.16..$	$-1 < q < 2$ ([4, 29], Thm. 1)	?	$-4 < q < -1$
$\vdots$				
d	$-(d-1)+o(1)$	$-(d-4) < q < 2$ ([4, 29], Thm. 1)	?	$-(d-1) < q < -(d-4)$

1.3. **Relation to volume.** It can perhaps be traced back to Kalton and Koldobsky's paper [24] that the volume of hyperplane sections of convex bodies can be expressed in terms of negative moments (of linear forms in vectors uniform on the body). Brzezinski's work [8] makes the same connection for sections of products of Euclidean balls by block subspaces and our recent work with Nayar [11] explores this further. In particular, as [10] extends Ball's cube slicing result from [5] (in the form of sharp Khinchin inequality (1) when $d = 3$), Theorem 2 can be viewed as a probabilistic extension of Oleszkiewicz and Pełczyński's polydisc slicing from [43]. In fact, this connection was the main motivation of this work. It is very intriguing that the phase transition occurs exactly at $q = -2$ which is when (1) recovers the result for volume from [43].

More specifically, let $\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$ be the unit disc in the complex plane. Oleszkiewicz and Pełczyński in [43] proved the following sharp inequality about extremal-volume (complex) hyperplane sections of the polydics $\mathbb{D}^n$ in $\mathbb{C}^n$: for every (complex) codimension 1 subspace H in $\mathbb{C}^n$, we have

$$\mathrm{vol}_{2n-2}(\mathbb{D}^n \cap H) \leqslant \mathrm{vol}_{2n-2}(\mathbb{D}^n \cap (1,1,0,\dots,0)^\perp), \tag{6}$$

$$\mathrm{vol}_{2n-2}(\mathbb{D}^n \cap H) \geqslant \mathrm{vol}_{2n-2}(\mathbb{D}^n \cap (1,0,\dots,0)^\perp). \tag{7}$$

Here $a^\perp = \{z \in \mathbb{C}^n, \langle a, z\rangle = 0\}$ is the (codimension 1) hyperplane orthogonal to a vector a in $\mathbb{C}^n$ and $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{C}^n$. If we let $U_1, \dots, U_n$ be independent random vectors, each uniform on $\mathbb{D}$ and let $a = (a_1, \dots, a_n)$ be a unit vector in $\mathbb{C}^n$, then

$$\mathrm{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp) = \frac{\pi^{n-1}}{2} \lim_{p\to 2-} (2-p)\mathbb{E}\left|\sum_{k=1}^n a_k U_k\right|^{-p}$$

(such formulae hold for arbitrary origin-symmetric convex sets, and this one follows immediately from Corollary 11 in [11]). Moreover, the moments of sums of vectors uniform on balls are proportional to sums of vectors uniform on spheres (in a slightly higher dimension).

Proposition 3 ([4], [29])**.** *Let $d \geqslant 3$ and let $\xi_1, \xi_2, \dots$ be independent random vectors uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$ and let $U_1, U_2, \dots$ be independent random vectors uniform on the unit Euclidean ball $\mathbb{B}^{d-2}$ in $\mathbb{R}^{d-2}$. For every $q > -(d-2)$, $n \geqslant 1$ and scalars $a_1, \dots, a_n$, we have*

$$\mathbb{E}\left|\sum_{k=1}^n a_k U_k\right|^q = \frac{d-2}{d-2+q}\mathbb{E}\left|\sum_{k=1}^n a_k \xi_k\right|^q$$

This identity can be seen in a number of ways, but essentially it follows from the folklore result that if a random vector $\xi = (\xi_1, \dots, \xi_d)$ is uniform on S^{d-1}, then its projection $(\xi_1, \dots, \xi_{d-2})$ onto $\mathbb{R}^{d-2}$ is uniform on $\mathbb{B}^{d-2}$. Specialised to $d = 4$ and combined with the previous formula, it yields

$$\mathrm{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp) = \pi^{n-1}\mathbb{E}\left|\sum_{k=1}^n a_k \xi_k\right|^{-2}$$

(see also [28] and [30] for generalisations to noncentral sections). Thus, the upper bound (6) is Theorem 2 at $q = -2$, that is $c_2(2) = c_{4,2}(2)$. Incidentally, the lower bound (7) follows immediately from Jensen's inequality (see, e.g. [8], or [28], as well as [11] for a stability result).

The sequel is devoted to proofs. First we provide some background and give a brief summary. Then we move to the proof of Theorem 1 (which is very short) and the rest is occupied with the proof of Theorem 2.

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2. Proofs of the main results

2.1. **Some background and outline.** Theorem 1 will follow easily from the main result of [4]. As for positive moments, the point is that the range $-(d-4) < q < 0$ still warrants *enough* convexity of the underlying moment functional, specifically the function $|x|^q$ (in fact, its C^∞ regularisation/approximation) is bisubharmonic.

When $d = 4$, as in Theorem 2, this range is empty, Schur convexity/concavity does not hold, and more subtle arguments are needed. We will employ a Fourier-analytic approach (pioneered by Haagerup for random signs in [19]). On its own however, this does not dispense of all cases. We extend an inductive argument of Nazarov and Podkorytov from [39] to our multidimensional setting and *all* negative moments (building on [10] with new ideas needed to go beyond the -1st moment). The Fourier-analytic approach relies on the following integral representation of Gorin and Favorov for negative moments.

Lemma 4 (Lemma 3 in [18])**.** *For a random vector X in $\mathbb{R}^d$ and $0 < p < d$, we have*

$$\mathbb{E}|X|^{-p} = K_{p,d} \int_{\mathbb{R}^d} \left(\mathbb{E}e^{i\langle t,X\rangle}\right)|t|^{p-d}\mathrm{d}t, \tag{8}$$

provided that the right hand side integral exists, where

$$K_{p,d} = 2^{-p}\pi^{-d/2}\frac{\Gamma\left(\frac{d-p}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}.$$

Of course, the Fourier transform (the characteristic function) goes hand in hand with independence. The trade-off is that when applied to sums of independent random vectors uniform on spheres, highly-oscillating integrands appear, more precisely, the Bessel functions. To recall, for integral $k \geqslant 0$ and real x, we use the notation

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = x(x+1)\dots(x+k-1)$$

for the rising factorial (Pochhammer symbol). Throughout,

$$J_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{t}{2}\right)^{2k+\nu}$$

is the Bessel function of the first kind with parameter $\nu > 0$. We also introduce the function

$$\mathrm{j}_\nu(t) = 2^\nu\Gamma(\nu+1)t^{-\nu}J_\nu(t) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(\nu+1)_k}\left(\frac{t}{2}\right)^{2k}.$$

Its importance stems from the fact that for a random vector ξ uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$ and a vector v in $\mathbb{R}^d$, we have

$$\mathbb{E}e^{i\langle v,\xi\rangle} = \mathrm{j}_{d/2-1}(|v|) \tag{9}$$

(see, e.g. the proof of Proposition 10 in [29]). This combined with Lemma 4 gives the following corollary.

Corollary 5. *For independent, rotationally invariant random vectors $X_1, \dots, X_n$ in $\mathbb{R}^d$ and $0 < p < d$, we have*

$$\mathbb{E}\left|\sum_{k=1}^n X_k\right|^{-p} = \kappa_{p,d}\int_0^\infty \prod_{k=1}^n \left(\mathbb{E}\ \mathrm{j}_{d/2-1}(t|X_k|)\right)t^{p-1}\mathrm{d}t, \tag{10}$$

provided that the right hand side integral exists, where

$$\kappa_{p,d} = 2^{1-p}\frac{\Gamma\left(\frac{d-p}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{p}{2}\right)}.$$

Proof. Let $\xi_1, \ldots, \xi_n$ be independent random vectors, each uniform on the unit Euclidean sphere S^{d-1}, chosen independently of the X_k. Then X_k has the same distribution as $|X_k|\xi_k$ and (8) together with (9) and integration in polar coordinates give

$$\begin{aligned}\mathbb{E}\left|\sum_{k=1}^n X_k\right|^{-p} &= K_{p,d}\int_{\mathbb{R}^d}\left(\prod_{k=1}^n \mathbb{E}e^{i\langle t,|X_k|\xi_k\rangle}\right)|t|^{p-d}\mathrm{d}t \\ &= K_{p,d}\int_{\mathbb{R}^d}\left(\prod_{k=1}^n \mathbb{E}\ \mathrm{j}_{d/2-1}(|t||X_k|)\right)|t|^{p-d}\mathrm{d}t \\ &= K_{p,d}|S^{d-1}|\int_0^\infty\left(\prod_{k=1}^n \mathbb{E}\ \mathrm{j}_{d/2-1}(t|X_k|)\right)t^{p-1}\mathrm{d}t,\end{aligned}$$

where $|S^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$ is the $(d-1)$-dimensional volume of the unit sphere in $\mathbb{R}^d$. □

2.2. **Proof of Theorem 1.** Theorem 1 is a straightforward corollary of the following stronger Schur-concavity result. For background on Schur-majorisation, we refer for example to [7].

Theorem 6. *Let $d \geqslant 5$ and let $\xi_1, \xi_2, \ldots$ be independent random vectors uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$. For every $n \geqslant 1$ and $0 < p \leqslant d-4$, the function*

$$(x_1, \ldots, x_n) \mapsto \mathbb{E}\left|\sum_{k=1}^n \sqrt{x_k}\xi_k\right|^{-p}$$

is Schur-concave on $\mathbb{R}_+^n$.

Proof. Thanks to Lebesgue's monotone convergence theorem, it suffices to show that for every $\delta > 0$, the theorem holds with $|\cdot|^{-p}$ replaced by the function $\Psi_\delta(x) = (|x|^2+\delta)^{-p/2}$. The gain is that Ψ_δ is C^∞ on $\mathbb{R}^d$. In view of the result of Baernstein II and Culverhouse from [4], it suffices to show that Ψ_δ is bisubharmonic, that is $\Delta\Delta\Psi_\delta \geqslant 0$ on $\mathbb{R}^d$. We approach this directly. We have,

$$\Delta\Delta\Psi_\delta(x) = p(p+2)\left(|x|^2+\delta\right)^{-\frac{p}{2}-4}(A|x|^4 + B|x|^2 + C),$$

where $A = (p-d+2)(p-d+4)$, $B = 2\delta(d+2)(-p+d-4)$ and $C = \delta^2 d(d+2)$. For $p < d-4$, plainly $A > 0$ and $B^2 - 4AC = 8\delta^2(d+2)(p+4)(p-d+4) < 0$. This shows that Ψ_δ is bisubharmonic on $\mathbb{R}^d$ for every $\delta > 0$. □

Remark 7. The crux of Baernstein II and Culverhouse's work is the observation that the bisubharmonicity of a continuous function Ψ on $\mathbb{R}^d$ on one hand is sufficient for the Schur-convexity of the corresponding moment functional from Theorem 6, $\mathbb{E}\Psi\left(\sum_{k=1}^n \sqrt{x_k}\xi_k\right)$ (and necessary when Ψ is radial), and on the other hand, it is equivalent to the convexity of the function

$$t \mapsto \mathbb{E}\Psi(v + \sqrt{t}\xi)$$

on $\mathbb{R}_+$ for every $v \in \mathbb{R}^d$. In the sequel, we will need to examine the behaviour of this function on $(0,1)$ for unit vectors v when $\Psi(x) = |x|^{-p}$ (see Section 3.1 below).

2.3. **Outline of the proof of Theorem 2.** Recall that here $d = 4$ and $\xi_1, \xi_2, \dots$ are independent random vectors uniform on the unit Euclidean sphere S^3 in $\mathbb{R}^4$. For notational convenience, we put $q = -p$, $0 < p < 3$ and set

$$C_2(p) = c_{4,2}(q)^q = \mathbb{E}\left|\frac{\xi_1 + \xi_2}{\sqrt{2}}\right|^{-p} = 2^{p/2}\frac{\Gamma(3-p)}{\Gamma\left(2 - \frac{p}{2}\right)\Gamma\left(3 - \frac{p}{2}\right)}, \tag{11}$$

$$C_\infty(p) = c_{4,\infty}(q)^q = \mathbb{E}\left|\frac{Z}{2}\right|^{-p} = 2^{p/2}\Gamma\left(2 - \frac{p}{2}\right), \tag{12}$$

where Z is a standard Gaussian random vector in $\mathbb{R}^4$ (consult (2) and (3) to justify the explicit expressions on the right hand sides). Moreover, let $C(p)$ be the best constant such that the following equivalent form of (1),

$$\mathbb{E}\left|\sum_{k=1}^n a_k\xi_k\right|^{-p} \leqslant C(p)\left(\sum_{k=1}^n a_k^2\right)^{-p/2} \tag{13}$$

holds for every $n \geqslant 1$ and every real scalars $a_1, \dots, a_n$.

Theorem 2 is a consequence of the next two results, where we break it up into two regimes.

Theorem 8. *For $0 < p \leqslant 2$, we have $C(p) = C_\infty(p)$.*

Theorem 9. *For $2 < p < 3$, we have $C(p) = C_2(p)$.*

As optimality is clear, for the proofs of these theorems, we need to show that (13) holds with the specified values of $C(p)$.

2.3.1. *Outline of the proof of Theorem 8.* Thanks to homogeneity, we can assume that the a_k are positive with $\sum a_k^2 = 1$. Using the Fourier-analytic formula for negative moments (10) and Hölder's inequality, we obtain

$$\begin{aligned}\mathbb{E}\left|\sum_{k=1}^n a_k\xi_k\right|^{-p} &= \kappa_{p,4}\int_0^\infty\left(\prod_{k=1}^n \mathrm{j}_1(a_k t)\right)t^{p-1}\mathrm{d}t \\ &\leqslant \kappa_{p,4}\prod_{k=1}^n\left(\int_0^\infty |\mathrm{j}_1(a_k t)|^{a_k^{-2}}t^{p-1}\mathrm{d}t\right)^{a_k^2} \\ &= \kappa_{p,4}\prod_{k=1}^n\left(a_k^{-p}F\left(p, a_k^{-2}\right)\right)^{a_k^2}.\end{aligned} \tag{14}$$

where the following function has emerged (after a change of variables in the last line)

$$F(p,s) = \int_0^\infty |\mathrm{j}_1(t)|^s t^{p-1}\mathrm{d}t, \qquad p, s > 0. \tag{15}$$

This integral is finite as long as $p < \frac{3s}{2}$ because $\mathrm{j}_1(t) = O(t^{-3/2})$ (see (21) below).

The next step is to maximise, individually, the terms in the product on the right hand side of (14), that is to look into $\sup_{s\geqslant 1} s^{p/2}F(p,s)$. Heuristically, if we aim at proving that the worst case is Gaussian, that is when $a_1 = \cdots = a_n = \frac{1}{\sqrt{n}}$ with $n \to \infty$, a natural candidate for this supremum is then given by $s \to \infty$, which would correspond to the inequality

$$
\begin{aligned}
s^{p/2} F(p,s) \leqslant \lim_{s\to\infty} s^{p/2} \int_0^\infty |\mathrm{j}_1(t)|^s t^{p-1} \mathrm{d}t &= \lim_{s\to\infty} \int_0^\infty |\mathrm{j}_1(t/\sqrt{s})|^s t^{p-1} \mathrm{d}t \\
&= \int_0^\infty e^{-t^2/8} t^{p-1} \mathrm{d}t
\end{aligned}
\tag{16}
$$

(the last line can be justified using $\mathrm{j}_1(t) = 1 - \frac{t^2}{8} + o(t^2) = e^{-t^2/8} + o(t^2)$). Were it true for all values of p and s, we would get

$$
\mathbb{E}\left|\sum_{k=1}^n a_k \xi_k\right|^{-p} \leqslant \kappa_{p,4} \int_0^\infty e^{-t^2/8} t^{p-1} \mathrm{d}t = C_\infty(p),
$$

finishing the proof. Unfortunately, the integral inequality (16) fails in certain ranges of p and s, where additional arguments and ideas are needed. This is how we will proceed.

Step 1: Inequality (16) *holds for all* $0 < p \leqslant 2$ *and* $s \geqslant 2$.

As above, this gives the following partial case of the theorem when all coefficients a_k are small.

Corollary 10. *When* $0 < p \leqslant 2$, *inequality* (13) *holds with* $C(p) = C_\infty(p)$ *for every* $n \geqslant 1$ *and all real numbers* $a_1, \ldots, a_n$ *with* $\max_{k\leqslant n} |a_k| \leqslant \frac{1}{\sqrt{2}} \left(\sum_{k=1}^n a_k^2\right)^{1/2}$.

Step 2: For $\frac{1}{4} \leqslant p \leqslant 2$, *we employ induction on* n *to cover the case* $\max_{k\leqslant n} |a_k| > \frac{1}{\sqrt{2}} \left(\sum_{k=1}^n a_k^2\right)^{1/2}$.

This will give the theorem when $p \geqslant \frac{1}{4}$. For the induction to work, (13) is strengthened, but the base of the induction fails for small p (roughly $p < 0.2$), hence the next two steps. Fortunately, when p is *small*, the integral inequality holds for a wider range of s.

Step 3: Inequality (16) *holds for all* $0 < p \leqslant \frac{1}{4}$ *and* $s \geqslant 1.3$.

Corollary 11. *When* $0 < p \leqslant \frac{1}{4}$, *inequality* (13) *holds with* $C(p) = C_\infty(p)$ *for every* $n \geqslant 1$ *and all real numbers* $a_1, \ldots, a_n$ *such that* $\max_{k\leqslant n} |a_k| \leqslant \sqrt{\frac{10}{13}} \left(\sum_{k=1}^n a_k^2\right)^{1/2}$.

Finally, when one of the coefficients a_k is *large*, the inequality holds for a different reason (we will use a sort of projection-type argument).

Step 4: When $0 < p \leqslant \frac{1}{4}$, *inequality* (13) *holds with* $C(p) = C_\infty(p)$ *for every* $n \geqslant 1$ *and all real numbers* $a_1, \ldots, a_n$ *with* $\max_{k\leqslant n} |a_k| > \sqrt{\frac{10}{13}} \left(\sum_{k=1}^n a_k^2\right)^{1/2}$.

2.3.2. *Outline of the proof of Theorem 9.* If we want to prove that the worst case is now $n = 2$ with $a_1 = a_2 = \frac{1}{\sqrt{2}}$, it is only natural to expect that $\sup_{s\geqslant 1} s^{p/2}F(p,s)$ is attained at $s = 2$, corresponding to the integral inequality

$$s^{p/2}F(p,s) \leqslant 2^{p/2}F(p,2). \tag{17}$$

We will proceed similarly, with only the first two steps sufficing, as the inductive base now holds in the entire range.

Step 1: Inequality (17) *holds for all* $2 < p < 3$ *and* $s \geqslant 2$.

Taking this statement for granted for now, we derive the following corollary.

Corollary 12. *When* $2 < p < 3$*, inequality* (13) *holds with* $C(p) = C_2(p)$ *for every* $n \geqslant 1$ *and all real numbers* $a_1, \ldots, a_n$ *with* $\max_{k\leqslant n}|a_k| \leqslant \frac{1}{\sqrt{2}}\left(\sum_{k=1}^n a_k^2\right)^{1/2}$.

Proof. Assuming $\sum a_k^2 = 1$ and applying (17) to the right hand side of (14) yields

$$\begin{aligned}\mathbb{E}\left|\sum_{k=1}^n a_k\xi_k\right|^{-p} \leqslant \kappa_{p,4}\cdot 2^{p/2}F(p,2) &= 2^{p/2}\kappa_{p,4}\int_0^\infty \mathrm{j}_1(t)^2t^{p-1}\mathrm{d}t \\ &= 2^{p/2}\mathbb{E}\left|\xi_1+\xi_2\right|^{-p} = C_2(p)\end{aligned}$$

(for the penultimate step, recall Corollary 5). □

Step 2: For $2 < p < 3$*, we employ induction on* n *to cover the case* $\max_{k\leqslant n}|a_k| > \frac{1}{\sqrt{2}}\left(\sum_{k=1}^n a_k^2\right)^{1/2}$.

To carry out these steps, we first establish a variety of indispensable technical estimates. After this has been done in the next section, we will conclude the proof in Sections 4 and 5.

3. Ancillary results

3.1. **Two-coefficient function.** By rotational invariance,

$$\mathbb{E}|a_1\xi_1 + a_2\sqrt{t}\xi_2|^{-p} = \mathbb{E}|a_1e_1 + a_2\xi_2|^{-p}.$$

We begin with some properties of the function $t \mapsto \mathbb{E}|a_1e_1 + a_2\xi_2|^{-p}$, particularly important in the inductive part of our proof. Recall the definition of the (Gaussian) hypergeometric function which shows up very naturally, as explained in the next lemma. For real parameters a, b, c, it is defined for $|z| < 1$ by the power series,

$${}_2F_1(a,b;c;z) = \sum_{k=0}^\infty \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}.$$

Lemma 13. *Let $d \geqslant 1$ and let ξ be a random vector uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$. Let $p < d-1$. Then*

$$\begin{aligned}\mathbb{E}|e_1 + \sqrt{t}\xi|^{-p} &= {}_2F_1\left(\frac{p}{2}, \frac{p-d+2}{2}; \frac{d}{2}; t\right) \\ &= \sum_{k=0}^{\infty} \frac{\left(\frac{p}{2}\right)_k \left(\frac{p-d+2}{2}\right)_k}{\left(\frac{d}{2}\right)_k} \frac{t^k}{k!}, \qquad 0 < t < 1.\end{aligned}$$

Proof. Fix $0 < t < 1$. Let $\theta = \langle e_1, \xi \rangle$ be the first coordinate of ξ. Thus

$$\begin{aligned}\mathbb{E}|e_1 + \sqrt{t}\xi|^{-p} &= \mathbb{E}(1 + 2\sqrt{t}\theta + t)^{-p/2} \\ &= (1+t)^{-p/2}\mathbb{E}\left(1 + \frac{2\sqrt{t}}{1+t}\theta\right)^{-p/2} \\ &= (1+t)^{-p/2}\sum_{k=0}^{\infty}\binom{-p/2}{2k}(\mathbb{E}\theta^{2k})\left(\frac{2\sqrt{t}}{1+t}\right)^{2k}.\end{aligned}$$

From (9),

$$\mathbb{E}\theta^{2k} = \frac{(2k)!}{2^{2k} \cdot k!(d/2)_k},$$

hence

$$\mathbb{E}|e_1 + \sqrt{t}\xi|^{-p} = (1+t)^{-p/2}\sum_{k=0}^{\infty}\frac{(p/2)_{2k}}{2^{2k}(d/2)_k}\frac{1}{k!}\left(\frac{4t}{(1+t)^2}\right)^k.$$

Since $(p/2)_{2k}\, 2^{-2k} = \left(\frac{p}{4}\right)_k \left(\frac{p+2}{4}\right)_k$, we get

$$\begin{aligned}\mathbb{E}|e_1 + \sqrt{t}\xi|^{-p} &= (1+t)^{-p/2}{}_2F_1\left(\frac{p}{4}, \frac{p+2}{4}; \frac{d}{2}; \frac{4t}{(1+t)^2}\right) \\ &= {}_2F_1\left(\frac{p}{2}, \frac{p-d+2}{2}; \frac{d}{2}; t\right),\end{aligned}$$

where the last identity follows from Kummer's quadratic transformations for the hypergeometric function ${}_2F_1$ (see, e.g. 15.3.26 in [1]). The desired power series expansion now follows from the definition of ${}_2F_1$. □

This in particular yields the explicit expression for $c_{d,2}(q)$ from (2).

Corollary 14. *For $d \geqslant 1$ and $p < d-1$, we have*

$$\mathbb{E}|\xi_1 + \xi_2|^{-p} = {}_2F_1\left(\frac{p}{2}, \frac{p-d+2}{2}; \frac{d}{2}; 1\right) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma(d-p-1)}{\Gamma\left(\frac{d-p}{2}\right)\Gamma\left(d - \frac{p}{2} - 1\right)}.$$

Proof. The expression on the right hand side follows from Gauss' summation identity (see, e.g. 15.1.20 in [1]). □

Remark 15. In addition to the proof of Lemma 13 presented above we would like to sketch a different argument, in the spirit of Lemma 1 from [4], which bypasses the explicit use of the hypergeometric function. Let $\Psi(x) = |x|^{-p}$. Since on the unit sphere $\xi \in S^{d-1}$ is the outer-normal, by the divergence theorem (for the usual Lebesgue *non*normalised surface integral),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^{d-1}}|e_1+\sqrt{t}\xi|^{-p}\mathrm{d}\xi &= \frac{1}{2\sqrt{t}}\int_{S^{d-1}}\Big\langle(\nabla\Psi)(e_1+\sqrt{t}\xi),\xi\Big\rangle\mathrm{d}\xi \\ &= \frac{1}{2\sqrt{t}}\int_{\mathbb{B}^d}\mathrm{div}_x\Big((\nabla\Psi)(e_1+\sqrt{t}x)\Big)\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{B}^d}(\Delta\Psi)(e_1+\sqrt{t}x)\mathrm{d}x\end{aligned}$$

for every $0<t<1$ (note that $e_1+\sqrt{t}x$ on B_2^d is away from the origin where Ψ is singular). Computing the Laplacian yields the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^{d-1}}|e_1+\sqrt{t}\xi|^{-p}\mathrm{d}\xi = \frac{p(p-d+2)}{2}\int_{B_2^d}|e_1+\sqrt{t}x|^{-p-2}\mathrm{d}x.$$

Writing the last integral using polar coordinates allows to compute the higher derivatives by simply iterating this identity. Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}|e_1+\sqrt{t}\xi|^{-p} &= \frac{p(p-d+2)}{2}\frac{1}{|S^{d-1}|}\int_{B_2^d}|e_1+\sqrt{t}x|^{-p-2}\mathrm{d}x \\ &= \frac{p(p-d+2)}{2}\frac{1}{|S^{d-1}|}\int_0^1\int_{S^{d-1}}r^{d-1}|e_1+\sqrt{tr^2}\xi|^{-p-2}\mathrm{d}\xi\end{aligned} \tag{18}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbb{E}|e_1+\sqrt{t}\xi|^{-p} =& \frac{p(p-d+2)}{2}\frac{(p+2)(p-d+4)}{2} \\ &\cdot\frac{1}{|S^{d-1}|}\int_0^1 r^{d+1}\int_{\mathbb{B}^d}|e_1+\sqrt{t}rx|^{-p-4}\mathrm{d}x\mathrm{d}r,\end{aligned}$$

etc. It then remains to evaluate these derivatives at $t=0$ to get the power-series expansion coefficients.

Corollary 16. *Let ξ be a random vector uniform on the unit Euclidean sphere S^3 in $\mathbb{R}^4$. Let $0<p\leqslant 2$. Then*

$$\mathbb{E}|e_1+\sqrt{t}\xi|^{-p}\leqslant 1-\frac{p(2-p)}{8}t-\frac{p^2(4-p^2)}{192}t^2, \qquad 0<t<1.$$

Proof. When $d=4$ and $0<p<2$, all the terms in the power series from Lemma 13 but the first one (which equals 1) are negative. Dropping all but the first three thus gives the desired bound. □

Corollary 17. *Let $d\geqslant 1$. Let ξ be a random vector uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$. Let $0<p\leqslant d-2$. Then for every vector v in $\mathbb{R}^d$ and $a>0$, we have*

$$\mathbb{E}|v+a\xi|^{-p}\leqslant\min\{|v|^{-p},a^{-p}\}.$$

Proof. By homogeneity and rotational invariance, we can assume without loss of generality that $v=e_1$ and $0<a<1$. From (18) we see that the function $a\mapsto\mathbb{E}|e_1+a\xi|^{-p}$ is nonincreasing, in particular $\mathbb{E}|e_1+a\xi|^{-p}\leqslant 1$. □

3.2. **Bounds for the inductive base.** We remark that in several places we need to use numerical values of some special functions such as j_1, Γ, $\psi = (\log\Gamma)'$ and will implicitly do so (to the required precision).

Based on tables left by Gauss, Deming and Colcord in [12] found the value of $\min_{x>0}\Gamma(x)$ correct up to the 19th decimal which we record here (although we will not require such precision).

Lemma 18 ([12])**.** *We have,*

$$\min_{x>0}\Gamma(x) = 0.8856031944108886887..,$$

uniquely occurring at $x_0 = 1.46163214496836226...$

To check the base of the induction from Step 2 in Section 2.3.1, we will need the following two-point inequality.

Lemma 19. *For every* $\frac{1}{8} \leqslant q \leqslant 1$ *and* $0 \leqslant t \leqslant 1$*, we have*

$$1 - \frac{q(1-q)}{2}t - \frac{q^2(1-q^2)}{12}t^2 \leqslant \Gamma(2-q)\left(2 - \left(\frac{3-t}{2}\right)^{-q}\right).$$

Proof. We let $Q_q(t), R_q(t)$ denote the left hand side and the right hand side respectively and set $h_q(t) = R_q(t) - Q_q(t)$. We examine its second derivative,

$$h_q''(t) = -2^q\Gamma(2-q)q(q+1)(3-t)^{-q-2} + \frac{q^2(1-q^2)}{6}$$

which is clearly decreasing in t. Therefore, for all $0 \leqslant t \leqslant 1$, $h_q''(t) \leqslant h_q''(0)$ and for $0 < q < 1$, with the aid of Lemma 18,

$$\begin{aligned} -\frac{3^{q+2}}{2^q \cdot q(1+q)}h_q''(0) &= \Gamma(2-q) - (3/2)^{q+1}q(1-q) \\ &> 0.88 - (3/2)^2 \cdot \frac{1}{4} = 0.3175. \end{aligned}$$

As a result, $h_q(t)$ is concave on $[0,1]$. To show that $h_q(t) \geqslant 0$ on $[0,1]$, it thus suffices to verify that (A) $h_q(0) \geqslant 0$ and (B) $h_q(1) \geqslant 0$, for all $\frac{1}{8} \leqslant q \leqslant 1$.

(A): $h_q(0) \geqslant 0$ is equivalent to $\Gamma(2-q)\,(2-(2/3)^q) \geqslant 1$, or after taking logarithms, $g(q) \geqslant f(q)$ with $g(q) = \log\Gamma(2-q)$, $f(q) = -\log 2 - \log(1 - \frac{1}{2}(\frac{2}{3})^q)$. Both f and g are clearly convex (note $f(q) = -\log 2 + \sum_{k=1}^{\infty}[\frac{1}{2}(\frac{2}{3})^q]^k/k$). For $\frac{1}{8} \leqslant q \leqslant 0.35$, we lower-bound g by its supporting tangent at $q = \frac{1}{8}$, $g(q) \geqslant \ell(q) = g(\frac{1}{8}) + g'(\frac{1}{8})(q - \frac{1}{8})$. Since $\ell(\frac{1}{8}) - f(\frac{1}{8}) > 0.0005$ and $\ell(0.35) - f(0.35) > 0.0003$, thanks to the convexity of f, we conclude that indeed $g(q) > f(q)$ for $\frac{1}{8} \leqslant q \leqslant 0.35$. For the remaining range $0.35 \leqslant q \leqslant 1$, we crudely have, using the monotonicity of f and Lemma 18,

$$f(q) \leqslant f(0.35) < -0.124 < \log(0.885) < \log\Gamma(2-q) = g(q).$$

(B): $h_q(1) \geqslant 0$ is equivalent to $\Gamma(2-q) \geqslant 1-\frac{q(1-q)}{2}-\frac{q^2(1-q^2)}{12}$. Taking the logarithms and using $\log(1-x) \leqslant -x$, $x<1$, it suffices to show that

$$f(q) = \log\Gamma(2-q) + \frac{q(1-q)}{2} + \frac{q^2(1-q^2)}{12}$$

is nonnegative. This in fact holds for all $0 \leqslant q \leqslant 1$. Indeed, $f(0) = f(1) = 0$ and for $0 \leqslant q \leqslant 1$,

$$f''(q) = \sum_{k=0}^{\infty} \frac{1}{(2-q+k)^2} - q^2 - \frac{5}{6} \leqslant \sum_{k=0}^{\infty} \frac{1}{(1+k)^2} - 1 - \frac{5}{6} = \frac{\pi^2}{6} - \frac{11}{6} < -0.1,$$

so the concavity of f finishes the argument. □

We emphasise that in part (B) of this proof, we have shown that when $t = 1$, the inequality in Lemma 19 holds for all $0 \leqslant q \leqslant 1$. This combined with Corollary 16 leads to the following result, important in the sequel in the proof of integral inequality (16).

Corollary 20. *Let ξ be a random vector uniform on the unit Euclidean sphere S^3 in $\mathbb{R}^4$. Let $0 < p \leqslant 2$. Then*

$$\mathbb{E}|e_1 + \xi|^{-p} \leqslant \Gamma\left(2 - \frac{p}{2}\right),$$

equivalently

$$\int_0^\infty |\mathrm{j}_1(t)|^2 t^{p-1} \mathrm{d}t \leqslant 2^{p-1}\Gamma(p/2). \tag{19}$$

Proof. To explain the equivalent form involving j_1, note that, $\mathbb{E}|e_1+\xi|^{-p} = \mathbb{E}|\xi + \xi'|^{-p}$, for an independent copy ξ' of ξ, thanks to rotational invariance. It remains to use (10) which gives $\mathbb{E}|\xi_1+\xi_2|^{-p} = \kappa_{p,4}\int_0^\infty |\mathrm{j}_1(t)|^2 t^{p-1}\mathrm{d}t$ and plug in the value of $\kappa_{p,4}$. □

3.3. **The integral inequality:** $0 < p \leqslant 2$.

We record for future use the following bounds

$$|\mathrm{j}_1(t)| \leqslant \exp\left(-\frac{t^2}{8} - \frac{t^4}{3\cdot 2^7}\right), \qquad 0 \leqslant t \leqslant 4, \tag{20}$$

$$|\mathrm{j}_1(t)| \leqslant (8/\pi)^{1/2} t^{-1}(t^2-1)^{-1/4}, \qquad t \geqslant 1, \tag{21}$$

where the first one appears as Lemma 3.1 in [43] (see also [8, Lemma 3.6] for the proof of a more general statement) and the second one can be found in Watson's treatise (see [48, p.447] as well as [14, Lemma 4.4]), which in particular gives

$$|\mathrm{j}_1(t)| \leqslant (8/\pi)^{1/2}\left(\frac{t_0^2}{t_0^2-1}\right)^{1/4} t^{-3/2}, \qquad t \geqslant t_0 \geqslant 1. \tag{22}$$

We define

$$H(p,s) = \int_0^\infty \left(e^{-st^2/8} - |\mathrm{j}_1(t)|^s\right) t^{p-1}\mathrm{d}t, \qquad 0 < p < 2,\ s > 1 \tag{23}$$

and immediately observe that after a change of variables one integral can be expressed in terms of the gamma function,

$$G(p,s) = \int_0^\infty e^{-st^2/8} t^{p-1} \mathrm{d}t = s^{-p/2} 2^{3p/2-1} \Gamma(p/2). \tag{24}$$

Recall (15), $F(p,s) = \int_0^\infty |\mathrm{j}_1(t)|^s t^{p-1}\mathrm{d}t$, so

$$H(p,s) = G(p,s) - F(p,s). \tag{25}$$

Then the crucial integral inequality (16) is equivalent to $H(p,s) \geqslant 0$.

Our main goal and result here is that the integral inequality $H(p,s) > 0$ holds in rather wide ranges of parameters (p,s) (however, it does not all for $0 < p < 2$ and $s > 1$ which, as already noted, would have been enough to deduce Theorem 8).

Lemma 21. *The inequality $H(p,s) > 0$ holds in the following cases*

(a) $0 < p \leqslant 2$ *and* $s \geqslant 2$*,*

(b) $0 < p \leqslant \frac{1}{4}$ *and* $s \geqslant 1.3$*.*

For the proof, we will need several rather intricate estimates on various integrals. The general idea we employ here follows [43] and is to first use the explicit bounds on j_1 from (20) and (22) to get $H > 0$ in certain but *not all* cases and then extend them by interpolating in s (exploiting the simple dependence of G on s). This is in contrast to several works, e.g. [8, 9, 10, 14, 26, 37] which heavily rely on the approach developed by Nazarov and Podkorytov in [39] to integral inequalities with oscillatory integrands. We also refer to recent papers [2] as well as [36] for connections between such integral inequalities and majorisation.

We begin by setting

$$\begin{aligned} U(p,s) = {} & \frac{4^p (2\pi \cdot 15^{1/2})^{-s/2}}{3s/2 - p} \\ & + 2^{3p/2-1} s^{-p/2} \left(\Gamma\left(p/2\right) - \frac{\Gamma(p/2+2)}{6s} + \frac{\Gamma(p/2+4)}{72 s^2} \right) \end{aligned} \tag{26}$$

which emerges in the next lemma (following Lemma 3.2 from [43]).

Lemma 22. *For $p < 3s/2$, we have*

$$F(p,s) < U(p,s).$$

Proof. Using (20) and (22) with $t_0 = 4$, we get

$$\begin{aligned} F(p,s) = \int_0^\infty |\mathrm{j}_1(t)|^s t^{p-1} \mathrm{d}t < \int_0^\infty \exp\left(-s\frac{t^2}{8} - s\frac{t^4}{3 \cdot 2^7} \right) t^{p-1} \mathrm{d}t \\ + \left(\frac{8}{15^{1/4} (2\pi)^{1/2}} \right)^s \frac{4^{p-3s/2}}{3s/2 - p}, \end{aligned}$$

valid for $p < \frac{3s}{2}$. After the change of variables $u = st^2/8$, the first integral becomes

$$2^{3p/2-1}s^{-p/2}\int_0^\infty e^{-\frac{u^2}{6s}}e^{-u}u^{p/2-1}\mathrm{d}u.$$

We estimate the first exponential using $e^{-x} \leqslant 1 - x + \frac{x^2}{2}$, $x \geqslant 0$, which gives the bound

$$\int_0^\infty \left(1 - \frac{u^2}{6s} + \frac{u^4}{72s^2}\right)e^{-u}u^{p/2-1}\mathrm{d}u = \Gamma\left(p/2\right) - \frac{\Gamma(p/2+2)}{6s} + \frac{\Gamma(p/2+4)}{72s^2}.$$

$\square$

Lemma 23. *The inequality*

$$U(p,s) < G(p,s)$$

holds in the following cases

(i) $0 < p \leqslant \frac{1}{4}$ *and* $s \geqslant \frac{17}{10}$*,*

(ii) $0 < p \leqslant \frac{4}{5}$ *and* $s \geqslant 2$*,*

(iii) $0 \leqslant p \leqslant 2$ *and* $s \geqslant \frac{8}{3}$*.*

Proof. Note that $U < G$ is equivalent to the following inequality (after canceling $\Gamma(p/2)$ on both sides, factoring out $\Gamma(p/2+2)$ and moving terms across using that $3s/2 - p > 0$),

$$(2\pi\cdot 15^{1/2})^{s/2}2^{-p/2}\left(\frac{3s}{2} - p\right)\frac{12s - \left(\frac{p}{2}+2\right)\left(\frac{p}{2}+3\right)}{144} > \frac{s^{p/2+2}}{\Gamma(p/2+2)}. \tag{27}$$

To shorten the notation, let $a = (2\pi)^{1/2}\cdot 15^{1/4}$ and

$$A(p,s) = 2^{-p/2}\left(\frac{3s}{2} - p\right)\frac{12s - \left(\frac{p}{2}+2\right)\left(\frac{p}{2}+3\right)}{144}$$

which is decreasing in p and increasing in s. In each of the cases we will simply replace A with its smallest possible value given the range of p and s, so we let $p_1 = \frac{1}{4}$, $s_1 = \frac{17}{10}$, $p_2 = \frac{4}{5}$, $s_2 = 2$ and $p_3 = 2$, $s_3 = \frac{8}{3}$ and have $A(p,s) \geqslant A_k$, where $A_k = A(p_k, s_k)$ for $k = 1,2,3$ in cases (i), (ii), (iii), respectively. Then it suffices to prove that

$$A_k a^s > \frac{s^{p/2+2}}{\Gamma(p/2)}.$$

We take the logarithm and consider

$$f(p,s) = s\log a + \log A_k - \left(\frac{p}{2}+2\right)\log s + \log\Gamma\left(\frac{p}{2}+2\right).$$

Our goal is to show that $f(p,s) > 0$. We observe that

$$\frac{\partial}{\partial p}f(p,s) = -\frac{1}{2}\log s + \psi\left(\frac{p}{2}+2\right) \leqslant -\frac{1}{2}\log s_k + \psi\left(\frac{p_k}{2}+2\right)$$

in each case respectively and the resulting numerical values on the right bounded above by -0.03, -0.04 and -0.05, $k = 1,2,3$. Similarly,

$$\frac{\partial}{\partial s}f(p,s) = \log a - \frac{p/2+2}{s} \geqslant \log a - \frac{p_k/2+2}{s_k}$$

with the right hand side bounded this time below by 0.34, 0.39 and 0.47, $k = 1, 2, 3$. Thus $f(p, s)$ is decreasing in p and increasing in s, so

$$f(p, s) \geqslant f(p_k, s_k)$$

and after plugging in the explicit numerical values, the right hand side is bounded below by 0.041, 0.049 and 0.032, $k = 1, 2, 3$, thus proving (i), (ii) and (iii). $\square$

The next two lemmas are vital for the interpolation argument.

Lemma 24. *For $\frac{4}{5} \leqslant p \leqslant 2$, we have*

$$F(p, 8/3) < e^{-p/6} G(p, 2).$$

Proof. Using (22) with $t_0 = 5$, we get

$$\int_5^\infty |\mathrm{j}_1(t)|^{8/3} t^{p-1} \mathrm{d}t \leqslant (8/\pi)^{4/3} (25/24)^{2/3} \frac{5^{p-4}}{4-p} \tag{28}$$

which for $p \leqslant 2$ gives

$$\int_5^\infty |\mathrm{j}_1(t)|^{8/3} t^{p-1} \mathrm{d}t \leqslant (8/\pi)^{4/3} (25/24)^{2/3} \frac{5^{p-4}}{2} = \frac{2}{3^{2/3} \cdot 5^{8/3} \pi^{4/3}} 5^p.$$

We divide the interval $[0, 5]$ into consecutive subintervals of the form $[\frac{k}{m}, \frac{k+1}{m}]$, for $k = 0, 1, \ldots, 5m-1$ with $m = 100$ and crudely bound

$$\begin{aligned} &\int_0^5 |\mathrm{j}_1(t)|^{8/3} t^{p-1} \mathrm{d}t \\ &\quad < \int_0^{1/m} t^{p-1} \mathrm{d}t + \frac{1}{m} \sum_{k=1}^{5m-1} \max\left\{ \left|\mathrm{j}_1\left(\frac{k}{m}\right)\right|^{8/3}, \left|\mathrm{j}_1\left(\frac{k+1}{m}\right)\right|^{8/3} \right\} \\ &\qquad\qquad \cdot \max\left\{ \left(\frac{k}{m}\right)^{p-1}, \left(\frac{k+1}{m}\right)^{p-1} \right\} \end{aligned} \tag{29}$$

(we have used that $|\mathrm{j}_1| < 1$ and that j_1 is monotone on $[0, 5]$; the latter is justified e.g. in [43], p. 290, in the proof of Proposition 1.1). Now, $\int_0^{1/m} t^{p-1} \mathrm{d}t = \frac{1}{pm^p} < \frac{1}{0.8m^p}$. A resulting bound on $e^{p/6} 2^{1-p} \int_0^\infty |\mathrm{j}_1(t)|^{8/3} t^{p-1} \mathrm{d}t$ is of the form

$$h(p) = \sum_k \lambda_k a_k^p$$

with explicit positive numbers λ_k, a_k. We check that $L(p) = \log h(p) < \log \Gamma(p/2) = R(p)$ for $0.8 \leqslant p \leqslant 2$ relying on the fact that both sides are clearly convex (recall that summation preserves log-convexity). Specifically, we divide the interval $[0.8, 2]$ into 12 consecutive subintervals $[u_i, u_{i+1}]$, $u_i = 0.8 + 0.1i$, $i = 0, 1, \ldots, 11$ and on each interval we lower-bound $R(p)$ by its tangent put at the middle $v_i = \frac{u_i + u_{i+1}}{2}$, $\ell_i(p) = R'(v_i)(p - v_i) + R(v_i)$ and then check that $\ell_i(p) > L(p)$ by checking the values at the end-points $p = u_i, u_{i+1}$, which are gathered in Table 2. $\square$

Lemma 25. *For $0 < p \leqslant \frac{1}{4}$, we have*

$$F(p, 1.3) < e^{2p/17} G(p, 1.7).$$

Proof. Fix $0 < p \leqslant \frac{1}{4}$. We break the integral on the left hand side into the sum of 4 integrals $A_1 + \cdots + A_4$ over $(0,1)$, $(1,5)$, $(5,10)$ and $(10,\infty)$. For the first one, we use (21),

$$|\mathrm{j}_1(t)|^{1.3} < \exp\left\{-\frac{13}{10}\left(\frac{t^2}{8} + \frac{t^4}{3\cdot 2^7}\right)\right\} < 1 - \frac{13}{80}t^2 + \frac{377}{38400}t^4, \qquad 0 < t < 1$$

(the last inequality obtained by taking the first terms in the power series expansion of the penultimate expression, which gives an upper bound as can be checked directly by differentiation). Integrating against t^{p-1} yields

$$A_1 \leqslant \frac{1}{p} - \frac{13}{80(p+2)} + \frac{377}{38400(p+4)} < \frac{1}{p} - \frac{13}{80(p+2)} + \frac{377}{38400\cdot 4}.$$

For the last one, we use (22) with $t_0 = 10$,

$$\begin{aligned} A_4 \leqslant \int_{10}^{\infty} \left((8/\pi)^{1/2}(100/99)^{1/4}t^{-3/2}\right)^{1.3} t^{p-1}\mathrm{d}t &= \frac{2^{53/20}}{11^{13/40}\cdot 5^{3/10}(3\pi)^{13/20}}\frac{10^p}{39-20p} \\ &\leqslant \frac{2^{53/20}}{11^{13/40}\cdot 5^{3/10}(3\pi)^{13/20}}\frac{10^p}{34}. \end{aligned}$$

For A_2 and A_3, we use Riemann sums. First, without any error term thanks to the monotonicity of j_1 on $(1,5)$,

$$\begin{aligned} A_2 &\leqslant \sum_{k=0}^{4m-1} \max\left\{\left|\mathrm{j}_1\left(1+\frac{k}{m}\right)\right|^{1.3}, \left|\mathrm{j}_1\left(1+\frac{k+1}{m}\right)\right|^{1.3}\right\} \int_{1+\frac{k}{m}}^{1+\frac{k+1}{m}} t^{p-1}\mathrm{d}t \\ &< \sum_{k=0}^{4m-1} \max\left\{\left|\mathrm{j}_1\left(1+\frac{k}{m}\right)\right|^{1.3}, \left|\mathrm{j}_1\left(1+\frac{k+1}{m}\right)\right|^{1.3}\right\} \frac{(1+k/m)^{p-1}}{m}. \end{aligned}$$

Second, on $(5,10)$, we choose the midpoints and bound the error simply using the supremum of the derivative via the crude (numerical) bound

$$\sup_{t\in[5,10]}\left|\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{j}_1(t)|^{1.3}\right| < 0.06$$

(since $\left|\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{j}_1(t)|^{1.3}\right| = 1.3|\mathrm{j}_1(t)|^{0.3}|\mathrm{j}_1'(t)|$ and $j_1'(t) = -2\frac{J_2(t)}{t} = 2\frac{J_0(t)}{t} - 4\frac{J_1(t)}{t^2}$, the function under the supremum can be expressed in terms of J_0 and J_1 and the supremum can be estimated by employing the precise polynomial-type approximations to J_0 and J_1 from [1], 9.4.3 and 9.4.6, pp.369–370). This leads to

$$\begin{aligned} A_3 &\leqslant \sum_{k=0}^{5m-1}\left|\mathrm{j}_1\left(5+\frac{k+1/2}{m}\right)\right|^{1.3}\int_{5+\frac{k}{m}}^{5+\frac{k+1}{m}} t^{p-1}\mathrm{d}t + 0.06\frac{1}{2m}\int_5^{10} t^{p-1}\mathrm{d}t \\ &< \sum_{k=0}^{5m-1}\left|\mathrm{j}_1\left(5+\frac{k+1/2}{m}\right)\right|^{1.3}\frac{(5+k/m)^{p-1}}{m} + \frac{3\cdot 5^p}{100m}. \end{aligned}$$

Table 2. Proof of Lemma 24: lower bounds on the differences at the end-points of the linear approximations ℓ_i to $R(p)$.

i	0	1	2	3	4	5	6	7	8	9	10	11
$10^3\cdot(\ell_i(u_i) - L(u_i))$	1	5	8	9	10	12	13	14	14	15	15	15
$10^3\cdot(\ell_i(u_{i+1}) - L(u_{i+1}))$	4	8	9	10	11	13	14	14	15	15	15	14

With hindsight, we choose $m = 200$. Adding these 4 estimates together (call the right-most hand sides of these bounds $B_1, \dots, B_4$) and multiplying through p, it suffices to show that $L(p) < R(p)$ for $0 < p \leqslant \frac{1}{4}$, where

$$L(p) = p \cdot (B_1 + \cdots + B_4), \qquad R(p) = (e^{2/17} 2^{3/2} 1.7^{-1/2})^p \Gamma\left(\frac{p}{2} + 1\right).$$

Plainly, $R(p)$ is convex (as being log-convex), whilst

$$L(p) = \frac{67}{80} + \frac{13}{40(p+2)} + \frac{377}{153600} p + c_1 \cdot p 10^p + c_2 \cdot p 5^p + \sum_i \lambda_i p a_i^p$$

with positive constant c_1, c_2, λ_i (specified above) and $a_i \geqslant 1$ (of the form $(1+k/m)$, $k \geqslant 0$). Thus, $L(p)$ is also convex and now we proceed similarly to what we did in the proof of Lemma 24. Note that $L(0) = R(0) = 1$. For $0 < p \leqslant 0.02$, we lower-bound, $R(p) \geqslant \ell_0(p) = 1 + R'(0)p$ and check that $\ell_0(0.02) - L(0.02) > 10^{-5} > 0$, to conclude $R(p) \geqslant L(p)$, $0 \leqslant p \leqslant 0.02$. We divide the remaining interval $(0.02, 0.25)$ into 6 intervals: $(0.02, 0.05)$, $(0.05, 0.1)$, $(0.1, 0.15)$, $(0.15, 0.2)$, $(0.2, 0.23)$, $(0.23, 0.25)$, denoted say (u_i, u_{i+1}), $i = 1, \dots, 6$, choose their midpoints $v_i = \frac{u_i + u_{i+1}}{2}$ and lower-bound $R(p)$ by its tangent $\ell_i(p) = R'(v_i)(p - v_i) + R(v_i)$ and check that $\ell_i(p) > L(p)$ at $p = u_i, u_{i+1}$ (see Table 3) to conclude that $R(p) > L(p)$ for all $u_i \leqslant p \leqslant u_{i+1}$ $i = 1, \dots, 6$, by convexity. $\square$

TABLE 3. Proof of Lemma 25: lower bounds on the differences at the end-points of the linear approximations ℓ_i to $R(p)$.

i	1	2	3	4	5	6
$10^4 \cdot (\ell_i(u_i) - L(u_i))$	.7	1	3	4	5	3
$10^4 \cdot (\ell_i(u_{i+1}) - L(u_{i+1}))$	2	3	4	3	3	2

We are ready to prove the main inequalities of this section.

Proof of Lemma 21. First we show (a). Lemma 22 combined with Lemma 23 (ii), (iii) gives (a) for all $0 < p \leqslant \frac{4}{5}$, $s \geqslant 2$, as well as all $0 < p \leqslant 2$ and $s \geqslant \frac{8}{3}$, respectively. It remains to handle the case $\frac{4}{5} < p < 2$, $2 \leqslant s \leqslant \frac{8}{3}$. We apply Hölder's inequality, Lemma 24 and (19) to get,

$$\begin{aligned} F(p,s) &\leqslant F(p,2)^{\frac{8-3s}{2}} F(p,8/3)^{\frac{3s-6}{2}} \\ &\leqslant \Big(G(p,2)\Big)^{\frac{8-3s}{2}} \Big(e^{-p/6} G(p,2)\Big)^{\frac{3s-6}{2}} \\ &= e^{-p\frac{s-2}{4}} 2^{p-1} \Gamma(p/2). \end{aligned}$$

By concavity, $\log s \leqslant \frac{s-2}{2} + \log 2$, $s \geqslant 2$, thus $e^{-p\frac{s-2}{4}} \leqslant s^{-p/2} 2^{p/2}$, $s \geqslant 2$, $p > 0$, which gives (a).

To show (b), we proceed similarly. Lemma 22 combined with Lemma 23 (i) gives (b) for all $0 < p \leqslant \frac{1}{4}$ and $s \geqslant 1.7$. In the remaining case $1.3 \leqslant s \leqslant 1.7$, from

Hölder's inequality, Lemma 23 (i) and Lemma 25, we obtain

$$\begin{aligned}
F(p,s) &\leqslant F(p,1.7)^{\frac{10s-13}{4}} F(p,1.3)^{\frac{17-10s}{4}} \\
&\leqslant \Big(G(p,1.7)\Big)^{\frac{10s-13}{4}} \Big(e^{2p/17}G(p,1.7)\Big)^{\frac{17-10s}{4}} \\
&= e^{\frac{p}{2}\frac{17-10s}{17}} 1.7^{-p/2} 2^{3p/2-1}\Gamma(p/2).
\end{aligned}$$

Thanks to concavity, $\log s \leqslant \frac{10}{17}s - 1 + \log 1.7$, $s \leqslant 1.7$, which gives $e^{\frac{p}{2}\frac{17-10s}{17}} 1.7^{-p/2} \leqslant s^{-p/2}$, whence (b). □

3.4. **The integral inequality:** $2 < p < 3$**.** We follow the general approach from the previous case $p < 2$. Recall (15), $F(p,s) = \int_0^\infty |\mathrm{j}_1(t)|^s t^{p-1} \mathrm{d}t$ and that the crucial integral inequality (17) reads $s^{p/2}F(p,s) \leqslant 2^{p/2}F(p,2)$. Thus here we let

$$\tilde{H}(p,s) = s^{-p/2}2^{p/2}F(p,2) - F(p,s), \qquad 2 < p < 3,\ s > 1. \tag{30}$$

Note that we can express $F(p,2)$ explicitly: using Corollary 5 and (11), we obtain

$$\begin{aligned}
F(p,2) = \int_0^\infty \mathrm{j}_1(t)^2 t^{p-1}\mathrm{d}t = \kappa_{p,4}^{-1}\mathbb{E}|\xi_1+\xi_2|^{-p} &= \kappa_{p,4}^{-1}2^{-p/2}C_2(p) \\
&= 2^{p-1}\frac{\Gamma\left(\frac{p}{2}\right)\Gamma(3-p)}{\left[\Gamma\left(2-\frac{p}{2}\right)\right]^2\Gamma\left(3-\frac{p}{2}\right)}.
\end{aligned}$$

In view of (30), we therefore set

$$\tilde{G}(p,s) = s^{-p/2}2^{3p/2-1}\Gamma(p/2)D(p) \tag{31}$$

with

$$D(p) = \frac{\Gamma(3-p)}{\left[\Gamma\left(2-\frac{p}{2}\right)\right]^2\Gamma\left(3-\frac{p}{2}\right)}, \tag{32}$$

so that

$$\tilde{H}(p,s) = \tilde{G}(p,s) - F(p,s).$$

The main result of this section is that integral inequality (17) also holds for all $s \geqslant 2$. We emphasise that $\tilde{H}(p,2) = 0$.

Lemma 26. *The inequality $\tilde{H}(p,s) > 0$ holds for all $2 < p < 3$ and $s \geqslant 2$.*

This will be established in a very much similar way to the previous section: crude pointwise bounds on j_1 will suffice to handle the case $s \geqslant \frac{8}{3}$ which will then be extended to $s \geqslant 2$ by interpolation.

Lemma 27. *With $D(p)$ defined in (32), the function $p \mapsto \log D(p)$ is increasing, convex and positive on $(2,3)$.*

Proof. Let $x = \frac{3-p}{2}$, $0 < x < \frac{1}{2}$. By the Legendre duplication formula (see, e.g. 6.1.18 in [1]),

$$D(p) = \frac{\Gamma(2x)}{\Gamma(x+\frac{1}{2})^2\Gamma(x+\frac{3}{2})} = \frac{2^{2x-1}\Gamma(x)}{\sqrt{\pi}\Gamma(x+\frac{1}{2})\Gamma(x+\frac{3}{2})}.$$

Thus the convexity of $\log D(p)$ on $(2,3)$ is equivalent to the convexity of

$$f(x) = \log\Gamma(x) - \log\Gamma\left(x+\frac{1}{2}\right) - \log\Gamma\left(x+\frac{3}{2}\right)$$

on $(0,\frac{1}{2})$. Using the series representation of $(\log\Gamma(z))'' = \sum_{n=0}^{\infty}(z+n)^{-2}$ (see, e.g. 6.4.10 in [1]), we get

$$\begin{aligned} f''(x) &= \sum_{n=0}^{\infty}\frac{1}{(x+n)^2} - \sum_{n=0}^{\infty}\frac{1}{(x+n+\frac{1}{2})^2} - \sum_{n=0}^{\infty}\frac{1}{(x+n+3/2)^2} \\ &= \frac{1}{x^2} - \frac{1}{(x+\frac{1}{2})^2} + \sum_{n=1}^{\infty}\frac{1}{(x+n)^2} - 2\sum_{n=1}^{\infty}\frac{1}{(x+n+\frac{1}{2})^2}. \end{aligned}$$

For $0 < x < \frac{1}{2}$,

$$\sum_{n=1}^{\infty}\frac{1}{(x+n)^2} - 2\sum_{n=1}^{\infty}\frac{1}{(x+n+\frac{1}{2})^2} > \sum_{n=1}^{\infty}\frac{1}{(\frac{1}{2}+n)^2} - 2\sum_{n=1}^{\infty}\frac{1}{(n+\frac{1}{2})^2} = -\frac{\pi^2}{2}+4,$$

thus

$$f''(x) > \frac{1}{x^2} - \frac{1}{(x+\frac{1}{2})^2} - \frac{\pi^2}{2} + 4.$$

The right hand side is clearly decreasing (e.g., by looking at the derivative), so for $0 < x < \frac{1}{2}$, it is at least $4 - 1 - \frac{\pi^2}{2} + 4 = 7 - \frac{\pi^2}{2}$ which is positive.

Moreover, $\frac{\mathrm{d}}{\mathrm{d}p}\log D(p)|_{p=2} = \frac{1-\gamma}{2} > 0$ ($\gamma = 0.57..$ is Euler's constant), so $D(p)$ is strictly increasing on $(2,3)$ with $D(2) = 1$. $\square$

Lemma 28. *For all $2 < p < 3$ and $s \geqslant \frac{8}{3}$, we have*

$$U(p,s) < \tilde{G}(p,s).$$

Proof. We let $a = (2\pi)^{1/2}\cdot 15^{1/4}$ and inserting the definitions of U from (26) and $\tilde{G}$ from (31), the desired inequality becomes

$$\frac{4^p a^{-s}}{3s/2-p} + s^{-p/2}2^{3p/2-1}\left(\Gamma\left(p/2\right) - \frac{\Gamma(p/2+2)}{6s} + \frac{\Gamma(p/2+4)}{72s^2}\right) < s^{-p/2}2^{3p/2-1}\Gamma\left(\frac{p}{2}\right)D(p),$$

equivalently,

$$\frac{2^{p/2+1}a^{-s}}{3s/2-p}s^{p/2+2} < s^2\Gamma\left(\frac{p}{2}\right)(D(p)-1) + \Gamma\left(\frac{p}{2}+2\right)\frac{12s-(p/2+2)(p/2+3)}{72}.$$

The right hand side is clearly increasing with s ($D(p) > 1$ by Lemma 27), whereas the left hand side is decreasing with s (for every fixed $2 < p < 3$), as can be checked by examining the derivative of $\log(a^{-s}s^{p/2+2})$. Therefore, it suffices to prove this inequality for $s = \frac{8}{3}$. Moreover, after replacing $\Gamma(\frac{p}{2})$ on the right hand side with 0.88 (see Lemma 18) and $\Gamma(\frac{p}{2}+2)$ with $\Gamma(3) = 2$, it suffices to prove that the function

$$f(p) = 0.88(8/3)^2(D(p)-1) + \frac{32-(p/2+2)(p/2+3)}{36} - b\frac{(16/3)^{p/2}}{4-p},$$

where $b = 2a^{-8/3}(8/3)^2$, is positive for $2 < p < 3$. We put

$$L(p) = b\frac{(16/3)^{p/2}}{4-p} + \frac{1}{36}(p/2+2)(p/2+3)$$

and

$$R(p) = 0.88(8/3)^2(D(p)-1) + \frac{8}{9}$$

which are both convex ($D(p)$ is even log-convex, Lemma 27). For $2 < p < \frac{5}{2}$, we use the tangent $\ell_1(p) = R(2) + R'(2)(p-2)$ as a lower bound, $R(p) > \ell_1(p)$ and check that at $p = 2$, $p = \frac{5}{2}$ the linear function ℓ_1 dominates L (the difference is 0.017.. and 0.076.., respectively), which then gives $R > \ell_1 > L$ on $(2, \frac{5}{2})$. Similarly, for $\frac{5}{2} < p < 3$, $R(p) > \ell_2(p) = R(5/2) + R'(5/2)(p-5/2)$, and $\ell_2 - L$ at $p = \frac{5}{2}$ and $p = 3$ is 1.19.. and 3.77.., respectively. This finishes the proof. $\square$

Lemma 29. *For all* $2 < p < 3$*, we have*

$$F(p, 8/3) < e^{-p/6}\tilde{G}(p, 2).$$

Proof. Consider

$$L(p) = \log F(p, 8/3), \qquad R(p) = \log\left(e^{-p/6}\tilde{G}(p,2)\right)$$

which are both convex (recall Lemma 27). Using that, we crudely bound $R(p)$ from below by tangents: $r_1(p) = R(2) + R'(2)(p-2)$ on $(2, 2.5)$ and $r_2(p) = R(2.5) + R'(2.5)(p-2.5)$ on $(2.5, 3)$ and then compare their values at the end points with upper bounds on L to conclude that $r_1 > L$ on $(2, 2.5)$ and $r_2 > L$ on $(2.5, 3)$. Estimates (28) and (29) added together (applied with $m = 100$ as in Lemma 24) yield

$$L(2) < 0.35, \quad L(2.5) < 0.56, \quad L(3) < 0.96,$$

whereas we check directly that

$$r_1(2) > 0.359, \quad r_1(2.5) > 0.58, \quad r_2(3) > 1.48.$$

Comparing these values finish the argument. $\square$

Proof of Lemma 26. The argument relies on Lemmas 22, 28 and 29, following almost verbatim the proof of Lemma 21 and thus we omit the details. $\square$

3.5. **Miscellaneous facts.** Our first result here is a straightforward extension of Lemma 8 from [29] to negative moments (see also Lemma 3 in [10]).

Lemma 30. *Let* $0 < p < 1$*. Let* $n, d \geqslant 1$ *and let* $X_1, \ldots, X_n$ *be independent rotationally invariant random vectors in* $\mathbb{R}^d$*. Then*

$$\mathbb{E}\left|\sum_{k=1}^n X_k\right|^{-p} = \beta_{p,d}\mathbb{E}\left|\sum_{k=1}^n \langle v_k, X_k\rangle\right|^{-p}$$

for arbitrary vectors $v_1, \ldots, v_n$ *in* $\mathbb{R}^d$*, where*

$$\beta_{p,d} = \frac{\sqrt{\pi}\Gamma\left(\frac{d-p}{2}\right)}{\Gamma\left(\frac{1-p}{2}\right)\Gamma\left(\frac{d}{2}\right)}.$$

Proof. Thanks to homogeneity, we can assume that the v_k are *unit*. Thanks to rotational invariance and independence, we can assume without loss of generality that $v_1 = \cdots = v_n = e_1$, but then it suffices to consider the case $n = 1$ (because sums of independent rotationally invariant random vectors are rotationally invariant). The latter can be easily justified in a number of ways.

For instance, it follows from a Fourier-analytic argument: we invoke (10), rewrite $\mathbb{E}\, \mathrm{j}_{d/2-1}(t|X_k|)$ as $\mathbb{E}e^{it\langle v_k, X_k\rangle}$ and apply (8) with $d = 1$ to $\sum \langle v_k, X_k\rangle$ which gives $\beta_{p,d} = \kappa_{p,d}/(2K_{p,1})$.

Alternatively, we can apply a standard embedding-type argument: if we take a random vector ξ uniform on the unit Euclidean sphere S^{d-1}, independent of the X_k, we have for every vector x in $\mathbb{R}^d$

$$\mathbb{E}|\langle x, \xi\rangle|^{-p} = \beta_{p,d}^{-1}|x|^{-p}$$

with

$$\beta_{p,d}^{-1} = \mathbb{E}|\langle e_1, \xi\rangle|^{-p} = \frac{\int_{-1}^{1} |t|^{-p}(1-t^2)^{\frac{d-3}{2}}\mathrm{d}t}{\int_{-1}^{1}(1-t^2)^{\frac{d-3}{2}}\mathrm{d}t} = \frac{\Gamma\left(\frac{1-p}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-p}{2}\right)}.$$

Applying this to $x = X_1$, taking the expectation over X_1 and noting that $\langle X_1, \xi\rangle$ has the same distribution as $\langle X_1, e_1\rangle$ finishes the argument. $\square$

Lemma 31. *For every $0 < q < 2$, we have*

$$\left(\frac{13}{20}\right)^q < \Gamma(2-q).$$

Proof. The function $f(q) = \log\Gamma(2-q) - q\log\frac{13}{20}$ is convex on $(0,2)$ with $f'(0) = \gamma - 1 - \log\frac{13}{20} > 0.007$. Thus f is strictly increasing and the lemma follows since $f(0) = 0$. $\square$

4. End of the proof of Theorem 8

To finish the proof of Theorem 8, we only need to justify Steps 1-4 from Section 2.3.1.

4.1. Step 1 and 3: Integral inequality.

Lemma 21 (a) and (b) gives Step 1 and 3, respectively.

4.2. Step 2: Induction.

First note that, by homogeneity, (13) with $C(p) = C_\infty(p)$ is equivalent to

$$\mathbb{E}\left|\xi_1 + \sum_{k=2}^{n} a_k\xi_k\right|^{-p} \leqslant C_\infty(p)\left(1 + \sum_{k=2}^{n} a_k^2\right)^{-p/2}.$$

For $p > 0$ and $x \geqslant 0$ we define

$$\phi_p(x) = (1+x)^{-p/2}$$

and

$$\Phi_p(x) = \begin{cases} \phi_p(x), & x \geqslant 1, \\ 2\phi_p(1) - \phi_p(2-x), & 0 \leqslant x \leqslant 1. \end{cases}$$

Crucially, $\Phi_p(x) \leqslant \phi_p(x)$ for all $x \geqslant 0$, as, geometrically, on $[0,1]$, the graph of $\Phi_p(x)$ is obtained from the graph of $\phi_p(x)$ on $[1,2]$ by reflecting it about $(1, \phi_p(1))$. By induction on n, we will show a strengthened version of the above with ϕ_p on the right hand side replaced by Φ_p.

Theorem 32. *Let $\frac{1}{4} \leqslant p \leqslant 2$. Let $\xi_1, \xi_2, \dots$ be independent random vectors uniform on the unit Euclidean sphere S^3 in $\mathbb{R}^4$. For every $n \geqslant 2$ and nonnegative numbers $a_2, \dots, a_n$, we have*

$$\mathbb{E}\left|\xi_1 + \sum_{k=2}^n a_k \xi_k\right|^{-p} \leqslant C_\infty(p) \Phi_p\left(\sum_{k=2}^n a_k^2\right). \tag{33}$$

Proof. For the inductive base, when $n = 2$, (33) becomes

$$\mathbb{E}|\xi_1 + \sqrt{t}\xi_2|^{-p} \leqslant 2^{p/2}\Gamma\left(2 - \frac{p}{2}\right)\Phi_p(t), \qquad t \geqslant 0,$$

where we have put $t = a_2^2$. By homogeneity and the fact that $\Phi_p \leqslant \phi_p$, the case $t \geqslant 1$ reduces to the case $0 \leqslant t \leqslant 1$ which in turn follows by combining Corollary 16 and Lemma 19 (applied to $q = p/2$, noting as usual that by rotational invariance, $\mathbb{E}|e_1 + \sqrt{t}\xi_2|^{-p} = \mathbb{E}|\xi_1 + \sqrt{t}\xi_2|^{-p}$).

For the inductive step, let $n \geqslant 2$ and suppose (33) holds for all $n-1$ nonnegative numbers $a_2, \dots, a_n$. To prove it for n nonnegative arbitrary numbers, say $a_2, \dots, a_n, a_{n+1}$, we let

$$x = a_2^2 + \cdots + a_n^2 + a_{n+1}^2$$

and consider 3 cases.

Case 1: $a_k > 1$ for some $2 \leqslant k \leqslant n+1$. Then $x > 1$, so $\Phi_p(x) = \phi_p(x)$ and our goal is to show

$$\mathbb{E}\left|\sum_{k=1}^{n+1} a_k \xi_k\right|^{-p} \leqslant C_\infty(p)\left(\sum_{k=1}^{n+1} a_k^2\right)^{-p/2} \tag{34}$$

where we put $a_1 = 1$. Let $a_1^*, \dots, a_{n+1}^*$ be a nonincreasing rearrangement of the sequence $a_1, \dots, a_{n+1}$ and set $a_k' = \frac{a_k^*}{a_1^*}$, $k = 1, \dots, n+1$. Thanks to homogeneity, to prove (34), it is enough to prove

$$\mathbb{E}\left|\sum_{k=1}^{n+1} a_k' \xi_k\right|^{-p} \leqslant C_\infty(p)\Phi_p\left(\sum_{k=2}^{n+1} a_k'^2\right)$$

which is handled by either of the next two cases because here $a_1' = 1$ and $a_k' \leqslant 1$ for all $k \geqslant 2$.

Case 2.1: $a_k \leqslant 1$ for all $2 \leqslant k \leqslant n+1$ and $x \geqslant 1$. Since $x \geqslant 1$, our goal is again (34) with $a_1 = 1$. We have,

$$\max_{k \leqslant n+1} a_k = 1 \leqslant \frac{1}{\sqrt{2}}\sqrt{1+x} = \frac{1}{\sqrt{2}}\left(\sum_{k=1}^{n+1} a_k^2\right)^{1/2},$$

so Corollary 10 finishes the inductive argument in this case.

Case 2.2: $a_k \leqslant 1$ for all $2 \leqslant k \leqslant n+1$ and $x < 1$. Fix vectors $v_2, \dots, v_{n+1}$ in $\mathbb{R}^4$ with $|v_k| = a_k$, for each $k = 2, \dots, n+1$. Then, plainly,

$$\mathbb{E}\left|\xi_1 + \sum_{k=2}^{n+1} a_k \xi_k\right|^{-p} = \mathbb{E}\left||e_1|\xi_1 + \sum_{k=2}^{n+1} |v_k|\xi_k\right|^{-p}$$

and thanks to Lemma 30, when $0 < p < 1$, the right hand side can be written as

$$\mathbb{E}\left||e_1|\xi_1 + \sum_{k=2}^{n+1} |v_k|\xi_k\right|^{-p} = \beta_{p,4}\mathbb{E}\left|\langle e_1, \xi_1\rangle + \sum_{k=2}^{n+1} \langle v_k, \xi_k\rangle\right|^{-p}.$$

If we let Q be a random orthogonal matrix, independent of the ξ_k and note that (ξ_n, ξ_{n+1}) has the same distribution as $(\xi_n, Q\xi_n)$, we obtain

$$\mathbb{E}\left|\langle e_1, \xi_1\rangle + \sum_{k=2}^{n+1} \langle v_k, \xi_k\rangle\right|^{-p} = \mathbb{E}_Q\mathbb{E}_\xi\left|\langle e_1, \xi_1\rangle + \sum_{k=2}^{n-1} \langle v_k, \xi_k\rangle + \left\langle v_n + Q^\top v_{n+1}, \xi_n\right\rangle\right|^{-p}.$$

Going back to the vector sum again via Lemma 30, we arrive at the identity

$$\mathbb{E}\left|\xi_1 + \sum_{k=2}^{n+1} a_k\xi_k\right|^{-p} = \mathbb{E}_Q\mathbb{E}_\xi\left|\xi_1 + \sum_{k=2}^{n-1} |v_k|\xi_k + |v_n + Q^\top v_{n+1}|\xi_n\right|^{-p}.$$

As it holds for all $0 < p < 1$ and both sides are clearly analytic in p wherever the expectations exists, so in $\{p \in \mathbb{C},\ \mathrm{Re}(p) < 3\}$ (as follows, e.g. from Morera's theorem by a standard argument), the same identity continues to hold for all $0 < p < 3$. Conditioned on the value of Q, the inductive hypothesis applied to the $n-1$ nonnegative numbers $|v_2|, \dots, |v_{n-1}|, |v_n + Q^\top v_{n+1}|$ yields

$$\mathbb{E}\left|\xi_1 + \sum_{k=2}^{n+1} a_k\xi_k\right|^{-p} \leqslant \mathbb{E}_Q C_\infty(p)\Phi_p\left(|v_2|^2 + \cdots + |v_{n-1}|^2 + |v_n + Q^\top v_{n+1}|^2\right).$$

Note that

$$|v_2|^2 + \cdots + |v_{n-1}|^2 + |v_n \pm Q^\top v_{n+1}|^2 = x \pm 2\left\langle v_n, Q^\top v_{n+1}\right\rangle,$$

so thanks to the symmetry of the distribution of Q, we can rewrite the right hand side as

$$C_\infty(p)\mathbb{E}_Q\frac{\Phi_p(x + 2\left\langle v_n, Q^\top v_{n+1}\right\rangle) + \Phi_p(x - 2\left\langle v_n, Q^\top v_{n+1}\right\rangle)}{2}.$$

The proof of the inductive step now follows from the following *extended* concavity property of Φ_p applied to $a_\pm = x \pm 2\left\langle v_n, Q^\top v_{n+1}\right\rangle$. $\square$

Lemma 33. *Let $p > 0$. For every $a_-, a_+ \geqslant 0$ with $\frac{a_- + a_+}{2} \leqslant 1$, we have*

$$\frac{\Phi_p(a_-) + \Phi_p(a_+)}{2} \leqslant \Phi_p\left(\frac{a_- + a_+}{2}\right).$$

Proof. This is Lemma 20 in [10] (stated there for *no* reason only for $0 < p < 1$, as the proof works for every $p > 0$ because it only uses the convexity of ϕ_p). □

4.3. **Step 4: Projection.** Let us say that $a_1 = \max_{k \leqslant n} |a_k|$, so $a_1 > \sqrt{\frac{10}{13}}$. *Projecting* onto this coefficient, that is applying Corollary 17 to $a = a_1$ and $v = \sum_{k=2}^n a_k \xi_k$ (conditioning on its value), we get

$$\mathbb{E}\left|\sum_{k=1}^n a_k \xi_k\right|^{-p} \leqslant a_1^{-p} \leqslant \left(\frac{13}{10}\right)^{p/2} \leqslant 2^{p/2}\Gamma\left(2 - \frac{p}{2}\right) = C_\infty(p),$$

where the last inequality results from Lemma 31 (applied to $q = p/2$). This finishes the proof of Theorem 8. □

5. End of the proof of Theorem 9

To finish the proof of Theorem 9, we only need to show here Steps 1 and 2 from Section 2.3.2.

5.1. **Step 1: Integral inequality.** Lemma 26 gives the desired claim.

5.2. **Step 2: Induction.** We repeat the entire inductive argument from Section 4.2 verbatim, replacing $\frac{1}{4} \leqslant p \leqslant 2$ with $2 < p < 3$ and $C_\infty(p)$ with $C_2(p)$. The only modification required is to check the inductive base which now amounts to verifying that

$$\mathbb{E}|\xi_1 + \sqrt{t}\xi_2|^{-p} \leqslant C_2(p)\Phi_p(t) = C_2(p)(2^{1-p/2} - (3-t)^{-p/2}), \qquad 0 \leqslant t \leqslant 1.$$

By Lemma 13, the left hand side is clearly increasing in t (when $2 < p < 3$ and $d = 4$ all the coefficients in the power series expansion therein are positive), whereas the right hand side is clearly decreasing in t. By the definition of $C_2(p)$, there is equality at $t = 1$. This finishes the whole proof.

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Appendix: Behaviour of the constants

We sketch an argument of the following proposition which justifies (5).

Proposition 34. *For every $d \geqslant 1$, the equation $c_{d,2}(q) = c_{d,\infty}(q)$ has a unique solution $q = q_d^*$ in $(-(d-1), 2)$. Moreover, $c_{d,2}(q) < c_{d,\infty}(q)$ for $-(d-1) < q < q_d^*$ and $c_{d,2}(q) > c_{d,\infty}(q)$ for $q_d^* < q < 2$. For $d \geqslant 5$, we have $q_d^* \in (-(d-1), -(d-2))$.*

Proof. Since the cases $1 \leqslant d \leqslant 4$ have been explicitly dealt with (see the discussion in the introduction), it is enough to analyse the case $d \geqslant 5$. Moreover, by the Schur-concavity result of [4] and [29], $c_{d,\infty}(q) < c_{d,2}(q)$ for every $0 < q < 2$, so we can further assume that $-(d-1) < q < 0$. We look into the sign of

$$h_d(q) = \log(c_{d,2}(q)^q) - \log(c_{d,\infty}(q)^q),$$

which can be equivalently recast as

$$h_d(q) = \log\left(2^{-\frac{q}{2}}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma(d+q-1)}{\Gamma\left(\frac{d+q}{2}\right)\Gamma\left(d+\frac{q}{2}-1\right)}\right) - \log\left(\left(\frac{2}{d}\right)^{q/2}\frac{\Gamma\left(\frac{d+q}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right)$$
$$= -q\log 2 + \frac{q}{2}\log d + \log\left(\frac{\Gamma\left(\frac{d}{2}\right)^2\Gamma(d+q-1)}{\Gamma\left(\frac{d+q}{2}\right)^2\Gamma\left(d+\frac{q}{2}-1\right)}\right).$$

Writing $x = \frac{q+d-1}{2} \in \left(0, \frac{d-1}{2}\right)$ and $\tilde{h}_d(x) = h_d(2x+1-d)$, we get (using the Legendre duplication formula $\Gamma(2x)\sqrt{\pi} = 2^{2x-1}\Gamma(x)\Gamma(x+1/2)$) that

$$\tilde{h}_d(x) = x\log d + \log\left(\frac{\Gamma(x)}{\Gamma\left(x+\frac{1}{2}\right)\Gamma\left(x+\frac{d-1}{2}\right)}\right) + \log\left(\frac{2^{d-2}\Gamma\left(\frac{d}{2}\right)^2}{\sqrt{\pi}d^{\frac{d-1}{2}}}\right).$$

We now make the following claims.

Claim 1. For all $0 < x < 1$, $\tilde{h}_d''(x) > 0.06$.

Claim 2. For every $d \geqslant 5$, $\inf_{1<x<\frac{d-1}{2}} \tilde{h}_d'(x) > 0$.

Claim 3. $\tilde{h}_d(\frac{1}{2}) < 0$.

The strict convexity from Claim 1, the simple observation that $\tilde{h}_d(0+) = +\infty$ and Claim 3 give that that $\tilde{h}_d$ has a unique zero, say x_0 in $(0, \frac{1}{2})$, is positive on $(0, x_0)$ and negative on $(x_0, \frac{1}{2})$. Claim 2 and the simple observation that $\tilde{h}_d(\frac{d-1}{2}) = 0$ gives that $\tilde{h}_d$ is negative on $\left[1, \frac{d-1}{2}\right)$. Convexity also gives that $\tilde{h}_d$ is negative on $(\frac{1}{2}, 1)$, for $h_d(\frac{1}{2})$ and $h_d(1)$ are negative. These give the desired behaviour of $c_{d,2}(q) - c_{d,\infty}(q)$ for $-(d-1) < q < 0$. Finally, it also follows from Claim 2 that $h_d'(0) > 0$ which gives $c_{d,2}(0) - c_{d,\infty}(0) > 0$. It remains to prove the claims. $\square$

Proof of Claim 1. Differentiating twice yields

$$\tilde{h}_d''(x) = \sum_{n=0}^{\infty}\left(\frac{1}{(x+n)^2} - \frac{1}{\left(x+n+\frac{1}{2}\right)^2} - \frac{1}{\left(x+n+\frac{d-1}{2}\right)^2}\right).$$

Note that the first two terms make up a decreasing function, thus for $0 < x < 1$ and $d \geqslant 5$ the right hand side is greater than

$$\sum_{n=0}^{\infty}\left(\frac{1}{(1+n)^2} - \frac{1}{\left(1+n+\frac{1}{2}\right)^2} - \frac{1}{(n+2)^2}\right) = 5 - \frac{\pi^2}{2} > 0,$$

which proves the claim. $\square$

Proof of Claim 2. Differentiating once yields

$$\tilde{h}_d'(x) = \log d + \left(\psi(x) - \psi\left(x+\frac{1}{2}\right)\right) - \psi\left(x+\frac{d-1}{2}\right)$$

where $\psi = (\log \Gamma)'$ as usual denotes the digamma function. By the well-known inequality $\psi(u) \leqslant \log u - \frac{1}{2u}$, $u > 0$ (see, e.g. 6.3.21 in [1]), we obtain

$$\tilde{h}_d'(x) \geqslant \log d - \log\left(x + \frac{d-1}{2}\right) + \frac{1}{2x+d-1} - \left(\psi\left(x+\frac{1}{2}\right) - \psi(x)\right).$$

Put $y = \frac{d-1}{2}$ and call the right hand side $F(x,y)$. Note that for every fixed $x > 1$,

$$\frac{\partial F}{\partial y}(x,y) = \frac{1}{y+1/2} - \frac{1}{x+y} - \frac{1}{2}\frac{1}{(x+y)^2} > \frac{1}{y+1/2} - \frac{1}{1+y} - \frac{1}{2}\frac{1}{(1+y)^2}$$

which is clearly positive for all $y > 0$. Therefore, for all $1 < x < y$,

$$\tilde{h}_d'(x) \geqslant F(x,y) > F(x,x).$$

It remains to prove that $f(x) = F(x,x) > 0$ for every $x > 1$. We have,

$$f(x) = \left(\log\left(1 + \frac{1}{2x}\right) + \frac{1}{4x}\right) - \left(\psi\left(x + \frac{1}{2}\right) - \psi(x)\right).$$

Note that each bracket is a decreasing function in x (for the second one, e.g. by taking the derivative). Thus, crudely, for $1 < x < 1.07$,

$$f(x) > \left(\log\left(1 + \frac{1}{2\cdot 1.07}\right) + \frac{1}{4\cdot 1.07}\right) - \left(\psi(1) - \psi\left(1 + \frac{1}{2}\right)\right) > 0.003.$$

For $x \geqslant 1.07$, using again $\psi(x+1/2) \leqslant \log(x+1/2) - \frac{1}{2x+1}$ as well as $\psi(x) \geqslant \log(x+1/2) - \frac{1}{x}$ (see [17]), we get

$$f(x) \geqslant \log\left(1 + \frac{1}{2x}\right) + \frac{1}{4x} - \left(\frac{1}{x} - \frac{1}{2x+1}\right)$$

It is elementary to verify that the right hand side is positive for $x \geqslant 1.07$ (it is in fact unimodal, e.g. by analysing its derivative). □

Proof of Claim 3. We have, $\tilde{h}_d\left(\frac{1}{2}\right) = \log\left(2^{d-2}d^{1-d/2}\Gamma\left(\frac{d}{2}\right)\right)$. Letting $u = d/2 \geqslant 5/2$ and using [23], we get

$$\tilde{h}_d\left(\frac{1}{2}\right) < \log\left(\sqrt{2\pi}2^{u-1}e^{-u+\frac{1}{12u}}u^{1/2}\right) \leqslant \log\left(\sqrt{2\pi}2^{u-1}e^{-u+\frac{1}{30}}u^{1/2}\right).$$

Denoting the right hand side by $f(u)$, we see that f is strictly concave. Since $f'(5/2) < -0.1$, f is decreasing for $u \geqslant 5/2$. Thus $f(5/2) < -0.04$ finishes the argument. □

Remark 35. We have,

$$q_d^* = -(d-1) + O(d)\exp\left(-\frac{1-\log 2}{2}d\right), \qquad d \to \infty. \tag{35}$$

As before, by Claim 1, to show $q_d^* < -(d-1) + 2\alpha_d$ for some $\alpha_d > 0$, it suffices to check that $\tilde{h}_d(\alpha_d) < 0$. With the choice of α_d from (35), this follows by using [23] and a simple bound $\Gamma(u) < \frac{1}{u}$, $0 < u < 1$.

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